Statistics and Applications {ISSN 2452-7395 (online)}
Volume 15 Nos. 1&2, 2017 (New Series), pp 117-146

Multi-Character Survey using Forced Quantitative Randomized Response Model with Two Independent Samples

Sukhjinder Singh Sidhu[1] **and Sarjinder Singh**[2]
[1]*Punjab Agricultural University, Ludhiana*
[2]*Texas A&M University—Kingsville, USA*

Received: January 16, 2017; Revised February 03, 2017; Accepted: February 14, 2017

Abstract

The forced quantitative randomized response (FQRR) model with two independent samples is not limited by the need to know the value of the forced question. FQRR is a maximally efficient randomized response design for estimating the population total of a sensitive variable. This paper proposes a set of alternative estimators for probability proportional to size with replacement (PPSWR) in multi-character survey, which elicits simultaneous information on multiple sensitive study variables. The proposed estimators are all minimally biased because they are suitable for situations between the optimum for usual estimators and the optimum for estimators based on multi-characters with no correlation. The Mean Square Error (MSE) expressions for the proposed estimators were derived under optimal sample size. The behavior of the proposed estimators was examined under super population model. Ann empirical study was also carried out to reveal the performance of the proposed estimators.

Keywords: Total estimation, Randomized response, Forced response, Sensitive multi-characteristics, Super population model, Mean square error.

1. Introduction

Warner (1965) developed an interviewing procedure designed to reduce or eliminate bias and called it randomized response technique (RRT). The use of a RRT protects against privacy violation, thereby diminishing the respondent's need to give socially desirable answers. Meta-analysis of 42 comparative studies showed that randomized response conditions resulted in more valid population estimates than direct question–answer conditions, where direct question–answer conditions is the umbrella term for research methods in which the (sensitive) question is asked directly of the respondent (Lensvelt-Mulders et al., 2005).

Eichhorn and Hayre (1983) proposed an ingenious method to collect information on quantitative characters rather than qualitative characters. According to this model, each respondent in the sample is requested to give the scrambled response $r_i = Y_i S$, where the Y_i are the real values of the quantitative variable and S is the scrambling variable. The forced qualitative model randomized models due to Liu and Chow (1976 a, b) and Stem and Steinhorst (1984) are special cases of the situation where the value of the forced quantitative randomized response is simply replaced by a forced 'yes' response.

Corresponding Author: Sarjinder Singh
E-mail: *dasarjinder.singh@tamuk.edu*

In the method of Bar-Lev, Bobovitch, and Boukai (2004), each respondent is required to rotate a spinner unobserved by the interviewer. If the spinner stops in the shaded area, the respondent reports the true response on the sensitive variable, say Y_i. If the spinner stops in the non-shaded area, the respondent reports the scrambled response, say Y_iS, where S is any scrambling variable of known distribution. The Bar-Lev, Bobovitch, and Boukai (2004) model assumes that $E(S)=\theta$ and $V(S)=\gamma^2$ are known.

Odumade and Singh (2008) generalized the forced quantitative randomized response model (GFQRR) of Gjestvang and Singh (2007) for estimating the population total of a sensitive variable under a unified setup. Each respondent in the simple random with replacement sample is provided with a randomization device, like a spinner, bearing three types of statements:

(i) Report your true income, say, X_i;

(ii) Report the scrambled response, X_iZ; and

(iii) Report a fixed value already printed on the card, say F, with proportions p_1, p_2, and p_3 respectively, such that $p_1+p_2+p_3=1$, and Z is a scrambling variable whose distribution is assumed known.

Odumade and Singh (2008) showed that the Bar-Lev, Bobovitch, and Boukai (2004), Eichhorn & Hayre (1983), and Liu & Chow (1976 a, b) models are special cases of the GFQRR model.

Odumade and Singh (2008) also newly considered two independent simple random samples with replacement sampling of sizes n_1 and n_2 such that $n_1+n_2=n$. Each respondent in the first simple random with replacement sample is provided with a randomization device, like a spinner, bearing three types of statements:

(i) Report your true income, say X_i;

(ii) Report the scrambled response, say X_iZ_1; and

(iii) Report a fixed value already printed on the card, say F_1, with proportions p_1, p_2 and p_3 respectively, such that $p_1+p_2+p_3=1$.

Each respondent in the second independent and non-overlapping simple random with replacement sample is provided with a randomization device, like a spinner, bearing three types of statements:

(i) Report your true income, say, X_i;

(ii) Report the scrambled response, say, F_1 ; and

(iii) Report a fixed value already printed on the card, say F_2 with proportions p_4, p_5 and p_6 respectively, such that $p_4+p_5+p_6=1$, and $p_3F_1=p_6F_2$, where Z_1 and Z_2 are any scrambling variables with known distributions.

Some of the study variables may be poorly correlated with the selection probabilities in sample surveys with many variables. In these cases, commonly used estimators available in the literature result in large variance. Rao (1966) provided alternative estimators when the study variable and size measure are unrelated and demonstrated that these alternative estimators are more efficient though biased. The Rao (1966) model is not commonly encountered in practice since the correlation is not always zero. Bansal and Singh (1985) developed a transformed estimator of population total suitable for the characteristics covering entire range of positive correlation. Amahia et al. (1989) suggested simple alternatives to the transformations in Bansal and Singh (1985).

Observing the simplicity and wide application of Greenberg et al. (1971), Sidhu et al. (2007, 2009) proposed a set of alternative estimators for probability proportional to size with replacement (PPSWR) corresponding to multi-character survey, which elicit simultaneous information on multiple sensitive study variables when the value of the unrelated question is known or unknown in advance. The PPSWR sampling scheme is more efficient than simple random sampling (SRS) when the correlation between the study variable and auxiliary variable is positive and high and the regression line passes through the origin. The above estimators are suitable when variables under study have high and positive correlation with selection probabilities. However, in case of sensitive multi-characteristics, when some have moderate and others have very low correlation with selection probabilities, multiple characteristic and RRT models need to be combined.

2. Proposed Strategy and Estimator

Let a finite population of N units denoted by $\Omega = (U_1, U_2, ..., U_N)$. Let Y be the sensitive variable and x be an auxiliary variable taking a known positive value x_i on Ω such that $X = \sum_{i=1}^{N} x_i$. Suppose a sample of n units is chosen using PPSWR sampling. Two independent and non-overlapping simple random samples using with replacement sampling of sizes n_1 and n_2 such that $n_1 + n_2 = n$ are chosen such that the estimator is not limited by the need to know the value of the forced question. S_1 and S_2 are any scrambling variables with known distributions. $E(S_1) = \theta_1$, $E(S_2) = \theta_2$, $V(S_1) = \gamma_1^2$ and $V(S_2) = \gamma_2^2$ are known.

Each respondent in the first simple random with replacement sample is provided with a randomization device, say, a deck-D1 of cards, bearing three types of statements:

(i) Report your true income, say, Y_i;

(ii) Report the scrambled response, say, $Y_i S_1$; and

(iii) Report a fixed value already printed on the card, say F_1, with proportions t_1, t_2 and t_3 respectively, such that $t_1 + t_2 + t_3 = 1$.

Mathematically, the distribution of the responses from the first sample is given by

$$Z_{1i} = \begin{cases} Y_i, & \text{with probability} = t_1 \\ Y_i S_1, & \text{with probability} = t_2 \\ F_1, & \text{with probability} = t_3 \end{cases} \tag{2.1}$$

Each respondent in the second independent and non-overlapping simple random with replacement sample is provided with a randomization device, say, a deck-D2 of cards, bearing three types of statements:

(i) Report your true income, say, Y_i;

(ii) Report the scrambled response, say, $Y_i S_2$; and

(iii) Report a fixed value already printed on the card, say, F_2, with proportions, t_4, t_5, and t_6, respectively, such that $t_4 + t_5 + t_6 = 1$.

Mathematically, the distribution of the responses from the second sample is given by

$$Z_{2i} = \begin{cases} Y_i, & \text{with probability} = t_4 \\ Y_i S_2, & \text{with probability} = t_5 \\ F_2, & \text{with probability} = t_6 \end{cases} \quad (2.2)$$

In this model F_1 and F_2 are fixed values such that $t_3 F_1 = t_6 F_2$. This model differs completely from existing RRT models because the randomization devices used in the two independent samples are dependent on each other. This model yields estimators:

$$E(Z_{1i}) = t_1 Y_i + t_2 Y_i \theta_1 + t_3 F_1 \quad (2.3)$$

and

$$E(Z_{2i}) = t_4 Y_i + t_5 Y_i \theta_2 + t_6 F_2 . \quad (2.4)$$

Subtracting, (2.3) and (2.4) give:

$$E(Z_{1i}) - E(Z_{2i}) = Y_i \left[(t_1 - t_4) + (t_2 \theta_1 - t_5 \theta_2) \right] \quad (2.5)$$

The new estimators also take into account the already known rough value of the correlation coefficient ρ between the characteristic under study, Y, and the measure of size, p. Some variables under study may have low positive correlation with selection probabilities and others may have high correlation. Under this scheme, the proposed set of estimators of population total for multi-character survey based on the method of moments is

$$(\hat{Y}_{GS})_k = \frac{\frac{1}{n_1} \sum_{i=1}^{n_1} \frac{Z_{1i}}{p_{ik}^*} - \frac{1}{n_2} \sum_{i=1}^{n_2} \frac{Z_{2i}}{p_{ik}^*}}{(t_1 - t_4) + (t_2 \theta_1 - t_5 \theta_2)} , \quad (2.6)$$

where the p_{ik}^* are transformations of selection probabilities available in literature:

$$p_{i0}^* = \frac{1}{N}, \text{ Rao (1966);} \quad (2.7)$$

$$p_{i1}^* = \left(1 + \frac{1}{N}\right)^{(1-\rho)} \left(1 + p_i\right)^{\rho} - 1, \text{ Bansal and Singh (1985);} \quad (2.8)$$

$$p_{i2}^* = \frac{(1-\rho)}{N} + \rho p_i \text{ , Amahia et al. (1989);} \quad (2.9)$$

$$p_{i3}^* = \left(\frac{1}{N}\right)^{(1-\rho)} p_i^{\rho} \ , \quad (2.10)$$

$$p_{i4}^* = \left[N(1-\rho) + \frac{\rho}{p_i} \right]^{-1} , \quad (2.11)$$

and

$$p_{i5}^* = \frac{\left(1 - \sqrt[3]{\rho}\right)}{N} + \sqrt[3]{\rho}\, p_i \text{ , Grewal et al. (1997).} \quad (2.12)$$

For $\rho = 0$, the p_{ik}^* for $k = 0, 1, 2, 3, 4, 5$ reduce to p_{i0}^* at (2.3). For $\rho = 1$, p_{ik}^* reduces to original selection probabilities, p_i .

The proposed estimators are minimally biased and suitable for situations between the optimum for usual estimators and the optimum for estimators based on multi-characters with no correlation according to the following theorems.

Theorem 2.1: The bias expression for the set of estimators of population total

$$(\hat{Y}_{GS})_k = \frac{\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{Z_{1i}}{p_{ik}^*} - \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{Z_{2i}}{p_{ik}^*}}{(t_1-t_4)+(t_2\theta_1-t_5\theta_2)}$$

is given by

$$B(\hat{Y}_{GS})_k = \sum_{j=1}^{N}\left(\frac{p_i}{p_{ik}^*}-1\right)Y_i. \tag{2.13}$$

Proof: Let E_1 be the expected value over all possible samples and E_2 be the expected value over the randomized device. Then the expected value of $(\hat{Y}_{GS})_k$ is given by

$$E(\hat{Y}_{GS})_k = E_1E_2(\hat{Y}_{GS})_k$$

$$= E_1E_2\left[\frac{\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{Z_{1i}}{p_{ik}^*} - \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{Z_{2i}}{p_{ik}^*}}{(t_1-t_4)+(t_2\theta_1-t_5\theta_2)}\right], \quad = E_1\left[\frac{\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{E_2(Z_{1i})}{p_{ik}^*} - \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{E_2(Z_{2i})}{p_{ik}^*}}{(t_1-t_4)+(t_2\theta_1-t_5\theta_2)}\right]$$

$$= \left[\frac{(p_1+p_2\theta_1)\sum_{i=1}^{N}\frac{Y_ip_i}{p_{ik}^*} + t_3F_1\sum_{i=1}^{N}\frac{p_i}{p_{ik}^*} - (p_4+p_5\theta_2)\sum_{i=1}^{N}\frac{Y_ip_i}{p_{ik}^*} - t_6F_2\sum_{i=1}^{N}\frac{p_i}{p_{ik}^*}}{(t_1-t_4)+(t_2\theta_1-t_5\theta_2)}\right] = \sum_{i=1}^{N}\frac{Y_ip_i}{p_{ik}^*}.$$

Now, $B(\hat{Y}_{GS})_k = E(\hat{Y}_{GS})_k - Y \quad = \sum_{i=1}^{N}\left[\frac{p_i}{p_{ik}^*}-1\right]Y_i$,

hence the theorem.

