The Problem of 36 Officers, Euler's Conjecture and Related Matters

Aloke Dey

NASI Senior Scientist Platinum Jubilee Fellow
Indian Statistical Institute, New Delhi

Orthogonal Latin Squares

A Latin square of order (or, side) s is an $s \times s$ matrix (array) with entries from a set of $s \geq 2$ distinct symbols (or, letters) such that each symbol appears in each row and each column precisely once. Two Latin squares of the same order are said to be orthogonal to each other if, when any of the squares is superimposed on the other, every ordered pair of symbols appears exactly once.
For example, consider the following pair of Latin squares of order $s = 4$:

$$L_1 = \begin{array}{cccc} A & C & D & B \\ B & D & C & A \\ C & A & B & D \\ D & B & A & C \end{array}, \quad L_2 = \begin{array}{cccc} \alpha & \delta & \beta & \gamma \\ \beta & \gamma & \alpha & \delta \\ \gamma & \beta & \delta & \alpha \\ \delta & \alpha & \gamma & \beta \end{array}.$$

Superimposing L_2 over L_1, one gets the following arrangement:

$$L = \begin{array}{cccc} A\alpha & C\delta & D\beta & B\gamma \\ B\beta & D\gamma & C\alpha & A\delta \\ C\gamma & A\beta & B\delta & D\alpha \\ D\delta & B\alpha & A\gamma & C\beta \end{array}.$$

Clearly, L_1 and L_2 are orthogonal to each other, because in L, each Latin alphabet appears with each Greek alphabet exactly once.
An arrangement like L is now called an Eulerian square, named after the legendary mathematician Leonhard Euler (1707–1783), who studied such objects in 1782 and also made a famous conjecture about their existence. Eulerian squares are also called Graeco-Latin squares in Statistics literature.

If in a set of Latin squares every pair is orthogonal, then the set is said to form a set of mutually orthogonal Latin squares (MOLS).
The number of MOLS of order s is bounded above by $s - 1$ and if this upper bound is attained, we say that there is a complete set of MOLS.
A complete set of MOLS of order s can be constructed if s is a prime or a prime power, i.e., if $s = p^q$ where p is a prime number and $q \geq 1$ is an integer. Such a complete set of MOLS was constructed by R. C. Bose (1938) and independently, by W. L. Stevens (1939).
It is not known yet whether the above condition, viz., s is a prime or prime power for the existence of a complete set of MOLS of order s, is necessary as well.

Clearly, in order to construct an Eulerian square, one needs a pair of orthogonal Latin squares. One of the most intriguing questions regarding orthogonal Latin squares is:

Can one construct a pair of orthogonal Latin squares of order s for every integer $s > 2$?

(One can easily see that such a pair does not exist for $s = 2$.)

In 1779, Euler started looking at the problem of finding Eulerian squares of every order. In fact, in his 1779 paper (which was published in 1782), Euler was able to construct an Eulerian square of every order s, where s is (i) either an odd integer or, (ii) a multiple of 4. Thus, the existence of Eulerian squares of all orders s where $s \equiv 0, 1, \text{ or } 3 \pmod 4$ was settled by Euler in 1782. The only case not settled till then was for orders $s \equiv 2 \pmod 4$. This brings us to the problem of 36 officers.

The Problem of 36 Officers

Here is the statement of the problem.

There are 36 army officers, 6 from each rank and 6 from each regiment. Is it possible to arrange these 36 officers in a 6×6 square arrangement such that each rank and each regiment shows up in each row and each column?

How did this problem arise in the first place? Folklore has the following 'explanation':

"It appears that the Emperor was to visit a garrison town in which six regiments were quartered and the commandant took into his head to arrange 36 officers in a square, one of each rank from each regiment, so that, whichever row or column the Emperor walked along, he would meet one officer of each of the six ranks and one from each of the six regiments".

In the IMS Bulletin of 1987, at the initiative of the then Editor, George Styan, a prize was offered for the first correct (or most plausible) answers to the following questions:

- Who was the Emperor?
- Which was the garrison town?
- Who was the commandant?

Three responses were received to these questions and the prize was given to S. C. Pearce who stated that Joseph II was the Emperor! The garrison town was probably St. Petersburg. Nothing is known about the commandant!

