Statistics and Applications {ISSN 2454-7395 (online)}
Volume 20, No. 1, 2022 (New Series), pp 213-227

Cost and Profit Analysis of State-dependent Feedback Queue with Impatient Customer Subject to Catastrophes

Ankita Roy Chowdhury and Indra
Department of Statistics and Operational Research,
Kurukshetra University, Kurukshetra, Haryana-136119, India

Received: 29 January 2021; Revised: 21 March 2021; Accepted: 23 April 2021

Abstract

The paper analyses a single-server Markovian queueing system having state dependent service rates with customer's balking and feedback subject to catastrophes. Using matrix-geometric solution method, we obtain steady-state solution for the system. Various measurable indicators have been evaluated with the assistance of Maple software and based on these measures; we have presented an expected cost and profit analysis.

Key words: Matrix-geometric method; Balking; Feedback; Catastrophes; State-dependent.

1. Introduction

In such an intense market condition, where attracting and finding a potential customer is very difficult, no one wants to bear the cost of customer loss. So, providing quality of service at faster rate is very important factor in this fast paced life. Considering parameters like balking, feedback and state-dependent service rate provides more pliability for optimal design and finds its applicability in communication network, production system, and in various congestion problems. Queueing analysis presents an optimal solution by providing suitable suggestions to reduce congestion.

In this paper, we have considered the parameters; balking, feedback, catastrophe and state dependent service altogether, to analyse the system performance. These parameters inordinately affect the system and its cost function.

There are many practical situations, where service rates depend on the size of the system. Such situations can be seen in hotels or restaurants during rush hours, where waiters and cooks work with a faster rate to cope up with the demand, or in hospitals for patients coming to the emergency ward *etc*. Many authors have contributed in the study of state dependent service rates. Davingon and Disney (1976) considered single server state dependent feedback queue. Doshi and Jangerman (1986) obtained some important performance measures for an M/G/1 queue where, balking depended on system size using supplementary variable technique. Abou-El-Ata (1991) extended the model of Ancker and Gafarian (1963) to study the state dependent finite queue with impatient customers. Such system may also get affected by balking, where customer doesn't want to join the system due to long queue.

Corresponding Author: Indra
E-mail: indra@kuk.ac.in

Queueing systems incorporating balking, feedback or both have attracted many researchers. They are useful in designing and managing systems like transmission of data, emergency ward of health sector where balking is common and chances of rework is more. The perception of customer impatience was first appeared in the work of Haight (1957). Tackas (1963) analysed a single server queue with feedback. For conceivable uses, the history and contributions of researchers on queueing systems with balking and feedback, one may see articles by Santhakumaran and Thangaraj (2000), Choudhury and Paul (2005), Kumar *et al.* (2013), Varalakshmi *et al.* (2018), Bouchentouf *et al.* (2019).

Queueing models with catastrophes have gained importance during last few decades because of its relevancy in many area *viz.* computers and telecommunications, health sector, production sector, disaster management. Queues with catastrophes have attracted many researchers due to the fact; they are very unpredictable in nature and force the customers to leave the system immediately. So, including them in modelling makes the model more pragmatic. Thangaraj and Vanitha (2009) obtained transient solution of M/M/1 feedback queue with catastrophe using continued fractions. Kumar *et al.* (2014) studied queueing systems subjected to catastrophes and customer's impatience and obtained time-dependent and steady-state probabilities when system is operational as well when under repair process. Bura and Bura (2015) analysed finite, single-server markovian-catastrophic queueing system with restorative effects.

The primary objectives of this paper are:

i. To obtain steady state solutions to aforesaid queueing system using matrix geometric method.
ii. To evaluate important performance measures such as mean number of customers in the system and in the queue, probability of ideal, probability of busy, mean balking rate *etc.* and to perform sensitivity analysis.
iii. To formulate an expected cost and profit functions based on measures obtained.
iv. Graphical representations showing effect of different parameters on expected cost and expected profit functions.

2. Model Assumptions and Descriptions

We consider a markovian queueing system of infinite capacity, where the arrivals and departures both follow Poisson process with mean inter-arrival time $\frac{1}{\lambda}$ and mean inter-service times $\frac{1}{\mu_1}$ or $\frac{1}{\mu_2}$ depending upon the system size. Arriving customer may join the queue with probability 'β' if he finds the server non-empty or balk with probability $'1-\beta'$ according to some predetermined norms. The server decides to operate with two different service rates; 'slow and fast' subjected to the length of the queue. If it finds the system size is less than or equal to the critical value 'r', it serves with a slower rate 'μ_1'; otherwise with a faster rate 'μ_2'. If customer, on service completion is satisfied by the service, the customer leaves the system with probability 'q'. On contrary the customer re-joins the queue with probability 'p' if one finds the service dissatisfactory. Occurrence of catastrophes ejects all the customers from the system instantly and system becomes inactive momentarily. Catastrophes occur according to Poisson process with rate of occurrence$'\ \xi'$, when the system is non-empty.