Theorem 2.2: The variance of the estimator $(\hat{Y}_{GS})_k$ is given by

$$V(\hat{Y}_{GS}) = \frac{1}{\{(t_1-t_4)+(t_2\theta_1-t_5\theta_2)\}^2}\left[\frac{T_1}{n_1}+\frac{T_2}{n_2}\right], \tag{2.14}$$

where

$$T_1 = \sum_{i=1}^{N}\frac{\sigma_{Z1i}^2 p_i}{p_{ik}^{*2}} + (t_1+t_2\theta_1)^2\sum_{i=1}^{N}p_i\left(\frac{Y_i}{p_{ik}^*}-\sum_{i=1}^{N}\frac{Y_ip_i}{p_{ik}^*}\right)^2 + (t_3F_1)^2\sum_{i=1}^{N}p_i\left(\frac{1}{p_{ik}^*}-\sum_{i=1}^{N}\frac{p_i}{p_{ik}^*}\right)^2$$

$$+2(t_3F_1)((t_1+t_2\theta_1)\sum_{i=1}^{N}p_i\left(\frac{1_i}{p_{ik}^*}-\sum_{i=1}^{N}\frac{p_i}{p_{ik}^*}\right)\left(\frac{Y_i}{p_{ik}^*}-\sum_{i=1}^{N}\frac{Y_ip_i}{p_{ik}^*}\right), \tag{2.15}$$

and $$T_2 = \sum_{i=1}^{N}\frac{\sigma_{Z2i}^2 p_i}{p_{ik}^{*2}} + (t_4 + t_5\theta_2)^2 \sum_{i=1}^{N} p_i\left(\frac{Y_i}{p_{ik}^*} - \sum_{i=1}^{N}\frac{Y_i p_i}{p_{ik}^*}\right)^2 + (t_6F_2)^2\sum_{i=1}^{N} p_i\left(\frac{1}{p_{ik}^*} - \sum_{i=1}^{N}\frac{p_i}{p_{ik}^*}\right)^2$$

$$+ 2(t_6F_2)((t_4 + t_5\theta_2)\sum_{i=1}^{N} p_i\left(\frac{1_i}{p_{ik}^*} - \sum_{i=1}^{N}\frac{p_i}{p_{ik}^*}\right)\left(\frac{Y_i}{p_{ik}^*} - \sum_{i=1}^{N}\frac{Y_i p_i}{p_{ik}^*}\right). \quad (2.16)$$

Also, for $k = 0,1,2,3,4,5$,

$$\sigma_{Z_1i}^2 = E(Z_{1i}^2) - (E(Z_{1i}))^2 = t_1Y_i^2 + t_2Y_i^2\theta_1^2(1 + C_{\gamma1}^2) + t_3F_1^2 - [t_1Y_i + t_2Y_i\theta_1 + t_3F_1]^2$$

$$= \left[t_1 + t_2\theta_1^2(1 + C_{\gamma_1}^2) - (t_1 + t_2\theta_1)^2\right]Y_i^2 + t_3(1 - t_3)F_1^2 - 2t_3F_1(t_1 + t_2\theta_1)Y_i\,, \quad (2.17)$$

and $$\sigma_{Z_2i}^2 = E(Z_{2i}^2) - (E(Z_{2i}))^2 = t_4Y_i^2 + t_5Y_i^2\theta_2^2(1 + C_{\gamma2}^2) + t_6F_2^2 - [t_4Y_i + t_5Y_i\theta_2 + t_6F_2]^2$$

$$= \left[t_4 + t_5\theta_2^2(1 + C_{\gamma_2}^2) - (t_4 + t_5\theta_2)^2\right]Y_i^2 + t_6(1 - t_6)F_2^2 - 2t_6F_1(t_4 + t_5\theta_2)Y_i \quad (2.18)$$

Proof. Let E_1 and E_2 denote the expected values defined earlier, and let V_1 and V_2 be the corresponding variances. Then

$$V(\hat{Y}_{GS})_k = E_1V_2(\hat{Y}_{GS})_k + V_1E_2(\hat{Y}_{GS})_k\,,$$

so $$V(\hat{Y}_{GS})_k = E_1V_2\left(\frac{\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{Z_{1i}}{p_{ik}^*} - \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{Z_{2i}}{p_{ik}^*}}{(t_1 - t_4) + (t_2\theta_1 - t_5\theta_2)}\right) + V_1E_2\left(\frac{\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{Z_{1i}}{p_{ik}^*} - \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{Z_{2i}}{p_{ik}^*}}{(t_1 - t_4) + (t_2\theta_1 - t_5\theta_2)}\right)$$

$$= E_1\left(\frac{\frac{1}{n_1^2}\sum_{i=1}^{n_1}\frac{V_2(Z_{1i})}{p_{ik}^{*2}} - \frac{1}{n_2^2}\sum_{i=1}^{n_2}\frac{V_2(Z_{2i})}{p_{ik}^{*2}}}{\{(t_1 - t_4) + (t_2\theta_1 - t_5\theta_2)\}^2}\right) + V_1\left(\frac{\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{E_2(Z_{1i})}{p_{ik}^*} - \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{E_2(Z_{2i})}{p_{ik}^*}}{(t_1 - t_4) + (t_2\theta_1 - t_5\theta_2)}\right)$$

$$= \frac{1}{\{(t_1 - t_4) + (t_2\theta_1 - t_5\theta_2)\}^2}\left[\frac{1}{n_1}\sum_{i=1}^{N}\frac{V_2(Z_{1i})p_i}{p_{ik}^{*2}} + \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{V_2(Z_{2i})p_i}{p_{ik}^{*2}}\right.$$

$$\left. + (t_1 + t_2\theta_1)^2V_1\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{y_i}{p_{ik}^*}\right) + (t_3F_1)^2V_1\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{p_{ik}^*}\right) + 2(t_3F_1)(t_1 + t_2\theta_1)C_1\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{y_i}{p_{ik}^*}, \frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{p_{ik}^*}\right)\right]$$

$$= \frac{1}{\{(t_1 - t_4) + (t_2\theta_1 - t_5\theta_2)\}^2}\left[\frac{1}{n_1}\sum_{i=1}^{N}\frac{\sigma_{Z_1i}^2 p_i}{p_{ik}^{*2}} + \frac{1}{n_2}\sum_{i=1}^{N}\frac{\sigma_{Z_1i}^2 p_i}{p_{ik}^{*2}}\right.$$

$$+ \left\{\frac{(t_1 + t_2\theta_1)^2}{n_1} + \frac{(t_3 + t_4\theta_2)^2}{n_2}\right\}\sum_{i=1}^{N} p_i\left(\frac{Y_i}{p_{ik}^*} - \sum_{i=1}^{N}\frac{Y_i p_i}{p_i^*}\right)^2$$

$$+ \left\{\frac{(t_3F_1)^2}{n_1} + \frac{(t_6F_2)^2}{n_2}\right\}\sum_{i=1}^{N} p_i\left(\frac{1}{p_{ik}^*} - \sum_{i=1}^{N}\frac{p_i}{p_i^*}\right)^2$$

$$+2\left\{\frac{(t_3F_1)(t_1+t_2\theta_1)}{n_1}+\frac{(t_6F_2)(t_4+t_5\theta_2)}{n_2}\right\}\sum_{i=1}^{N}p_i\left(\frac{Y_i}{p_{ik}^*}-\sum_{i=1}^{N}\frac{Y_ip_i}{p_i^*}\right)\left(\frac{1}{p_{ik}^*}-\sum_{i=1}^{N}\frac{p_i}{p_i^*}\right)\Bigg]\Bigg],$$

which proves the theorem.

Note that the proposed estimator $(\hat{Y}_{GS})_k$ is free from both forced fixed quantitative responses F_1 and F_2, and hence it can be used in practice. For any survey, values of F_1 and F_2 can be fixed. Reporting a fixed value does not compromise the privacy of a respondent. For a given value of F_1 and F_2, this paper compares different transformations of the selection probabilities. The minimum variance is calculated based on the optimum values of the sample sizes n_1 and n_2, for simplicity.

Theorem 2.3: The minimum variance of the estimator $(\hat{Y}_{GS})_k$, for $k=0,1,2,3,4,5,$ is given by

$$Min\,V\left(\hat{Y}_{GS}\right)_k=\frac{\left(\sqrt{T_1}+\sqrt{T_2}\right)^2}{n\{(t_1-t_4)+(t_2\theta_1-t_5\theta_2)\}^2}. \quad (2.19)$$

Proof. Note that $n=n_1+n_2$, thus the optimum values of n_1 and n_2 are given by

$$\frac{dV(\hat{Y}_{GS})_k}{dn_1}=0, \text{ or equivalently,}$$

$$-\frac{T_1}{n_1^2}+\frac{T_2}{n_2^2}=0 \quad (2.20)$$

Thus, $\dfrac{n_1}{n_2}=\dfrac{\sqrt{T_1}}{\sqrt{T_2}}$. (2.21)

Substituting (2.21) into (2.20), we get (2.19), which proves the theorem.

3. Super Population Model

The efficiency comparison of the proposed estimators $(\hat{Y}_{SG})_k$ for $k=0,1,2,3,4,5$ in relation to those given by Rao (1966) for $k=0$, cannot be handled theoretically. Therefore, the expected values of the MSE of the estimators were found under the super-population model suggested by Cochran (1977). For this model:

$$Y_i=\beta p_i+e_i, \quad (3.1)$$

where the e_i, $i=1,2,...,N$, are the error terms such that

$$E(e_i/p_i)=0,\ E(e_ie_j/p_ip_j)=0, \text{ and } E(e_i^2/p_i)=ap_i^g; a>0, g\geq 0. \quad (3.2)$$

Here, the $E(e_i^2/p_i)$, $i=1,2,...,N$, are the residual variances of Y for given p_i. The expected value of the residual variance in the super population model is given by

$$E(ap_i^g)=\frac{a}{N}\sum_{i=1}^{N}p_i^g. \quad (3.3)$$

Now, $MSE\ (\hat{Y}_{GS})_k=V(\hat{Y}_{GS})_k+\left(B(\hat{Y}_{GS})_k\right)^2$ (3.4)

Taking the expected value under the model (3.1) on both sides of (3.4) yields

$$E_m\left[MSE(\hat{Y}_{GS})_k\right]=E_m\left[V(\hat{Y}_{GS})_k\right]+E_m\left[B(\hat{Y}_{GS})_k\right]^2$$

$$= \frac{\left[\sqrt{E_m(T_1)} + \sqrt{E_m(T_2)}\right]^2}{n\{(t_1 - t_4) + (t_2\theta_1 - t_5\theta_2)\}^2} + \beta^2\left(\sum_{i=1}^{N}\frac{p_i^2}{p_{ik}^*} - 1\right)^2 + a\sum_{i=1}^{N}\left(\frac{p_i}{p_{ik}^*} - 1\right)^2 p_i^g \quad , \tag{3.5}$$

where

$$E_m(T_1) = \sum_{i=1}^{N}\frac{E_m(\sigma_{Z1i}^2)p_i}{p_{ik}^{*2}} + (t_1 + t_2\theta_1)^2\left[\beta^2\left\{\sum_{i=1}^{N}\frac{p_i^3}{p_{ik}^{*2}} - \left(\sum_{i=1}^{N}\frac{p_i^2}{p_{ik}^*}\right)^2\right\} + a\sum_{i=1}^{N}\frac{p_i^{g+1}(1-p_i)}{p_{ik}^{*2}}\right]$$

$$+ (t_3F_1)\sum_{i=1}^{N}p_i\left(\frac{1}{p_{ik}^*} - \sum_{i=1}^{N}\frac{p_i}{p_{ik}^*}\right)^2 + 2(t_3F_1)(t_1 + t_2\theta_1)\beta\sum_{i=1}^{N}p_i\left(\frac{1}{p_{ik}^*} - \sum_{i=1}^{N}\frac{p_i}{p_{ik}^*}\right)\left(\frac{p_i}{p_{ik}^*} - \sum_{i=1}^{N}\frac{p_i^2}{p_{ik}^*}\right), \tag{3.6}$$

and

$$E_m(T_2) = \sum_{i=1}^{N}\frac{E_m(\sigma_{Z2i}^2)p_i}{p_{ik}^{*2}} + (t_3 + t_4\theta_2)^2\left[\beta^2\left\{\sum_{i=1}^{N}\frac{p_i^3}{p_{ik}^{*2}} - \left(\sum_{i=1}^{N}\frac{p_i^2}{p_{ik}^*}\right)^2\right\} + a\sum_{i=1}^{N}\frac{p_i^{g+1}(1-p_i)}{p_{ik}^{*2}}\right]$$

$$+ (t_6F_2)\sum_{i=1}^{N}p_i\left(\frac{1}{p_{ik}^*} - \sum_{i=1}^{N}\frac{p_i}{p_{ik}^*}\right)^2 + 2(t_6F_2)(t_3 + t_4\theta_2)\beta\sum_{i=1}^{N}p_i\left(\frac{1}{p_{ik}^*} - \sum_{i=1}^{N}\frac{p_i}{p_{ik}^*}\right)\left(\frac{p_i}{p_{ik}^*} - \sum_{i=1}^{N}\frac{p_i^2}{p_{ik}^*}\right), \tag{3.7}$$

with

$$E_m(\sigma_{Z1i}^2) = \left[t_1 + t_2\theta_1^2\left(1 + C_{\gamma1}^2\right) - \left(t_1 + t_2\theta_1\right)^2\right]\left[\beta^2 p_i^2 + ap_i^g\right] + t_3\left(1 - t_3\right)F_1^2 - 2t_3F_1\left(t_1 + t_2\theta_1\right)\beta p_i \tag{3.8}$$

and

$$E_m(\sigma_{Z2i}^2) = \left[t_4 + t_5\theta_2^2\left(1 + C_{\gamma2}^2\right) - \left(t_4 + t_5\theta_2\right)^2\right]\left[\beta^2 p_i^2 + ap_i^g\right] + t_6\left(1 - t_6\right)F_2^2 - 2t_6F_2\left(t_4 + t_5\theta_2\right)\beta p_i \,. \tag{3.9}$$