The commandant, of course, had set himself an impossible task as, the solution to the problem is provided by a 6×6 Eulerian square, which was later shown to be non-existent. Euler (1782) himself could not find an Eulerian square of order 6; he proceeded to 'show' the non-existence of such a square using an argument that is not entirely correct in method but correct in its conclusion. Having failed to construct an Eulerian square of order 6, Euler went on to make the following conjecture.

EULER'S CONJECTURE: There does not exist a pair of orthogonal Latin squares of order $s \equiv 2 (\mathrm{mod}\ 4)$, or equivalently, no Eulerian square of order $s \equiv 2 \pmod 4$ exists.

G. Tarry in 1900, by an exhaustive and laborious search showed the impossibility when $s = 6$. A shorter proof of the non-existence of an Eulerian square of order 6, based on coding theory was given by D. R. Stinson (1984). J. Peterson (1901) and P. Wernicke (1910) made erroneous attempts to prove Euler's conjecture as did MacNeish (1922). The arguments used by Peterson and MacNeish were shown to be false by F. W. Levi (1942) and the falsity of Wernicke's argument was shown by MacNeish (1922). Two leading statisticians, R. A. Fisher and F. Yates, in 1934 published a list of all possible Latin squares of order 6 and concluded as below:

(Fisher & Yates, 1934) Euler's conclusion that no Graeco-Latin 6×6 square exists is easily verified from the 12 types of 6×6 Latin squares exemplified in this paper.

The MacNeish-Mann Conjecture

For an integer s, let $N(s)$ denote the maximum number of MOLS of order s. Then, as seen earlier, $N(s) = s - 1$, if s is a prime or a prime power. A challenging problem is to determine the value of $N(s)$ when s is neither a prime nor a prime power. One of the earliest results in this direction is due to H. F. MacNeish (1922); this was generalized somewhat and put on an algebraic foundation by H. B. Mann (1942). Let $s = p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m}$ be the prime-power decomposition of s, where $p_1, \ldots, p_m$ are distinct primes and $n_1, \ldots, n_m$ are positive integers. Define

$$n(s) = min\{p_1^{n_1}, p_2^{n_2} \ldots, p_m^{n_m}\} - 1.$$

MacNeish (1922) showed that $N(s) \geq n(s)$. MacNeish went further to conjecture that $n(s)$ is also the upper bound on $N(s)$ and therefore, $N(s) = n(s)$. This is the MacNeish-Mann conjecture.

Note that had the MacNeish-Mann conjecture been true, it would have shown the truth of Euler's conjecture as, by the MacNeish-Mann conjecture, $N(s) = 1$ if $s \equiv 2 \pmod 4$. However, E. T. Parker (1959a) showed that the MacNeish-Mann conjecture is false.

The first result casting serious doubts on the truth of Euler's conjecture is due to R. C. Bose and S. S. Shrikhande (1959) who were able to construct an Eulerian square of order $s = 22$. In the same year, Parker (1959b) proved another result, an application of which yielded a pair of orthogonal Latin squares of order 10 (or, an Eulerian square of order 10). This is shown next, where the symbols of both the Latin squares are $0, 1, 2, \ldots, 9$.

A Pair of Orthogonal Latin squares of order 10

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

The above Eulerian square of order 10 was first obtained by Parker using a UNIVAC computer and appears to be the first attempt to use computers for solving a combinatorial problem.

Once Eulerian squares of orders 10 and 22 were found, more doubts about the validity of Euler's conjecture arose as both 10 and 22 are congruent to 2 mod 4. Further results casting serious doubts on the truth of Euler's conjecture were provided by Bose and Shrikhande (1960). That the Euler's conjecture is false for *all* orders $s = 4t + 2,\ t > 1$ was shown by Bose, Shrikhande and Parker (1960). Their result is stated below.

There exists at least two MOLS of side $s \equiv 2 \pmod 4, s \neq 6$.

Coupling the above result with those of Euler (1782), one has the following result.

There exists at least two MOLS of side $s > 2, s \neq 6$.