The infinitesimal generator matrix Q of the system is given by:

$$
\boldsymbol{Q} = \begin{array}{c} \left(\begin{array}{cccccccccc}
-\lambda & \lambda & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(q\mu_1+\xi) & -(q\mu_1+\beta\lambda+\xi) & \beta\lambda & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\xi & q\mu_1 & -(q\mu_1+\beta\lambda+\xi) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\xi & 0 & 0 & q\mu_1 & -(q\mu_1+\beta\lambda+\xi) & \beta\lambda & 0 & \cdots & \cdots \\
\xi & 0 & 0 & \cdots & q\mu_1 & -(q\mu_1+\beta\lambda+\xi) & \beta\lambda & \cdots & \cdots \\
\xi & 0 & 0 & \cdots & \cdots & q\mu_2 & -(q\mu_2+\beta\lambda+\xi) & \beta\lambda & \cdots \\
\vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}\right) \end{array}
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ \cdots \\ r-1 \\ r \\ r+1 \\ \cdots \\ \cdots \end{array}
$$

Let $n(t) \equiv$ number of customers in the system at time 't'. Let 'n' be the stationary random variable for the number of the customers in the system. We define π_i={n = i}= $\lim_{t\to\infty} P\{n(t) = i\}$, where $i \in \mathbb{W}$ and π_i represents the stationary probability of i customers in the system. The stationary probability vector is given by,

$$\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \dots\dots\dots\dots\dots, \pi_r, \pi_{r+1}, \dots\dots\dots) \tag{1}$$

The steady-state probabilities π_i are related geometrically to each other as $\pi_i = \pi_r R^{i-r}\ \forall i \geq r$. Here, R is called the rate element and for this system it is given by:

$$R = \frac{(\beta\lambda + q\mu_2 + \xi) - \sqrt{(\beta\lambda + q\mu_2 + \xi)^2 - 4\beta\lambda q\mu_2}}{2q\mu_2} \tag{2}$$

The steady-state probabilities are obtained by solving the following equations

$$\boldsymbol{\pi Q = 0} \tag{3}$$

$$\boldsymbol{\pi e = 1} \tag{4}$$

3. Performance Measures

We calculate some performance indicators using the probabilities; obtained by employing equation (3) and equation (4), for the system as follows.

i) "Expected number of customers in the system:"

$$\boldsymbol{MNS = \sum_{n=1}^{r} n\pi_n + \sum_{n=r}^{\infty} n\pi_r R^{n-r}} \tag{5}$$

ii) "Expected number of customers in the queue:"

$$\boldsymbol{MNQ} = \sum_{n=1}^{r-1} \boldsymbol{n\pi_{n+1}} + \sum_{n=r}^{\infty} \boldsymbol{n\pi_r R^{n+1-r}} \tag{6}$$

iii) Mean Balking Rate (B.R):

$$\boldsymbol{B.R} = (\mathbf{1} - \boldsymbol{\beta})\lambda(\mathbf{1} - \boldsymbol{\pi_0}) \tag{7}$$

iv) Probability that the server is busy:

$$\boldsymbol{P_b} = (\mathbf{1} - \boldsymbol{\pi_0}) \tag{8}$$

v) Probability that the server is ideal:

$$\boldsymbol{P_I} = \boldsymbol{\pi_0} \tag{9}$$

vi) Expected waiting time in the system:

$$\boldsymbol{MWS} = \frac{\boldsymbol{MNS}}{\boldsymbol{\lambda}} \tag{10}$$

vii) Expected waiting time in the queue:

$$\boldsymbol{MWQ} = \boldsymbol{MWS} - \frac{\mathbf{1}}{\boldsymbol{\mu_2}} \tag{11}$$

Special Case

If we put $\beta = 1, q = 1, \xi = 0$ and consider only one service rate throughout *i.e.* μ, then the rate element reduces to

$$\boldsymbol{R} = \frac{\boldsymbol{\lambda}}{\boldsymbol{\mu}}$$

and $\boldsymbol{\pi_n}$is given by:

$$\boldsymbol{\pi_n} = \boldsymbol{R^n}(\mathbf{1} - \boldsymbol{R})$$

which is same as the probability of *n* customers in the system, for classical M/M/1 queue.

Particular Cases

We obtain stationary probabilities when *r* = 1 and *r* =2 in the following section.