The expected value of the bias under the super population model is

$$E_m\left[B(\hat{Y}_{GS})_k\right] = \beta\sum_{i=1}^{N}\left(\frac{p_i}{p_{ik}^*} - 1\right)p_i \,. \tag{3.10}$$

The model expected relative bias is given by

$$E_m\left[RB(\hat{Y}_{GS})_k\right] = \left(\frac{\beta\sum_{i=1}^{N}\left(\frac{p_i}{p_{ik}^*} - 1\right)p_i}{\beta}\right)\times 100\% = \left(\sum_{i=1}^{N}\frac{p_i^2}{p_{ik}^*} - 1\right)\times 100\% \quad . \tag{3.11}$$

4. Simulation Study

The six different estimators were compared for several situations through simulation study with an empirical investigation under super population model. Note the relative efficiency depends on parameters g, ρ, $C_{\gamma1}$, $C_{\gamma2}$, θ_1,, θ_2, t_1, t_2, $t_3 = 1 - t_1 - t_2$, t_4, t_5, and $t_6 = 1 - t_4 - t_5$. The parameters t_1, t_2, t_4 and t_5 were chosen to ensure that t_3 and t_6 were positive and practical. In any real survey the coefficient of variations of the scrambling variables C_{γ_1} and C_{γ_2} are expected to lie between 0.1 and 0.9, but these were chosen close to 10% for consistent and practicable data sets, following Cochran (1977).

Literature search indicates the proportions t_j, $j = 1,2,3,4,5,6$, are best chosen in the same range as the expected prevalence of the sensitive topic in the population (Clark and Desharnais, 1998). These proportions cannot be varied indefinitely because the protection of privacy is in the core of the theory behind randomized response methods. When any t_j, $j = 1,2,3,4,5,6$, approaches one, anonymity protection becomes nil. Maintenance of privacy and efficiency of the design are therefore in conflict with each other (Chaudhuri and Mukerjee, 1988). As Soeken and McReady (1982) showed, however, one value of t_j can be chosen between 0.75 and 0.8 without interfering with the perceived grade of anonymity. The average t_j across studies was 0.67 (Lensvelt-Mulders et al., 2005).

An extra advantage of using a forced response method is that the perceived protection of the respondents can be manipulated. The values of θ_1 and θ_2 are chosen by the investigator based on the nature of the sensitive variable under study. In this study, $\theta_1 = 2$, $\theta_2 = 1.5$, $C_{\gamma 1} = \gamma_1 / \sqrt{\theta_1} = 0.1$,, $C_{\gamma 2} = \gamma_2 / \sqrt{\theta_2} = 0.2$, $t_1 = 0.65$, $t_2 = 0.25$, $t_3 = 0.1$, $t_4 = .25$, $t_5 = .65$ and $t_6 = 0.1$ were chosen by investigators. The forced response value $F_1 = 20$ was fixed, and the value of $F_2 = t_6^{-1} t_3 F_1$ was computed for the entire simulation study, although there is lot of flexibility in the choice of this parameter, depending on the survey.

$\text{NITR} = 500$ populations, each of size $N = 1000$ units, were generated from different distributions as shown in Table 4.1. The sample size $n = 50$ was fixed in the entire simulation study because the value of percent relative efficiency does not change much due change in sample size.

The percent relative efficiency of the proposed estimators, $(\hat{Y}_{GS})_k$ for $k = 1,2,3,4,5,6$ with respect to the modified Rao (1966) estimator for $k = 0$, were examined for different values of the correlation coefficient $\rho =$ 0.1, 0.3, 0.5, 0.7, 0.9; $g =$ 0, 1, 2; $\beta = 0.5$, 1.0, 1.5 and $a = 1.0$. The percent relative efficiency, $\text{RE}(k)$, $k = 1, 2, 3, 4, 5, 6$, of the k-th transformation with respect to the $k = 0$ transformation is given by

$$RE(k) = RE(0,k) = \frac{\sum_{IT=1}^{NITR} E_m\left[MSE(\hat{Y}_{GS})_0\right]_{IT}}{\sum_{IT=1}^{NITR} E_m\left[MSE(\hat{Y}_{GS})_k\right]_{IT}} \times 100 . \tag{4.1}$$

The averaged percent relative bias $RB(k)$, for $k = 1,2,3,4,5,6$, was computed as

$$RB(k) = \frac{1}{NITR} \sum_{IT=1}^{NITR} E_m\left[RB(\hat{Y}_{GS})_k\right]_{IT} \tag{4.2}$$

Different distributions, shown in Table 4.1, were used to compute the selection probabilities

$$p_i = \frac{x_i}{\sum_{i=1}^{N} x_i} . \tag{4.3}$$

Table 4.1: **Density Functions for Various Probability Distributions.**

Sr. No.	Distribution	Density function	Range	R-function
1	Uniform	$f(x)=\frac{1}{45}$	$5 \le x \le 50$	runif
2	Exponential	$f(x)=e^{-x}$	$0 \le x < \infty$	rexp
3	Chi-square with $\nu = 6$	$f(x)=\frac{1}{2^{\nu/2}\Gamma_{\nu/2}} e^{-x/2} x^{(\nu-2)/2}$	$0 \le x < \infty$	rgamma
4	Gamma with *p=2*	$f(x)=\frac{1}{\Gamma_p} e^{-x} x^{p-1}$	$0 \le x < \infty$	rgamma
5	Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2}$; $\mu = 2$, $\sigma = 0.3$	$-\infty < x < \infty$	rnorm
6	Log Normal	$f(x)=\frac{1}{x\sqrt{2\pi}} e^{-\{\log(x)\}^2/2}$	$0 < x < \infty$	rlnorm
7	Beta with p=3, q=2	$f(x)=\frac{1}{\beta(p,q)} x^{p-1}(1-x)^{q-1}$	$0 \le x \le 1$	rbeta

The results obtained from these computations for different sampling schemes are given in tables in Appendix A. The R-code used in the simulation study is given in the Appendix B.

4.1 Uniform distribution results with $g = 0,1,2$, and $\beta = 0.5,1.01.5$

Percent relative efficiency for the uniform distribution:

- if $\rho = 0.1$, then RE(1)=104.00%, RE(2)=104.00%, RE(3)=102.00%, RE(4)=99.60%, and RE(5)=108.80%;
- if $\rho = 0.3$, then RE(1)=108.70%, RE(2)=108.70%, RE(3)=103.30%, RE(4)=96.50%, and RE(5)=102.90%; and
- if $\rho = 0.5$, then RE(1)=108.30%, RE(2)=108.30%, RE(3)=100.00%, RE(4)=90.60%, and RE(5)=95.00%.

Percent relative bias for the uniform distribution:

- If $\rho = 0.1$, then RB(0) lies between 22.3% and 22.4%, RB(1) lies between 18.1% and 18.2%, RB(2) lies between 18.1% and 18.2%, RB(3) lies between 19.10% and 19.20%, RB(4) = 20.10%, and RB(5) = 7.0%.

Considering percent relative efficiency and percent bias, the fifth transformation p_{i5}^* by Grewal et al. (1997) performed better than the other transformation for the uniform distribution.

4.2 Exponential distribution results with $g = 0,1,2$, and $\beta = 0.5, 1.0 1.5$

Percent relative efficiency for the exponential distribution:

- if $\rho = 0.1$, then RE(1) lies between 115.90% and 116.10%, RE(2) lies between 115.90% and 116.10%, RE(3) lies between 107.30% and 107.40%, RE(4) lies between 90.20% and 91.20%, and RE(5) lies between 131.10% and 131.20%;
- if $\rho = 0.3$, then RE(1) lies between 131.30% and 131.50%, RE(2) lies between 131.30% and 131.50%, RE(3)=112.70%, RE(4) lies between 55.70% and 58.50%, and RE(5) lies between 115.20% and 115.40%;
- if $\rho = 0.5$, then RE(1) lies between 129.60% and 129.80%, RE(2) lies between 129.60% and 129.80%, RE(3)=100.00%, RE(4) lies between 33.40% and 34.90%, and RE(5) lies between 95.8% and 95.90%.

Percent relative bias for the exponential distribution:

- irrespective of the value of ρ between 0.1 and 0.5, the value of RB(0), R(1), RB(2), RB(3) and RB(4) are very high and are unacceptable;
- however, the fifth transformation p_{i5}^* again shows a reasonable RB(5) value between 9.4% and 9.5% for $\rho = 0.3$.

For low correlation values like $\rho = 0.3$, the fifth transformation p_{i5}^* due to Grewal et al. (1997) again performs better than the other transformations in the exponential distribution case.

4.3 Chi-square distribution results with $g = 0,1,2$, and $\beta = 0.5, 1.0 1.5$

Percent relative efficiency for the Chi-square distribution:

- if $\rho = 0.1$, then RE(1) lies between 105.50% and 105.60%, RE(2) lies between 105.50% and 105.60%, RE(3)=102.6%, RE(4)=99.50%, and RE(5) lies between 110.00% and 110.10%;
- if $\rho = 0.3$, then RE(1) lies between 110.00% and 110.90%, RE(2) lies between 110.80% and 110.90%, RE(3)=104.10%, RE(4)=95.70%, and RE(5) lies between 102.50% and 102.50%;
- if $\rho = 0.5$, then RE(1) lies between 109.20% and 109.30%, RE(2) lies between 109.20% and 109.30%, RE(3)=100.00%, RE(4) lies between 88.50% and 89.00%, and RE(5) lies between 93.80% and 93.90%.

Percent relative bias for the Chi-square distribution:

- once again, irrespective of the value of ρ, the value of RB(0) remains quite high;
- if $\rho = 0.5$ then RB(1) and RB(2) are below 10%;
- if $\rho = 0.3$, then the fifth transformation p_{i5}^* shows relative bias of between 3.5% and 3.6%.

The fifth transformation p_{i5}^* due to Grewal et al. (1997) showed both relative efficiency of 102.50% and negligible relative bias of 3.6%, and thus outperformed the other transformations in the Chi-square distribution case.

4.4 Gamma distribution results with $g = 0{,}1{,}2$, and $\beta = 0.5{,}1.0{,}1.5$

Percent relative efficiency for the gamma distribution:

- if $\rho = 0.1$, then RE(1) lies between 108.20% and 108.30%, RE(2) lies between 108.20% and 108.30%, RE(3) is between 103.8% and 103.90%, RE(4)=99.00%, and RE(5) lies between 115.20% and 115.40%;
- if $\rho = 0.3$, then RE(1)=116.10%, RE(2)=116.10%, RE(3)=106.20%, RE(4) is between 91.70% and 91.90%, and RE(5)=105.10%.

Percent relative bias for the gamma distribution:

- again the value of RB(5) remains 5.2% for $\rho = 0.3$.

For situations with a low correlation value like $\rho = 0.3$, the fifth transformation p_{i5}^{*} performed better than its competitors in the gamma distribution case.