The methods of Bose, Shrikhande and Parker to prove the falsity of Euler's conjecture used, among other things, a combinatorial arrangement called balanced incomlete block designs. Let $\mathcal{V}$ be a finite set of v objects (or, treatments, using the terminology of statistical design of experiments) and $\mathcal{B}$, a collection of k-subsets of $\mathcal{V}$, where $2 \leq k < v$; these subsets are called blocks. The pair $(\mathcal{V}, \mathcal{B})$ is a balanced incomplete block (BIB) design if (i) every treatment appears in r blocks and (ii) each pair of treatments occurs together in λ blocks. If $|\mathcal{B}| = b$, where $|\cdot|$ denotes the cardinality of a set, then the integers v, b, r, k, λ are called the parameters of a BIB design.

Although BIB designs were first used as experimental designs in 1936, such objects were known even in the 19th century. Kirkman (1850) solved the following problem, originally proposed by Woolhouse (1844):

A school mistress is in the habit of taking 15 girls of her school for a daily morning walk in 5 batches of 3 girls each, so that each girl has 2 companions. Is it possible to find an arrangement so that for 7 consecutive days, no girl walks with any of her companions in any batch more than once?

The solution of the above problem (called the Kirkman's schoolgirl problem) has a one-one correspondence with the solution of a BIB design and such a BIB design is also called a Kirkman Triple System, KTS(15). A KTS(15) is shown next, where the schoolgirls are labeled $1, 2, \ldots, 15$:

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
1,6,11	1,8,10	1,3,9	1,2,5	2,3,6	1,7,14	1,12,13
2,7,12	2,9,11	2,13,14	3,10,12	5,7,13	3,5,11	2,4,10
3,8,13	3,4,7	4,5,8	4,11,13	8,9,12	4,6,12	5,6,9
4,9,14	5,12,14	6,7,10	6,8,14	10,11,14	9,10,13	7,8,11
5,10,15	6,13,15	11,12,15	7,9,15	1,4,15	2,8,15	3,14,15

It is easily seen that the above plan is a BIB design with parameters $v = 15, b = 35, r = 7, k = 3, \lambda = 1$ when triplets of girls are treated as blocks. A solution of Kirkman's Triple System KTS (m) for all $m \equiv 3 \pmod 6$ was provided by Raychoudhuri and Wilson (1971).

J. Steiner (1853) proposed the problem of arranging n objects in triplets (called Steiner's triple systems) such that every pair of objects appears in exactly one set. We now recognize that such triples are in fact BIB designs with block size 3.

A Slice of History

The literature on Latin squares is at least three centuries old, one of the earliest references being a monograph *Koo-Soo-Ryak* by Choi Seok-Jeong (1646–1715), who used orthogonal Latin squares of order 9 to construct a magic square and stated that he cannot find orthogonal Latin squares of order 10. Recall that a (traditional) magic square of order $n \geq 2$ with magic constant $e = n(n^2+1)/2$ is an $n \times n$ matrix $A = (a_{ij})$ with entries $1, 2, \ldots, n^2$, such that:
(i) $\sum_{i=1}^{n} a_{ij} = e, 1 \leq j \leq n$,
(ii) $\sum_{j=1}^{n} a_{ij} = e, 1 \leq i \leq n$,
(iii) $\sum_{i=1}^{n} a_{ii} = e$, and
(iv) $\sum_{i=1}^{n} a_{i,(n-i+1)} = e$.

Orthogonal Latin squares can be used to construct magic squares. As an example, consider the following pair of orthogonal Latin squares of order 4, written with symbols 0, 1, 2, 3:

$$L_1 = \begin{matrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{matrix}, \quad L_2 = \begin{matrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \end{matrix}.$$

Superimposing one square over the other, one gets the following square:

$$L = \begin{matrix} 00 & 11 & 22 & 33 \\ 23 & 32 & 01 & 10 \\ 31 & 20 & 13 & 02 \\ 12 & 03 & 30 & 21 \end{matrix}$$

Replacing the element (ij) of L by $4i + j + 1$, one obtains the following magic square of order 4 ($e = 34$):

$$M = \begin{array}{rrrr} 1 & 6 & 11 & 16 \\ 12 & 15 & 2 & 5 \\ 14 & 9 & 8 & 3 \\ 7 & 4 & 13 & 10 \end{array}$$

Euler's interest in this area also probably originated from the connection of Eulerian squares to magic squares. Euler, in a paper entitled "De quadratis magicis" and read before the Academy of Sciences at St. Petersburg on October 17, 1776, constructed magic squares of orders 3, 4 and 5 from orthogonal Latin squares. He could not construct an Eulerian square of order 6 which prompted him to make his conjecture.