4.5 Normal distribution results with $g = 0{,}1{,}2$, and $\beta = 0.5{,}1.0{,}1.5$

For the normal distribution, with any value of ρ between 0.1 and 0.9, inclusive, there is no benefit to using any transformation considering percent relative efficiency. However, percent relative bias can be reduced from 1% to 0.1%, which are both negligible values. In nutshell, in case of the normal distribution, all six transformations performed equally well.

4.6 Log-normal distribution results with $g = 0{,}1{,}2$, and $\beta = 0.5{,}1.0{,}1.5$

Percent relative efficiency for the log-normal distribution:

- if $\rho = 0.1$, then RE(1) lies between 120.40% and 120.80%, RE(2) lies between 120.40% and 120.80%, RE(3) lies between 108.20% and 108.40%, RE(4) lies between 98.30% and 99.30%, and RE(5) lies between 127.90% and 128.10%;
- if $\rho = 0.3$, then RE(1) lies between 132.70% and 133.00%, RE(2) lies between 132.70% and 133.00%, RE(3) lies between 112.60% and 112.80%, RE(4) lies between 86.60% and 86.70%, and RE(5) lies between 107.70% and 107.80%;
- if $\rho = 0.5$, then RE(1) lies between 125.50% and 125.70%, RE(2) lies between 125.50% and 125.80%, RE(3) lies between 100.00% and 100.00%, RE(4) lies between 69.90% and 70.10%, and RE(5) lies between 88.20% and 88.30%.

Percent relative bias for the log-normal distribution:

- the value RB(0) to RE(4) are quite unacceptable for low values of the correlation coefficient;
- however, RE(5) is an acceptable percent relative bias value for $\rho = 0.3$ and onwards.

Once again, the fifth transformation p_{i5}^{*} performed quite well for the log-normal distribution with the value of the correlation coefficient around $\rho = 0.3$.

4.7 Beta distribution results with $g = 0{,}1{,}2$, and $\beta = 0.5{,}1.0{,}1.5$

Percent relative efficiency for the beta distribution:

- if $\rho = 0.1$, then RE(1)=102.00%, RE(2)=102.00%, RE(3)=101.00%, RE(4)=99.80%, and RE(5) is between 104.20% and 104.30%;
- if $\rho = 0.3$, then RE(1) is between 104.20% and 104.30%, RE(2) is between 104.20% and 104.30%, RE(3)=101.00%, RE(4)=99.80%, and RE(5)=101.20%;

- if $\rho = 0.5$, then RE(1) is between 103.90% and 104.00%, RE(2)=104.00%, RE(3)=100.00%, RE(4) is between 95.20% and 95.30, and RE(5) is between 97.20% and 97.30%;

Percent relative bias for the log-normal distribution:

- if $\rho = 0.1$, then RB(0) is between 11.10% and 11.20%, RB(1) is between 9.1% and 9.2%, RB(2) is between 9.1% and 9.2%, RB(3) is between 9.5% and 9.6%., RB(4)=10%, and RB(5) is between 3.5% and 3.6%;
- if $\rho = 0.3$, then RB(0) is between 11.10% and 11.20%, RB(1) is between 5.7% and 5.8%, RB(2) is between 5.7% and 5.8%, RB(3) is between 6.7% and 6.8%, RB(4)=7.8%, and RB(5)=1.5%.

5. Discussion of Results

This study considered results only up to correlation value of ρ less than or equal to 0.5 because in multi-character survey only study variables that have low correlation with the selection probabilities are of interest. In practice there is no situation when the value of ρ is likely to be zero, and hence such an assumption showed higher values of RB(0) in the entire simulation study.

This empirical study concludes that the first two transformations p_{1i}^* and p_{2i}^* performed equally well considering the mean square error and percent relative bias. The relative efficiency of the estimator with the third transformation p_{3i}^* generally remained less than that with p_{2i}^*, but more than 100%. The transformation p_{4i}^* was least efficient and even showed relative efficiency less than 100% for all seven distributions. The most efficient and reliable performance was observed in the fifth transformation p_{5i}^*. Hence, when using forced randomized response model for multi-characteristics survey, the use of fifth transformation p_{5i}^* is recommended.

Acknowledgements

The authors are thankful to Dr. Hukum Chandra and a referee for comments on the original version of the manuscript. The authors would also like to thank Dr. Polly Allred, Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX for help in editing the original manuscript.

References

Amahia, G.N., Chaubey, Y.P. and Rao, T.J. (1989). Efficiency of a new estimator in PPS sampling for multiple characteristics. *Journal of Statistical Planning and Inference,* **21(1)**, 72-84.

Bansal, M.L and Singh, R. (1985). An alternative estimator for multiple characteristics in PPS sampling. *Journal of Statistical Planning and Inference,* **11(3)**, 313-20.

Bar-Lev, S.K., Bobovitch, E. and Boukai, B. (2004). A note on randomized response models for quantitative data. *Metrika,* **60(3)**, 255-260.

Chaudhuri, A. and Mukerjee, R. (1988). *Randomized Response: Theory and Techniques*. Marcel Dekker Inc., New York (ISBN-13: 978-0824777852).

Clark, S. J. and Desharnais, R. A. (1998). Honest answers to embarrassing questions: Detecting cheating in the randomized response model. *Psychological Methods,* **3(2)**, 160-168.

Cochran, W.G. (1977). *Sampling Techniques, 3rd Ed.,* John Wiley and Sons, New York (ISBN-13: 978-0471162407).

Eichhorn, B.H. and Hayre, L.S. (1983). Scrambled randomized response methods for obtaining sensitive quantitative data. *Journal of Statistical Planning and Inference,* **7(4)**, 307-316.

Gjestvang, C.R. and Singh, S. (2007). Forced quantitative randomized response model: A new device. *Metrika,* **66(2)**, 243-256.

Greenberg, B.G., Kuelber, R.R., Abernathy, .J.R. and Horvitz, D.G. (1971). Application of the randomized response technique in obtaining quantitative data. *Journal of the American Statistical Association,* **66(334)**, 243-50.

Grewal, I.S., Bansal, M.L. and Singh, S. (1997). An alternative estimator for multiple characteristics using randomized response technique in PPS sampling. *Aligarh Journal of Statistics,* **19**, 51-65.

Lensvelt-Mulders, G.J.L.M., Hox, J.J., van der Heijden, P.G.M. and Maas, C.J.M. (2005). Meta-analysis of randomized response research: Thirty-five years of validation, *Sociological Methods & Research* **33(3)**, 319–348.

Liu, P.T. and Chow, L.P. (1976a). A new discrete quantitative randomized response model. *Journal of the American Statistical Association,* **71(353)**, 72-73.

Liu, P.T. and Chow L.P. (1976b). The efficiency of the multiple trial randomized response technique. *Biometrics,* **32(3)**, 607-618.

Odumade, O. and Singh, S. (2008). Generalized forced quantitative response model: A unified approach. *Journal of the Indian Society of Agricultural Statistics,* **62(3)**,244-252.

Rao, J.N.K. (1966). Alternative estimators in PPS sampling for multiple characteristics. *Sankhyā: The Indian Journal of Statistics, Series A,* **28(1)** : 47-60.

Soeken, K.L. and Macready, G.B. (1982). Respondents' perceived protection when using randomized response. *Psychological Bulletin* **92(2)**, 487–498.

Sidhu, S.S., Bansal, M.L., Kim J.M. and Singh, S. (2009). Unrelated question model in sensitive multi-character surveys. *Communications of the Korean Statistical Society,* **16(1)**,169–183.

Sidhu, S.S., Bansal, M.L. and Singh. S. (2007). Estimation of sensitive multi-characters using unknown value of unrelated question. *Applied Mathematical Sciences,* **1(37)**, 1803-1820.

Stem, D.E. and Steinhorst, R.K. (1984). Telephone interview and mail questionnaire applications of the randomized response model. *Journal of the American Statistical Association,* **79(387)**, 555-564.

Warner, S.L. (1965). Randomized response: A survey technique for eliminating evasive answer bias. *Journal of the American Statistical Association,* **60(309)**, 63-69.

Appendix A

Table 4.2.1.0: Uniform distribution results with *g* = 0

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
0	0.5	0.10	104.0	104.0	102.0	99.6	108.8	22.3	18.2	18.2	19.2	20.1	7.0
0	0.5	0.30	108.7	108.7	103.3	96.5	102.8	22.2	11.3	11.3	13.4	15.5	3.0
0	0.5	0.50	108.3	108.3	100.0	90.8	95.0	22.4	6.2	6.2	8.6	11.2	1.3
0	0.5	0.70	101.3	101.3	91.6	83.4	86.2	22.3	2.5	2.5	4.6	6.7	0.4
0	0.5	0.90	84.8	84.8	78.7	75.2	76.4	22.3	0.4	0.4	1.3	2.2	0.1
0	1.0	0.10	104.0	104.0	102.0	99.6	108.8	22.4	18.2	18.2	19.2	20.1	7.0
0	1.0	0.30	108.7	108.7	103.3	96.5	102.9	22.3	11.3	11.3	13.5	15.6	3.0
0	1.0	0.50	108.3	108.3	100.0	90.8	95.0	22.3	6.2	6.2	8.6	11.2	1.3
0	1.0	0.70	101.3	101.3	91.6	83.4	86.1	22.3	2.5	2.5	4.6	6.7	0.4
0	1.0	0.90	84.8	84.8	78.7	75.2	76.4	22.3	0.4	0.4	1.3	2.2	0.1
0	1.5	0.10	104.0	104.0	102.0	99.6	108.8	22.3	18.1	18.1	19.2	20.1	7.0
0	1.5	0.30	108.7	108.7	103.3	96.5	102.8	22.2	11.3	11.3	13.4	15.6	3.0
0	1.5	0.50	108.3	108.3	100.0	90.8	95.0	22.3	6.2	6.2	8.6	11.1	1.3
0	1.5	0.70	101.3	101.3	91.6	83.3	86.1	22.3	2.5	2.5	4.6	6.7	0.4
0	1.5	0.90	84.8	84.8	78.7	75.2	76.4	22.3	0.4	0.3	1.3	2.2	0.1

Table 4.2.1.1: Uniform distribution results with *g* = 1

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
1	0.5	0.10	104.0	104.0	102.0	99.6	108.8	22.3	18.1	18.1	19.2	20.1	7.0
1	0.5	0.30	108.7	108.7	103.3	96.5	102.9	22.4	11.4	11.4	13.5	15.7	3.0
1	0.5	0.50	108.3	108.3	100.0	90.8	95.0	22.3	6.2	6.2	8.6	11.2	1.3
1	0.5	0.70	101.3	101.3	91.6	83.4	86.2	22.3	2.5	2.5	4.6	6.7	0.4
1	0.5	0.90	84.8	84.8	78.7	75.2	76.4	22.4	0.4	0.4	1.3	2.2	0.1
1	1.0	0.10	104.0	104.0	102.0	99.6	108.8	22.3	18.2	18.2	19.2	20.1	7.0
1	1.0	0.30	108.7	108.7	103.3	96.5	102.9	22.3	11.4	11.4	13.5	15.6	3.0
1	1.0	0.50	108.3	108.3	100.0	90.8	95.0	22.3	6.2	6.2	8.6	11.2	1.3
1	1.0	0.70	101.3	101.3	91.6	83.4	86.2	22.3	2.5	2.5	4.6	6.7	0.4
1	1.0	0.90	84.8	84.8	78.7	75.2	76.4	22.3	0.4	0.3	1.3	2.2	0.1
1	1.5	0.10	104.0	104.0	102.0	99.6	108.8	22.3	18.1	18.1	19.1	20.1	7.0
1	1.5	0.30	108.7	108.7	103.3	96.5	102.8	22.2	11.3	11.3	13.4	15.6	3.0
1	1.5	0.50	108.3	108.3	100.0	90.8	95.0	22.4	6.2	6.2	8.6	11.2	1.3
1	1.5	0.70	101.3	101.3	91.6	83.4	86.2	22.3	2.5	2.5	4.6	6.7	0.4
1	1.5	0.90	84.8	84.8	78.7	75.2	76.4	22.3	0.4	0.4	1.3	2.2	0.1