For over a century, no progress was made on Euler's conjecture, though it was not totally neglected by mathematicians of that time. In 1842, Gauss and Schumacher corresponded about a work of Clausen, who apparently established the impossibility of an Eulerian square when $s = 6$ and conjectured the impossibility when $s = 2 \pmod 4$. This work was never published!

Latin square amulets go back to medieval Islam (c1200) and a magic square of the famous Arab sufi, Ahmad ibn Ali ibn Yusuf al-Buni indicates the knowledge of a pair of orthogonal Latin squares of order 4. A new edition of J. Ozanam's four-volume treatise "Récréations mathématiques et physiques ...", published in 1723 had the following puzzle:

There are 16 playing cards of four denominations, ace (A), king (K), queen (Q) and jack (J) from each of the four suits, spade, heart, diamond and club. Is it possible to arrange these 16 cards in a 4×4 square such that each denomination and each suit appears in each row, each column and (additionally) on the two diagonals exactly once?

Here is a solution to this problem.

A♣	Q♠	J♦	K♥
K♦	J♥	Q♣	A♠
Q♥	A♦	K♠	J♣
J♠	K♣	A♥	Q♦

It is not hard to see that the above solution is given by the Eulerian square of order 4, shown in the beginning of the talk.

In 1896, E. H. Moore published an influential paper "Tactical Memoranda I–III" in the American Journal of Mathematics. In Memorandum II of this paper, Moore used finite fields of order s to construct a complete set of MOLS of order s, a result rediscovered much later by Bose (1938) and independently by Stevens (1939). Clearly, neither Bose nor Stevens were aware of the work of Moore.

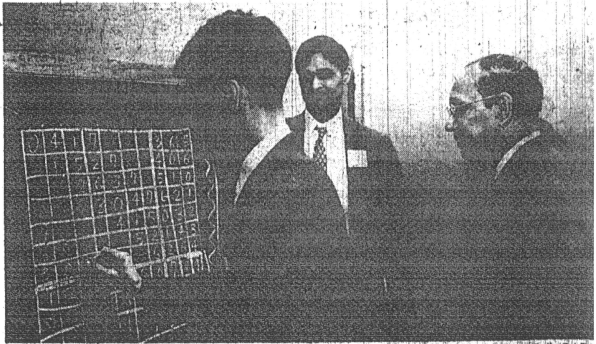
The final results of Bose, Shrikhande and Parker on the falsity of Euler's conjecture for all orders $s = 2 \pmod 4$, $s > 6$, were announced in the annual meeting of the American Mathematical Society, held in New York during the last week of April, 1959. This major result was reported on the front page of the Sunday edition of the New York Times of April 26, 1959 with the header

Major Mathematical Conjecture Propounded 177 Years Ago Is Disproved.

The New York Times in the report also made the following remark:

The three mathematicians who finally cracked the problem are now known among their colleagues as EULER'S SPOILERS.

Major Mathematical Conjecture Propounded 177 Years Ago Is Disprove



The New York Tim

Discussing their solution to problem, from left: Dr. E. T. Parker, Prof. S. S. Shrikhande and Prof. R. C. Bos

By JOHN A. OSMUNDSEN

Another major mathematical problem—this one 177 years old—has been solved. Its solution was reported at the 557th meeting of the American Mathematical Association, which ended at the New Yorker Hotel yesterday. It was the second such achievement to come out of the meeting, something attending mathematicians called "extremely rare." The solution to the first problem, known as Frobenius' conjecture, was reported by Prof. John G. Thompson, a 26-year-old mathematician from DePaul University in Chicago. It dealt with so-called "group theory" and had puzzled mathematicians for more than fifty years. The second problem had resisted attempts at solution ever since Leonhard Euler (pronounced "oiler") stated it in a memoir in 178 It became famous as Euler conjecture. The three math maticians who finally cracke the problem are now know among their colleagues

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