Table 4.2.1.2: Uniform distribution results with g = 2

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
2	0.5	0.10	104.0	104.0	102.0	99.6	108.8	22.3	18.2	18.2	19.2	20.1	7.0
2	0.5	0.30	108.7	108.7	103.3	96.5	102.9	22.3	11.3	11.3	13.5	15.6	3.0
2	0.5	0.50	108.3	108.3	100.0	90.8	95.0	22.3	6.2	6.2	8.6	11.1	1.3
2	0.5	0.70	101.3	101.3	91.6	83.4	86.2	22.3	2.5	2.5	4.6	6.7	0.4
2	0.5	0.90	84.8	84.8	78.7	75.2	76.4	22.3	0.4	0.4	1.3	2.2	0.1
2	1.0	0.10	104.0	104.0	102.0	99.6	108.8	22.3	18.2	18.2	19.2	20.1	7.0
2	1.0	0.30	108.7	108.7	103.3	96.5	102.9	22.3	11.3	11.3	13.5	15.6	3.0
2	1.0	0.50	108.3	108.3	100.0	90.8	95.0	22.3	6.2	6.2	8.6	11.1	1.3
2	1.0	0.70	101.3	101.3	91.6	83.4	86.2	22.2	2.5	2.5	4.5	6.7	0.4
2	1.0	0.90	84.8	84.8	78.7	75.3	76.4	22.3	0.4	0.3	1.3	2.2	0.1
2	1.5	0.10	104.0	104.0	102.0	99.6	108.8	22.3	18.1	18.1	19.1	20.1	7.0
2	1.5	0.30	108.7	108.7	103.3	96.5	102.9	22.3	11.3	11.3	13.5	15.6	3.0
2	1.5	0.50	108.3	108.3	100.0	90.8	95.0	22.3	6.2	6.2	8.6	11.1	1.3
2	1.5	0.70	101.3	101.3	91.6	83.4	86.1	22.3	2.5	2.5	4.6	6.7	0.4
2	1.5	0.90	84.8	84.8	78.7	75.2	76.3	22.4	0.4	0.4	1.3	2.2	0.1

Table 4.2.2.0: Exponential distribution results with g = 0

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
0.0	0.5	0.1	116.0	116.0	107.4	90.5	131.2	99.7	69.7	69.7	82.5	89.7	21.8
0.0	0.5	0.3	131.5	131.5	112.7	57.3	115.4	100.1	37.4	37.3	54.5	70.1	9.5
0.0	0.5	0.5	129.7	129.7	100.0	34.3	95.9	99.8	19.3	19.3	32.9	49.9	4.4
0.0	0.5	0.7	111.2	111.2	67.4	21.8	74.1	99.5	8.1	8.1	16.6	29.9	1.7
0.0	0.5	0.9	70.7	70.7	25.8	14.3	46.8	100.0	1.4	1.4	4.7	10.0	0.3
0.0	1.0	0.1	116.0	116.0	107.4	90.8	131.3	100.2	69.9	69.9	82.9	90.2	21.9
0.0	1.0	0.3	131.4	131.4	112.7	56.8	115.2	99.8	37.2	37.2	54.4	69.9	9.5
0.0	1.0	0.5	129.6	129.6	100.0	33.4	95.8	99.7	19.3	19.2	32.9	49.9	4.4
0.0	1.0	0.7	111.3	111.4	67.2	21.7	74.2	99.4	8.1	8.1	16.6	29.8	1.7
0.0	1.0	0.9	70.7	70.7	25.7	14.2	46.8	99.6	1.4	1.4	4.6	10.0	0.3
0.0	1.5	0.1	116.0	116.0	107.4	91.2	131.2	100.1	69.8	69.8	82.8	90.1	21.9
0.0	1.5	0.3	131.4	131.4	112.7	57.0	115.3	100.0	37.3	37.3	54.5	70.0	9.5
0.0	1.5	0.5	129.6	129.6	100.0	34.8	95.9	99.5	19.3	19.2	32.8	49.8	4.4
0.0	1.5	0.7	111.3	111.3	67.2	21.4	74.1	99.9	8.1	8.1	16.7	30.0	1.7
0.0	1.5	0.9	70.7	70.7	26.1	14.7	46.8	99.6	1.4	1.4	4.6	10.0	0.3

Table 4.2.2.1: Exponential distribution results with *g* = 1

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
1.0	0.5	0.1	116.1	116.1	107.4	91.0	131.3	100.4	70.1	70.0	83.1	90.4	21.9
1.0	0.5	0.3	131.3	131.3	112.7	57.3	115.2	99.6	37.2	37.2	54.3	69.7	9.4
1.0	0.5	0.5	129.7	129.7	100.0	34.4	95.9	99.8	19.3	19.2	32.9	49.9	4.4
1.0	0.5	0.7	111.3	111.3	67.1	20.8	74.2	99.6	8.1	8.1	16.6	29.9	1.7
1.0	0.5	0.9	70.7	70.7	25.9	14.4	46.8	99.9	1.4	1.4	4.7	10.0	0.3
1.0	1.0	0.1	116.0	116.0	107.4	90.6	131.3	99.9	69.8	69.8	82.7	89.9	21.9
1.0	1.0	0.3	131.5	131.5	112.7	55.7	115.3	100.1	37.4	37.3	54.5	70.1	9.5
1.0	1.0	0.5	129.7	129.7	100.0	34.2	95.9	99.7	19.3	19.2	32.9	49.8	4.4
1.0	1.0	0.7	111.3	111.3	67.2	20.8	74.2	99.4	8.1	8.1	16.6	29.8	1.7
1.0	1.0	0.9	70.7	70.7	25.9	14.4	46.8	99.8	1.4	1.4	4.6	10.0	0.3
1.0	1.5	0.1	116.0	116.0	107.4	90.2	131.2	99.5	69.6	69.6	82.4	89.6	21.9
1.0	1.5	0.3	131.3	131.3	112.7	57.8	115.3	99.6	37.2	37.2	54.3	69.7	9.5
1.0	1.5	0.5	129.8	129.8	100.0	34.3	95.9	100.2	19.3	19.3	33.0	50.1	4.4
1.0	1.5	0.7	111.2	111.2	67.2	21.0	74.2	99.8	8.1	8.1	16.6	29.9	1.7
1.0	1.5	0.9	70.7	70.7	25.7	14.3	46.8	99.5	1.4	1.4	4.6	10.0	0.3

Table 4.2.2.2: Exponential distribution results with *g* = 2

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
1.0	1.5	0.9	70.7	70.7	25.7	14.3	46.8	99.5	1.4	1.4	4.6	10.0	0.3
2.0	0.5	0.1	116.0	116.0	107.4	90.9	131.2	100.3	70.0	69.9	82.9	90.3	21.9
2.0	0.5	0.3	131.3	131.3	112.7	57.9	115.3	99.5	37.2	37.2	54.2	69.7	9.5
2.0	0.5	0.5	129.8	129.8	100.0	34.7	95.9	99.9	19.3	19.3	32.9	49.9	4.4
2.0	0.5	0.7	111.3	111.3	67.2	21.4	74.2	99.7	8.1	8.1	16.6	29.9	1.7
2.0	0.5	0.9	70.7	70.7	25.4	13.9	46.8	99.8	1.4	1.4	4.6	10.0	0.3
2.0	1.0	0.1	115.9	115.9	107.3	90.4	131.1	99.5	69.6	69.5	82.3	89.6	21.8
2.0	1.0	0.3	131.5	131.5	112.7	58.5	115.3	99.8	37.3	37.3	54.4	69.8	9.5
2.0	1.0	0.5	129.6	129.6	100.0	34.9	95.8	99.3	19.2	19.2	32.7	49.6	4.4
2.0	1.0	0.7	111.2	111.3	67.2	21.8	74.1	100.4	8.1	8.1	16.7	30.1	1.7
2.0	1.0	0.9	70.8	70.8	25.6	14.2	46.8	99.3	1.4	1.4	4.6	9.9	0.3
2.0	1.5	0.1	116.0	116.0	107.4	91.1	131.3	99.8	69.8	69.8	82.6	89.8	21.9
2.0	1.5	0.3	131.4	131.4	112.7	57.5	115.3	99.9	37.3	37.3	54.4	69.9	9.5
2.0	1.5	0.5	129.7	129.7	100.0	33.8	95.9	99.6	19.3	19.2	32.8	49.8	4.4
2.0	1.5	0.7	111.4	111.4	67.1	21.2	74.1	99.9	8.1	8.1	16.7	30.0	1.7
2.0	1.5	0.9	70.7	70.7	25.8	14.4	46.8	99.8	1.4	1.4	4.6	10.0	0.3

Table 4.2.3.0: Chi-square distribution results with g = 0

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
0	0.5	0.1	105.5	105.5	102.6	99.5	110.0	33.2	25.5	25.5	28.1	29.9	8.7
0	0.5	0.3	110.8	110.8	104.1	95.7	102.5	33.3	14.7	14.7	19.2	23.3	3.6
0	0.5	0.5	109.3	109.3	100.0	88.9	93.8	33.4	7.6	7.6	11.9	16.7	1.5
0	0.5	0.7	100.7	100.7	90.1	80.3	84.4	33.3	3.0	3.0	6.2	10.0	0.5
0	0.5	0.9	83.0	83.0	75.4	71.3	73.6	33.3	0.4	0.4	1.7	3.3	0.1
0	1.0	0.1	105.5	105.5	102.6	99.5	110.0	33.2	25.5	25.5	28.1	29.9	8.7
0	1.0	0.3	110.8	110.8	104.1	95.7	102.5	33.3	14.7	14.7	19.2	23.3	3.6
0	1.0	0.5	109.3	109.3	100.0	88.9	93.9	33.2	7.6	7.6	11.9	16.6	1.5
0	1.0	0.7	100.7	100.7	90.2	80.4	84.4	33.4	3.0	3.0	6.2	10.0	0.5
0	1.0	0.9	83.0	83.0	75.5	71.3	73.7	33.3	0.4	0.4	1.7	3.3	0.1
0	1.5	0.1	105.5	105.5	102.6	99.5	110.0	33.3	25.6	25.6	28.1	30.0	8.7
0	1.5	0.3	110.9	110.9	104.1	95.7	102.5	33.4	14.8	14.8	19.3	23.4	3.6
0	1.5	0.5	109.2	109.2	100.0	89.0	93.8	33.3	7.6	7.6	11.9	16.6	1.5
0	1.5	0.7	100.7	100.7	90.1	80.3	84.4	33.3	3.0	3.0	6.2	10.0	0.5
0	1.5	0.9	83.0	83.0	75.5	71.3	73.7	33.2	0.4	0.4	1.7	3.3	0.1

Table 4.2.3.1: Chi-square distribution results with g = 1

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
1	0.5	0.1	105.5	105.5	102.6	99.5	110.0	33.2	25.5	25.5	28.0	29.9	8.7
1	0.5	0.3	110.8	110.8	104.1	95.7	102.5	33.3	14.7	14.7	19.2	23.3	3.6
1	0.5	0.5	109.2	109.2	100.0	88.9	93.8	33.3	7.6	7.6	11.9	16.6	1.5
1	0.5	0.7	100.7	100.7	90.1	80.3	84.5	33.3	3.0	3.0	6.2	10.0	0.5
1	0.5	0.9	83.0	83.0	75.4	71.3	73.7	33.2	0.4	0.4	1.7	3.3	0.1
1	1.0	0.1	105.5	105.5	102.6	99.5	110.0	33.2	25.5	25.5	28.1	29.9	8.7
1	1.0	0.3	110.8	110.8	104.1	95.7	102.5	33.3	14.8	14.8	19.2	23.3	3.6
1	1.0	0.5	109.2	109.3	100.0	88.9	93.8	33.3	7.6	7.6	11.9	16.6	1.5
1	1.0	0.7	100.7	100.7	90.2	80.4	84.4	33.4	3.0	3.0	6.2	10.0	0.5
1	1.0	0.9	83.0	83.0	75.4	71.1	73.6	33.3	0.4	0.4	1.7	3.3	0.1
1	1.5	0.1	105.6	105.6	102.6	99.5	110.1	33.4	25.7	25.7	28.2	30.1	8.7
1	1.5	0.3	110.8	110.8	104.1	95.7	102.5	33.2	14.7	14.7	19.1	23.2	3.5
1	1.5	0.5	109.3	109.3	100.0	88.9	93.8	33.4	7.6	7.6	11.9	16.7	1.5
1	1.5	0.7	100.7	100.7	90.2	80.4	84.5	33.3	3.0	3.0	6.1	10.0	0.5
1	1.5	0.9	83.0	83.0	75.4	71.2	73.6	33.4	0.4	0.4	1.7	3.3	0.1

Table 4.2.3.2: Chi-square distribution results with *g* = 2

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
2	0.5	0.1	105.5	105.5	102.6	99.5	110.0	33.3	25.6	25.6	28.1	30.0	8.7
2	0.5	0.3	110.9	110.9	104.1	95.7	102.5	33.5	14.8	14.8	19.3	23.4	3.6
2	0.5	0.5	109.3	109.3	100.0	88.9	93.8	33.3	7.6	7.6	11.9	16.7	1.5
2	0.5	0.7	100.7	100.7	90.2	80.4	84.5	33.3	3.0	3.0	6.2	10.0	0.5
2	0.5	0.9	82.9	82.9	75.3	71.1	73.5	33.5	0.4	0.4	1.7	3.4	0.1
2	1.0	0.1	105.5	105.5	102.6	99.5	110.1	33.3	25.6	25.6	28.1	30.0	8.7
2	1.0	0.3	110.8	110.8	104.1	95.7	102.5	33.4	14.7	14.7	19.2	23.4	3.6
2	1.0	0.5	109.3	109.3	100.0	88.9	93.8	33.3	7.6	7.6	11.9	16.7	1.5
2	1.0	0.7	100.7	100.7	90.2	80.4	84.5	33.3	3.0	3.0	6.1	10.0	0.5
2	1.0	0.9	83.0	83.0	75.3	71.1	73.6	33.4	0.4	0.4	1.7	3.3	0.1
2	1.5	0.1	105.5	105.5	102.6	99.5	110.1	33.2	25.5	25.5	28.1	29.9	8.7
2	1.5	0.3	110.8	110.8	104.1	95.7	102.5	33.1	14.7	14.7	19.1	23.2	3.5
2	1.5	0.5	109.3	109.3	100.0	88.9	93.8	33.3	7.6	7.6	11.9	16.7	1.5
2	1.5	0.7	100.7	100.7	90.2	80.4	84.5	33.3	3.0	3.0	6.1	10.0	0.5
2	1.5	0.9	83.0	83.0	75.3	71.1	73.6	33.3	0.4	0.4	1.7	3.3	0.1

Table 4.2.4.0: Gamma distribution results with *g* = 0

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
0	0.5	0.1	108.2	108.2	103.8	99.0	115.3	49.9	37.3	37.3	41.9	44.9	12.4
0	0.5	0.3	116.1	116.1	106.2	91.8	105.1	49.9	21.1	21.1	28.3	34.9	5.2
0	0.5	0.5	114.2	114.2	100.0	80.1	93.1	49.8	10.9	10.9	17.4	24.9	2.3
0	0.5	0.7	102.6	102.6	84.9	67.4	79.9	49.9	4.4	4.4	9.0	15.0	0.8
0	0.5	0.9	77.8	77.8	62.7	55.5	63.7	50.0	0.6	0.6	2.5	5.0	0.1
0	1.0	0.1	108.2	108.2	103.9	99.0	115.3	50.0	37.4	37.4	42.0	45.0	12.4
0	1.0	0.3	116.1	116.1	106.2	91.8	105.1	50.0	21.1	21.1	28.3	35.0	5.2
0	1.0	0.5	114.3	114.3	100.0	79.9	93.1	50.0	10.9	10.9	17.5	25.0	2.3
0	1.0	0.7	102.6	102.6	84.8	67.3	79.9	49.9	4.4	4.4	9.0	15.0	0.8
0	1.0	0.9	77.8	77.8	62.6	55.4	63.7	49.7	0.6	0.6	2.5	5.0	0.1
0	1.5	0.1	108.2	108.2	103.9	99.0	115.3	49.9	37.4	37.4	41.9	44.9	12.4
0	1.5	0.3	116.1	116.1	106.2	91.7	105.1	49.9	21.1	21.1	28.3	34.9	5.2
0	1.5	0.5	114.3	114.3	100.0	80.0	93.1	50.1	11.0	11.0	17.5	25.1	2.3
0	1.5	0.7	102.6	102.6	84.8	67.3	79.8	50.0	4.4	4.4	9.0	15.0	0.8
0	1.5	0.9	77.7	77.7	62.7	55.6	63.6	50.0	0.7	0.6	2.5	5.0	0.1

Table 4.2.4.1: Gamma distribution results with *g* = 1

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
1	0.5	0.1	108.3	108.3	103.9	99.0	115.4	50.2	37.5	37.5	42.1	45.1	12.5
1	0.5	0.3	116.1	116.1	106.2	91.9	105.1	50.1	21.1	21.1	28.4	35.0	5.2
1	0.5	0.5	114.3	114.3	100.0	79.8	93.2	50.0	10.9	10.9	17.5	25.0	2.3
1	0.5	0.7	102.6	102.6	84.8	67.3	79.9	50.0	4.4	4.4	9.0	15.0	0.8
1	0.5	0.9	77.7	77.7	62.6	55.5	63.6	50.2	0.7	0.7	2.5	5.0	0.1
1	1.0	0.1	108.2	108.2	103.8	99.0	115.2	49.9	37.3	37.3	41.9	44.9	12.4
1	1.0	0.3	116.1	116.1	106.2	91.7	105.1	50.0	21.1	21.1	28.4	35.0	5.2
1	1.0	0.5	114.3	114.3	100.0	80.1	93.1	49.9	11.0	11.0	17.5	25.0	2.3
1	1.0	0.7	102.6	102.7	84.8	67.0	79.8	50.3	4.4	4.4	9.0	15.1	0.8
1	1.0	0.9	77.8	77.8	62.6	55.4	63.6	49.9	0.7	0.6	2.5	5.0	0.1
1	1.5	0.1	108.2	108.2	103.9	99.0	115.3	50.0	37.4	37.4	42.0	45.0	12.5
1	1.5	0.3	116.1	116.1	106.2	91.8	105.1	50.0	21.1	21.1	28.4	35.0	5.2
1	1.5	0.5	114.3	114.3	100.0	80.0	93.2	50.0	11.0	10.9	17.5	25.0	2.3
1	1.5	0.7	102.6	102.6	84.8	67.5	79.8	49.9	4.4	4.4	9.0	15.0	0.8
1	1.5	0.9	77.8	77.8	62.5	55.3	63.6	49.9	0.7	0.6	2.5	5.0	0.1

Table 4.2.4.2: Gamma distribution results with *g* = 2

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
2	0.5	0.1	108.2	108.2	103.9	99.0	115.3	49.9	37.4	37.3	41.9	44.9	12.4
2	0.5	0.3	116.1	116.1	106.2	91.8	105.1	49.8	21.1	21.1	28.3	34.9	5.2
2	0.5	0.5	114.3	114.3	100.0	80.1	93.2	49.9	10.9	10.9	17.5	25.0	2.3
2	0.5	0.7	102.6	102.6	84.9	67.4	79.9	49.8	4.4	4.4	9.0	15.0	0.8
2	0.5	0.9	77.8	77.8	62.6	55.4	63.7	49.8	0.6	0.6	2.5	5.0	0.1
2	1.0	0.1	108.2	108.2	103.9	99.0	115.3	49.9	37.4	37.4	41.9	44.9	12.4
2	1.0	0.3	116.1	116.1	106.2	91.8	105.1	49.8	21.1	21.1	28.3	34.9	5.2
2	1.0	0.5	114.3	114.3	100.0	80.1	93.1	49.9	10.9	10.9	17.5	25.0	2.3
2	1.0	0.7	102.6	102.6	84.9	67.3	79.9	49.8	4.4	4.4	9.0	15.0	0.8
2	1.0	0.9	77.8	77.8	62.7	55.5	63.7	49.8	0.6	0.6	2.5	5.0	0.1
2	1.5	0.1	108.2	108.2	103.9	99.0	115.3	50.0	37.4	37.4	42.0	45.0	12.5
2	1.5	0.3	116.1	116.1	106.2	91.8	105.1	49.9	21.1	21.1	28.3	34.9	5.2
2	1.5	0.5	114.2	114.3	100.0	80.2	93.1	49.8	10.9	10.9	17.5	24.9	2.3
2	1.5	0.7	102.6	102.6	84.9	67.4	79.9	50.1	4.4	4.4	9.0	15.0	0.8
2	1.5	0.9	77.8	77.8	62.6	55.5	63.6	49.8	0.6	0.6	2.5	5.0	0.1

Table 4.2.5.0: Normal distribution results with *g* = 0

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
0	0.5	0.1	100.2	100.2	100.1	100.0	100.3	1.0	0.8	0.8	0.9	0.9	0.3
0	0.5	0.3	100.3	100.3	100.1	99.9	100.0	1.0	0.5	0.5	0.6	0.7	0.1
0	0.5	0.5	100.3	100.3	100.0	99.7	99.7	1.0	0.3	0.3	0.4	0.5	0.0
0	0.5	0.7	99.9	99.9	99.7	99.5	99.4	1.0	0.1	0.1	0.2	0.3	0.0
0	0.5	0.9	99.4	99.4	99.3	99.2	99.1	1.0	0.0	0.0	0.1	0.1	0.0
0	1.0	0.1	100.2	100.2	100.1	100.0	100.3	1.0	0.8	0.8	0.9	0.9	0.3
0	1.0	0.3	100.3	100.3	100.1	99.9	100.0	1.0	0.5	0.5	0.6	0.7	0.1
0	1.0	0.5	100.3	100.3	100.0	99.7	99.7	1.0	0.3	0.3	0.4	0.5	0.0
0	1.0	0.7	99.9	99.9	99.7	99.5	99.4	1.0	0.1	0.1	0.2	0.3	0.0
0	1.0	0.9	99.4	99.4	99.3	99.2	99.1	1.0	0.0	0.0	0.1	0.1	0.0
0	1.5	0.1	100.2	100.2	100.1	100.0	100.3	1.0	0.8	0.8	0.9	0.9	0.3
0	1.5	0.3	100.3	100.3	100.1	99.9	100.0	1.0	0.5	0.5	0.6	0.7	0.1
0	1.5	0.5	100.3	100.3	100.0	99.7	99.7	1.0	0.3	0.3	0.4	0.5	0.0
0	1.5	0.7	99.9	99.9	99.7	99.5	99.4	1.0	0.1	0.1	0.2	0.3	0.0
0	1.5	0.9	99.4	99.4	99.3	99.2	99.1	1.0	0.0	0.0	0.1	0.1	0.0

Table 4.2.5.1: Normal distribution results with *g* = 1

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
1	0.5	0.1	100.2	100.2	100.1	100.0	100.3	1.0	0.8	0.8	0.9	0.9	0.3
1	0.5	0.3	100.3	100.3	100.1	99.9	100.0	1.0	0.5	0.5	0.6	0.7	0.1
1	0.5	0.5	100.3	100.3	100.0	99.7	99.7	1.0	0.3	0.3	0.4	0.5	0.0
1	0.5	0.7	99.9	99.9	99.7	99.5	99.4	1.0	0.1	0.1	0.2	0.3	0.0
1	0.5	0.9	99.4	99.4	99.3	99.2	99.1	1.0	0.0	0.0	0.1	0.1	0.0
1	1.0	0.1	100.2	100.2	100.1	100.0	100.3	1.0	0.8	0.8	0.9	0.9	0.3
1	1.0	0.3	100.3	100.3	100.1	99.9	100.0	1.0	0.5	0.5	0.6	0.7	0.1
1	1.0	0.5	100.3	100.3	100.0	99.7	99.7	1.0	0.3	0.3	0.4	0.5	0.0
1	1.0	0.7	99.9	99.9	99.7	99.5	99.4	1.0	0.1	0.1	0.2	0.3	0.0
1	1.0	0.9	99.4	99.4	99.3	99.2	99.1	1.0	0.0	0.0	0.1	0.1	0.0
1	1.5	0.1	100.2	100.2	100.1	100.0	100.3	1.0	0.8	0.8	0.9	0.9	0.3
1	1.5	0.3	100.3	100.3	100.1	99.9	100.0	1.0	0.5	0.5	0.6	0.7	0.1
1	1.5	0.5	100.3	100.3	100.0	99.7	99.7	1.0	0.3	0.3	0.4	0.5	0.0
1	1.5	0.7	99.9	99.9	99.7	99.5	99.4	1.0	0.1	0.1	0.2	0.3	0.0
1	1.5	0.9	99.4	99.4	99.3	99.2	99.1	1.0	0.0	0.0	0.1	0.1	0.0

Table 4.2.5.2: Normal distribution results with *g* = 2

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
2	0.5	0.1	100.2	100.2	100.1	100.0	100.3	1.0	0.8	0.8	0.9	0.9	0.3
2	0.5	0.3	100.3	100.3	100.1	99.9	100.0	1.0	0.5	0.5	0.6	0.7	0.1
2	0.5	0.5	100.3	100.3	100.0	99.7	99.7	1.0	0.3	0.3	0.4	0.5	0.0
2	0.5	0.7	99.9	99.9	99.7	99.5	99.4	1.0	0.1	0.1	0.2	0.3	0.0
2	0.5	0.9	99.4	99.4	99.3	99.2	99.1	1.0	0.0	0.0	0.1	0.1	0.0
2	1.0	0.1	100.2	100.2	100.1	100.0	100.3	1.0	0.8	0.8	0.9	0.9	0.3
2	1.0	0.3	100.3	100.3	100.1	99.9	100.0	1.0	0.5	0.5	0.6	0.7	0.1
2	1.0	0.5	100.3	100.3	100.0	99.7	99.7	1.0	0.3	0.3	0.4	0.5	0.0
2	1.0	0.7	99.9	99.9	99.7	99.5	99.4	1.0	0.1	0.1	0.2	0.3	0.0
2	1.0	0.9	99.4	99.4	99.3	99.2	99.1	1.0	0.0	0.0	0.1	0.1	0.0
2	1.5	0.1	100.2	100.2	100.1	100.0	100.3	1.0	0.8	0.8	0.9	0.9	0.3
2	1.5	0.3	100.3	100.3	100.1	99.9	100.0	1.0	0.5	0.5	0.6	0.7	0.1
2	1.5	0.5	100.3	100.3	100.0	99.7	99.7	1.0	0.3	0.3	0.4	0.5	0.0
2	1.5	0.7	99.9	99.9	99.7	99.5	99.4	1.0	0.1	0.1	0.2	0.3	0.0
2	1.5	0.9	99.4	99.4	99.3	99.2	99.1	1.0	0.0	0.0	0.1	0.1	0.0

Table 4.2.6.0: Log-Normal distribution results with *g* = 0

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
0	0.5	0.1	120.7	120.7	108.3	98.3	128.1	171.1	93.5	93.3	134.7	154.0	23.5
0	0.5	0.3	132.9	132.9	112.7	86.6	107.8	167.5	42.8	42.7	80.0	117.2	9.4
0	0.5	0.5	125.5	125.6	100.0	70.1	88.3	169.8	20.4	20.3	45.1	84.9	4.1
0	0.5	0.7	103.5	103.5	75.6	54.3	69.3	166.5	7.9	7.9	21.3	50.0	1.4
0	0.5	0.9	66.5	66.5	48.8	41.9	49.2	168.2	1.2	1.2	5.6	16.8	0.2
0	1.0	0.1	120.6	120.6	108.3	98.3	128.1	170.9	93.2	93.1	134.4	153.8	23.4
0	1.0	0.3	132.9	133.0	112.7	86.6	107.8	170.1	42.9	42.8	80.8	119.1	9.4
0	1.0	0.5	125.6	125.6	100.0	70.0	88.3	169.8	20.4	20.3	45.2	84.9	4.1
0	1.0	0.7	103.5	103.5	75.6	54.3	69.3	173.2	7.9	7.9	21.6	52.0	1.4
0	1.0	0.9	66.4	66.5	48.8	41.9	49.2	168.3	1.2	1.2	5.6	16.8	0.2
0	1.5	0.1	120.5	120.6	108.3	98.3	128.0	167.4	92.8	92.7	132.1	150.7	23.4
0	1.5	0.3	133.0	133.0	112.7	86.6	107.7	173.2	43.1	43.0	81.7	121.3	9.4
0	1.5	0.5	125.5	125.6	100.0	70.0	88.2	166.9	20.4	20.3	44.9	83.4	4.1
0	1.5	0.7	103.5	103.5	75.6	54.3	69.2	167.2	7.9	7.9	21.4	50.2	1.4
0	1.5	0.9	66.5	66.5	48.8	42.0	49.3	170.5	1.2	1.2	5.6	17.1	0.2

Table 4.2.6.1: Log-Normal distribution results with g = 1

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
1	0.5	0.1	120.6	120.7	108.3	98.3	128.1	170.6	93.4	93.2	134.3	153.5	23.4
1	0.5	0.3	132.9	133.0	112.7	86.7	107.8	169.5	42.9	42.8	80.5	118.7	9.4
1	0.5	0.5	125.6	125.6	100.0	69.9	88.3	168.9	20.4	20.4	45.2	84.5	4.1
1	0.5	0.7	103.5	103.6	75.6	54.3	69.2	169.5	7.9	7.9	21.4	50.8	1.4
1	0.5	0.9	66.5	66.5	48.7	41.9	49.2	169.8	1.2	1.2	5.6	17.0	0.2
1	1.0	0.1	120.4	120.4	108.2	98.3	127.9	166.1	92.2	92.0	131.1	149.5	23.3
1	1.0	0.3	133.0	133.0	112.7	86.6	107.8	167.4	42.9	42.8	80.0	117.2	9.4
1	1.0	0.5	125.5	125.5	100.0	70.0	88.3	169.3	20.4	20.4	45.2	84.7	4.1
1	1.0	0.7	103.5	103.5	75.6	54.4	69.3	171.1	7.9	7.9	21.4	51.3	1.4
1	1.0	0.9	66.5	66.5	48.7	41.8	49.2	168.9	1.2	1.2	5.6	16.9	0.2
1	1.5	0.1	120.8	120.8	108.4	98.3	128.1	175.1	94.1	93.9	137.3	157.6	23.5
1	1.5	0.3	132.8	132.8	112.7	86.7	107.8	167.1	42.7	42.6	79.7	117.0	9.3
1	1.5	0.5	125.7	125.8	100.0	69.9	88.2	170.6	20.5	20.5	45.5	85.3	4.1
1	1.5	0.7	103.5	103.5	75.6	54.3	69.2	172.2	7.9	7.9	21.6	51.7	1.4
1	1.5	0.9	66.5	66.5	48.8	41.9	49.2	169.2	1.2	1.2	5.6	16.9	0.2

Table 4.2.6.2: Log-Normal distribution results with g = 2

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
2	0.5	0.1	120.6	120.6	108.3	98.3	128.0	169.5	93.1	92.9	133.5	152.5	23.4
2	0.5	0.3	133.0	133.0	112.8	86.6	107.8	173.0	43.0	42.9	81.5	121.1	9.4
2	0.5	0.5	125.5	125.6	100.0	70.1	88.3	168.3	20.4	20.3	44.9	84.1	4.1
2	0.5	0.7	103.5	103.5	75.6	54.4	69.3	167.1	7.9	7.9	21.3	50.1	1.4
2	0.5	0.9	66.5	66.5	49.0	42.1	49.4	166.0	1.2	1.2	5.6	16.6	0.2
2	1.0	0.1	120.6	120.6	108.3	98.3	128.0	170.0	93.2	93.0	133.9	153.0	23.4
2	1.0	0.3	132.8	132.8	112.7	86.6	107.8	170.2	42.8	42.7	80.6	119.1	9.3
2	1.0	0.5	125.6	125.6	100.0	70.0	88.3	168.8	20.4	20.3	45.0	84.4	4.1
2	1.0	0.7	103.5	103.5	75.7	54.5	69.3	167.4	7.9	7.9	21.3	50.2	1.4
2	1.0	0.9	66.4	66.4	48.7	41.9	49.2	169.8	1.2	1.2	5.6	17.0	0.2
2	1.5	0.1	120.6	120.7	108.3	98.3	128.0	170.6	93.5	93.3	134.4	153.6	23.4
2	1.5	0.3	132.7	132.7	112.6	86.6	107.7	169.3	42.7	42.6	80.3	118.5	9.3
2	1.5	0.5	125.6	125.6	100.0	70.0	88.3	170.4	20.4	20.4	45.3	85.2	4.1
2	1.5	0.7	103.5	103.5	75.6	54.4	69.3	168.8	7.9	7.9	21.4	50.6	1.4
2	1.5	0.9	66.4	66.5	48.8	41.9	49.2	168.1	1.2	1.2	5.6	16.8	0.2

Table 4.2.7.0: Beta distribution results with *g* = 0

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
0	0.5	0.1	102.0	102.0	101.0	99.8	104.2	11.1	9.1	9.1	9.6	10.0	3.6
0	0.5	0.3	104.2	104.2	101.6	98.2	101.2	11.1	5.8	5.8	6.8	7.8	1.5
0	0.5	0.5	104.0	104.0	100.0	95.3	97.2	11.1	3.1	3.1	4.3	5.6	0.6
0	0.5	0.7	100.4	100.4	95.7	91.2	92.6	11.1	1.3	1.3	2.3	3.3	0.2
0	0.5	0.9	91.9	91.9	88.3	86.1	87.0	11.1	0.2	0.2	0.7	1.1	0.0
0	1.0	0.1	102.0	102.0	101.0	99.8	104.2	11.1	9.1	9.1	9.5	10.0	3.5
0	1.0	0.3	104.3	104.3	101.6	98.2	101.2	11.1	5.8	5.8	6.8	7.8	1.5
0	1.0	0.5	104.0	104.0	100.0	95.2	97.2	11.1	3.1	3.1	4.3	5.5	0.6
0	1.0	0.7	100.4	100.4	95.7	91.1	92.6	11.1	1.3	1.3	2.3	3.3	0.2
0	1.0	0.9	91.8	91.8	88.3	86.0	86.9	11.1	0.2	0.2	0.7	1.1	0.0
0	1.5	0.1	102.0	102.0	101.0	99.8	104.2	11.1	9.1	9.1	9.5	10.0	3.5
0	1.5	0.3	104.2	104.2	101.6	98.2	101.2	11.1	5.7	5.7	6.7	7.8	1.5
0	1.5	0.5	104.0	104.0	100.0	95.2	97.3	11.1	3.1	3.1	4.3	5.5	0.6
0	1.5	0.7	100.4	100.4	95.7	91.1	92.6	11.1	1.3	1.3	2.3	3.3	0.2
0	1.5	0.9	91.9	91.9	88.3	86.1	87.0	11.1	0.2	0.2	0.7	1.1	0.0

Table 4.2.7.1: Beta distribution results with *g* = 1

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
1	0.5	0.1	102.0	102.0	101.0	99.8	104.2	11.1	9.1	9.1	9.6	10.0	3.6
1	0.5	0.3	104.2	104.2	101.6	98.2	101.2	11.1	5.7	5.7	6.7	7.8	1.5
1	0.5	0.5	104.0	104.0	100.0	95.2	97.3	11.1	3.1	3.1	4.3	5.6	0.6
1	0.5	0.7	100.4	100.4	95.7	91.2	92.6	11.1	1.3	1.3	2.3	3.3	0.2
1	0.5	0.9	91.9	91.9	88.3	86.1	87.0	11.1	0.2	0.2	0.7	1.1	0.0
1	1.0	0.1	102.0	102.0	101.0	99.8	104.2	11.2	9.1	9.1	9.6	10.0	3.6
1	1.0	0.3	104.3	104.3	101.6	98.2	101.2	11.1	5.8	5.8	6.8	7.8	1.5
1	1.0	0.5	104.0	104.0	100.0	95.3	97.2	11.1	3.1	3.1	4.3	5.6	0.6
1	1.0	0.7	100.4	100.4	95.7	91.1	92.6	11.1	1.3	1.3	2.3	3.3	0.2
1	1.0	0.9	91.9	91.9	88.3	86.1	87.0	11.1	0.2	0.2	0.7	1.1	0.0
1	1.5	0.1	102.0	102.0	101.0	99.8	104.3	11.2	9.2	9.2	9.6	10.0	3.6
1	1.5	0.3	104.2	104.2	101.6	98.2	101.2	11.1	5.7	5.7	6.7	7.8	1.5
1	1.5	0.5	104.0	104.0	100.0	95.3	97.2	11.1	3.1	3.1	4.3	5.6	0.6
1	1.5	0.7	100.4	100.4	95.7	91.1	92.6	11.1	1.3	1.3	2.3	3.3	0.2
1	1.5	0.9	91.9	91.9	88.4	86.2	87.0	11.1	0.2	0.2	0.7	1.1	0.0

Table 4.2.7.2: Beta distribution results with *g* = 2

g	β	ρ	RE(1)	RE(2)	RE(3)	RE(4)	RE(5)	RB(0)	RB(1)	RB(2)	RB(3)	RB(4)	RB(5)
2	0.5	0.1	102.0	102.0	101.0	99.8	104.2	11.1	9.1	9.1	9.6	10.0	3.6
2	0.5	0.3	104.2	104.2	101.6	98.3	101.2	11.1	5.7	5.7	6.7	7.8	1.5
2	0.5	0.5	104.0	104.0	100.0	95.2	97.3	11.1	3.1	3.1	4.3	5.6	0.6
2	0.5	0.7	100.4	100.4	95.7	91.1	92.6	11.1	1.3	1.3	2.3	3.3	0.2
2	0.5	0.9	91.9	91.9	88.3	86.0	87.0	11.1	0.2	0.2	0.7	1.1	0.0
2	1.0	0.1	102.0	102.0	101.0	99.8	104.2	11.1	9.1	9.1	9.6	10.0	3.6
2	1.0	0.3	104.3	104.3	101.6	98.2	101.2	11.1	5.8	5.8	6.8	7.8	1.5
2	1.0	0.5	104.0	104.0	100.0	95.2	97.2	11.1	3.2	3.1	4.3	5.6	0.6
2	1.0	0.7	100.4	100.4	95.7	91.1	92.6	11.1	1.3	1.3	2.3	3.3	0.2
2	1.0	0.9	91.9	91.9	88.3	86.1	87.0	11.1	0.2	0.2	0.7	1.1	0.0
2	1.5	0.1	102.0	102.0	101.0	99.8	104.2	11.2	9.2	9.2	9.6	10.0	3.6
2	1.5	0.3	104.3	104.3	101.6	98.2	101.2	11.2	5.8	5.8	6.8	7.8	1.5
2	1.5	0.5	103.9	104.0	100.0	95.3	97.2	11.1	3.1	3.1	4.3	5.6	0.6
2	1.5	0.7	100.4	100.4	95.7	91.1	92.6	11.1	1.3	1.3	2.3	3.3	0.2
2	1.5	0.9	91.8	91.8	88.3	86.0	86.9	11.1	0.2	0.2	0.7	1.1	0.0

Appendix B

R-codes used for simulation

```
for (g in seq(0.0, 2, 1.0))
{
for (beta in seq(0.5, 1.5, 0.5))
{
for (rho in seq(0.1,0.91,0.2))
{
rb0u<-c()
re1u<-c()
rb1u<-c()
re2u<-c()
rb2u<-c()
re3u<-c()
rb3u<-c()
re4u<-c()
rb4u<-c()
re5u<-c()
rb5u<-c()
nitr<-500
for(jjj in seq(1,nitr,1))
{
np<-1000
ns<-50
#Uniform distribution
#x<-runif(np, 5, 50)
#Exponential distribution
#x<-rexp(np)
#Chisquare distribution
#  ax<-3
#  bx<-2
#  x<-rgamma(np,ax,bx)
#Gamma distribution
#  ax<-2
#  bx<-1.0
#  x<-rgamma(np,ax,bx)
#Normal Distribution
#x<-rnorm(np, 2, 0.2)
#LogNormal Distribution
#x<-rlnorm(np)
#Beta distribution
x<-rbeta(np, 3, 2)
#***************************
pi<-x/sum(x)
pi0<-(1/np)*pi^0
pi1<-(1+1/np)^(1-rho)*(1+pi)^rho-1
pi2<-(1-rho)/np+rho*pi
pi3<-(1/np)^(1-rho)*pi^rho
pi4<-1/(np*(1-rho)+rho/pi)
```

```
pi5<-(1-rho^(1/3))/np + rho^(1/3)*pi
t1<-0.65
t2<-0.25
t3<-1-t1-t2
t4<-0.25
t5<-0.65
t6<-1-t4-t5
th1<-2.0
th2<-1.5
cg1<-0.10
cg2<-0.20
a<-1.0
f1<-20
f2<-t3*f1/t6
#cat(t1,t2,t3,t4,t5, t6,th1,th2,cg1,cg2,'\n')

emsigi1sq<-(t1+t2*th1^2*(1+cg1^2)-(t1+t2*th1)^2)*(beta^2+pi^2+a*pi^g)+t3*(1-t3)*f1^2-
2*t3*f1*(t1+t2*th1)*beta*pi
#print(c(emsigi1sq))
emsigi2sq<-(t4+t5*th2^2*(1+cg2^2)-(t4+t5*th2)^2)*(beta^2+pi^2+a*pi^g)+t6*(1-t6)*f2^2-
2*t6*f2*(t4+t5*th2)*beta*pi
#print(c(emsigi2sq))

#**********************************PI0*************************
term10<-sum(emsigi1sq*pi/pi0^2) + (t1+t2*th1)^2*(beta^2*(sum(pi^3/pi0^2)-
(sum(pi^2/pi0))^2 + a*sum(pi^(g+1)*(1-pi)/pi0^2))) + (t3*f1)*sum(pi* (1/pi0-
sum(pi/pi0))^2) + 2*(t3*f1)*(t1+t2*th1)*beta*sum(pi*(1/pi0-sum(pi/pi0))*(pi/pi0-
sum(pi^2/pi0)))

term20<-sum(emsigi2sq*pi/pi0^2) + (t4+t5*th2)^2*(beta^2*(sum(pi^3/pi0^2)-
(sum(pi^2/pi0))^2 + a*sum(pi^(g+1)*(1-pi)/pi0^2))) + (t6*f2)*sum(pi* (1/pi0-
sum(pi/pi0))^2) + 2*(t6*f2)*(t4+t5*th1)*beta*sum(pi*(1/pi0-sum(pi/pi0))*(pi/pi0-
sum(pi^2/pi0)))

v0<-(sqrt(term10)+sqrt(term20))^2/(ns*(t1-t4+(t2*th1-t5*th2))^2)+beta^2*(sum(pi^2/pi0)-
1)^2+a*sum((pi/pi0-1)^2*pi^g)
rb0<-100*(sum(pi^2/pi0)-1)

rb0u[jjj]<-rb0

#*****************************************************************

#*********************************PI1*******************************
term11<-sum(emsigi1sq*pi/pi1^2) + (t1+t2*th1)^2*(beta^2*(sum(pi^3/pi1^2)-
(sum(pi^2/pi1))^2 + a*sum(pi^(g+1)*(1-pi)/pi1^2))) + (t3*f1)*sum(pi* (1/pi1-
sum(pi/pi1))^2) + 2*(t3*f1)*(t1+t2*th1)*beta*sum(pi*(1/pi1-sum(pi/pi1))*(pi/pi1-
sum(pi^2/pi1)))

term21<-sum(emsigi2sq*pi/pi1^2) + (t4+t5*th2)^2*(beta^2*(sum(pi^3/pi1^2)-
(sum(pi^2/pi1))^2 + a*sum(pi^(g+1)*(1-pi)/pi1^2))) + (t6*f2)*sum(pi* (1/pi1-
```

```
sum(pi/pi1))^2)  +  2*(t6*f2)*(t4+t5*th1)*beta*sum(pi*(1/pi1-sum(pi/pi1))*(pi/pi1-
sum(pi^2/pi1)))

v1<-(sqrt(term11)+sqrt(term21))^2/(ns*(t1-t4+(t2*th1-t5*th2))^2)+beta^2*(sum(pi^2/pi1)-
1)^2+a*sum((pi/pi1-1)^2*pi^g)
re1<-v0*100/v1
rb1<-100*(sum(pi^2/pi1)-1)

re1u[jjj]<-re1
rb1u[jjj]<-rb1

#*********************************PI2********************************
term12<-sum(emsigi1sq*pi/pi2^2)  +  (t1+t2*th1)^2*(beta^2*(sum(pi^3/pi2^2)-
(sum(pi^2/pi2))^2  +  a*sum(pi^(g+1)*(1-pi)/pi2^2)))  +  (t3*f1)*sum(pi*  (1/pi2-
sum(pi/pi2))^2)  +  2*(t3*f1)*(t1+t2*th1)*beta*sum(pi*(1/pi2-sum(pi/pi2))*(pi/pi2-
sum(pi^2/pi2)))

term22<-sum(emsigi2sq*pi/pi2^2)  +  (t4+t5*th2)^2*(beta^2*(sum(pi^3/pi2^2)-
(sum(pi^2/pi2))^2  +  a*sum(pi^(g+1)*(1-pi)/pi2^2)))  +  (t6*f2)*sum(pi*  (1/pi2-
sum(pi/pi2))^2)  +  2*(t6*f2)*(t4+t5*th1)*beta*sum(pi*(1/pi2-sum(pi/pi2))*(pi/pi2-
sum(pi^2/pi2)))

v2<-(sqrt(term12)+sqrt(term22))^2/(ns*(t1-t4+(t2*th1-t5*th2))^2)+beta^2*(sum(pi^2/pi2)-
1)^2 + a*sum((pi/pi2-1)^2*pi^g)

re2<-v0*100/v2
rb2<-100*(sum(pi^2/pi2)-1)

re2u[jjj]<-re2
rb2u[jjj]<-rb2

#*********************************PI3********************************
term13<-sum(emsigi1sq*pi/pi3^2)  +  (t1+t2*th1)^2*(beta^2*(sum(pi^3/pi3^2)-
(sum(pi^2/pi3))^2  +  a*sum(pi^(g+1)*(1-pi)/pi3^2)))  +  (t3*f1)*sum(pi*  (1/pi3-
sum(pi/pi3))^2)  +  2*(t3*f1)*(t1+t2*th1)*beta*sum(pi*(1/pi3-sum(pi/pi3))*(pi/pi3-
sum(pi^2/pi3)))

term23<-sum(emsigi2sq*pi/pi3^2)  +  (t4+t5*th2)^2*(beta^2*(sum(pi^3/pi3^2)-
(sum(pi^2/pi3))^2  +  a*sum(pi^(g+1)*(1-pi)/pi3^2)))  +  (t6*f2)*sum(pi*  (1/pi3-
sum(pi/pi3))^2)  +  2*(t6*f2)*(t4+t5*th1)*beta*sum(pi*(1/pi3-sum(pi/pi3))*(pi/pi3-
sum(pi^2/pi3)))

v3<-(sqrt(term13)+sqrt(term23))^2/(ns*(t1-t4+(t2*th1-t5*th2))^2)+beta^2*(sum(pi^2/pi3)-
1)^2+a*sum((pi/pi3-1)^2*pi^g)
re3<-v0*100/v3
rb3<-100*(sum(pi^2/pi3)-1)

re3u[jjj]<-re3
rb3u[jjj]<-rb3
```

```
#*********************************PI4*******************************
term14<-sum(emsigi1sq*pi/pi4^2) + (t1+t2*th1)^2*(beta^2*(sum(pi^3/pi4^2)-(sum(pi^2/pi4))^2 + a*sum(pi^(g+1)*(1-pi)/pi4^2))) + (t3*f1)*sum(pi* (1/pi4-sum(pi/pi4))^2) + 2*(t3*f1)*(t1+t2*th1)*beta*sum(pi*(1/pi4-sum(pi/pi4))*(pi/pi4-sum(pi^2/pi4)))

term24<-sum(emsigi2sq*pi/pi4^2) + (t4+t5*th2)^2*(beta^2*(sum(pi^3/pi4^2)-(sum(pi^2/pi4))^2 + a*sum(pi^(g+1)*(1-pi)/pi4^2))) + (t6*f2)*sum(pi* (1/pi4-sum(pi/pi4))^2) + 2*(t6*f2)*(t4+t5*th1)*beta*sum(pi*(1/pi4-sum(pi/pi4))*(pi/pi4-sum(pi^2/pi4)))

v4<-(sqrt(term14)+sqrt(term24))^2/(ns*(t1-t4+(t2*th1-t5*th2))^2)+beta^2*(sum(pi^2/pi4)-1)^2+a*sum((pi/pi4-1)^2*pi^g)
re4<-v0*100/v4
rb4<-100*(sum(pi^2/pi4)-1)

re4u[jjj]<-re4
rb4u[jjj]<-rb4

#*********************************PI5*******************************
term15<-sum(emsigi1sq*pi/pi5^2) + (t1+t2*th1)^2*(beta^2*(sum(pi^3/pi5^2)-(sum(pi^2/pi5))^2 + a*sum(pi^(g+1)*(1-pi)/pi5^2))) + (t3*f1)*sum(pi* (1/pi5-sum(pi/pi5))^2) + 2*(t3*f1)*(t1+t2*th1)*beta*sum(pi*(1/pi5-sum(pi/pi5))*(pi/pi5-sum(pi^2/pi5)))

term25<-sum(emsigi2sq*pi/pi5^2) + (t4+t5*th2)^2*(beta^2*(sum(pi^3/pi5^2)-(sum(pi^2/pi5))^2 + a*sum(pi^(g+1)*(1-pi)/pi5^2))) + (t6*f2)*sum(pi* (1/pi5-sum(pi/pi5))^2) + 2*(t6*f2)*(t4+t5*th1)*beta*sum(pi*(1/pi5-sum(pi/pi5))*(pi/pi5-sum(pi^2/pi5)))

v5<-(sqrt(term15)+sqrt(term25))^2/(ns*(t1-t4+(t2*th1-t5*th2))^2)+beta^2*(sum(pi^2/pi5)-1)^2 + a*sum((pi/pi5-1)^2*pi^g)

re5<-v0*100/v5
rb5<-100*(sum(pi^2/pi5)-1)
re5u[jjj]<-re5
rb5u[jjj]<-rb5
}

rb0u<-round(mean(rb0u),2)
re1u<-round(mean(re1u),2)
rb1u<-round(mean(rb1u),2)

re2u<-round(mean(re2u),2)
rb2u<-round(mean(rb2u),2)

re3u<-round(mean(re3u),2)
rb3u<-round(mean(rb3u),2)

re4u<-round(mean(re4u),2)
```

```
rb4u<-round(mean(rb4u),2)

re5u<-round(mean(re5u),2)
rb5u<-round(mean(rb5u),2)
cat(g, beta, rho, re1u, re2u, re3u, re4u, re5u,rb0u, rb1u, rb2u, rb3u, rb4u, rb5u,'\n')
}
}
}
```