Case-I: When *r* = 1

The infinitesimal generator matrix Q of the system is given by:

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & \dots & \dots & \dots \\ (q\mu_1 + \xi) & -(q\mu_1 + \beta\lambda + \xi) & \beta\lambda & 0 & \dots & \dots \\ \xi & q\mu_2 & -(q\mu_2 + \beta\lambda + \xi) & \beta\lambda & 0 & \dots \\ \vdots & \vdots & \vdots & \dots & \dots & \dots \end{pmatrix}$$

Using (3) and (4) we have,

$$\pi_0 = \frac{q\mu_1(1-R)+\xi}{q\mu_1(1-R)+\lambda+\xi} \tag{12}$$

$$\pi_1 = \frac{\lambda(1-R)}{q\mu_1(1-R)+\lambda+\xi} \tag{13}$$

The other steady state probabilities are obtained by $\pi_i = \pi_1 R^{i-1}\ \forall\ i \geq 2$

Case-II: When *r* = 2

The infinitesimal generator matrix Q of the system is given by:

$$\boldsymbol{Q} = \begin{pmatrix} -\boldsymbol{\lambda} & \boldsymbol{\lambda} & \boldsymbol{0} & \dots & \dots & \dots \\ (\boldsymbol{q\mu_1}+\boldsymbol{\xi}) & -(\boldsymbol{q\mu_1}+\boldsymbol{\beta\lambda}+\boldsymbol{\xi}) & \boldsymbol{\beta\lambda} & \dots & \dots & \dots \\ \boldsymbol{\xi} & \boldsymbol{q\mu_1} & -(\boldsymbol{q\mu_1}+\boldsymbol{\beta\lambda}+\boldsymbol{\xi}) & \dots & \dots & \dots \\ \boldsymbol{\xi} & \boldsymbol{0} & \boldsymbol{q\mu_2} & -(\boldsymbol{q\mu_2}+\boldsymbol{\beta\lambda}+\boldsymbol{\xi}) & \boldsymbol{\beta\lambda} & \dots \\ \vdots & \vdots & \vdots & \dots & \dots & \dots \end{pmatrix}$$

Using (3) and (4) we have,

$$\pi_0 = \frac{(q\mu_1+\xi)(q\mu_1+\beta\lambda+\xi-q\mu_2 R)+\beta\lambda\xi(1-R)^{-1}}{(q\mu_1+\xi+\lambda)(q\mu_1+\beta\lambda+\xi-q\mu_2 R)+\beta\lambda\xi(1-R)^{-1}+\beta\lambda^2(1-R)^{-1}} \tag{14}$$

$$\pi_1 = \frac{(q\mu_1+\beta\lambda+\xi-q\mu_2 R)\lambda}{(q\mu_1+\xi+\lambda)(q\mu_1+\beta\lambda+\xi-q\mu_2 R)+\beta\lambda\xi(1-R)^{-1}+\beta\lambda^2(1-R)^{-1}} \tag{15}$$

$$\pi_2 = \frac{\beta\lambda^2}{(q\mu_1+\xi+\lambda)(q\mu_1+\beta\lambda+\xi-q\mu_2 R)+\beta\lambda\xi(1-R)^{-1}+\beta\lambda^2(1-R)^{-1}} \tag{16}$$

The remaining probabilities are obtained by $\pi_i = \pi_2 R^{i-2}\ \forall\ i \geq 3$.

4. Cost Model and Profit Model

Constructing an expected cost function for a system which not only get affected by varying arrival and service rates but also by balking, feedback, and catastrophes is very difficult. Here, we confine ourselves in determining the optimum value of 'r' which minimizes the cost. Let C_1 be the cost associated with a customer present in the queue, C_2 be the cost associated with a customer when server is busy, C_3 be the cost associated with a customer loss, and C_4 be the cost associated with server when it is ideal. So, we have the expected cost function as,

$$\text{Total Expected Cost } (\mathbf{TEC}) = \boldsymbol{C_1} * \mathbf{MNQ} + \boldsymbol{C_2} * \boldsymbol{P_b} + \boldsymbol{C_3} * \mathbf{B.R} + \boldsymbol{C_4} * \boldsymbol{P_I} \tag{17}$$

Similarly, for an expected profit function, we have

$$\text{Total Expected Profit } (\mathbf{TEP}) = \boldsymbol{\rho} * \mathbf{MNS} - \mathbf{TEC} \tag{18}$$

where $\boldsymbol{\rho}$ is the revenue.

Though the cost function may appeared to be simple but it is highly non-linear and complex in nature which makes it difficult in optimizing the value of '*r*'. In order to arrive at a decision, we carry out sensitivity analyses by substituting different values for the parameters.

5. Sensitivity Analysis

Sensitivity analyses have been performed to compare the systems $r = 1$ and $r = 2$, by changing values of the parameters involved. For calculation, let C_1 =100, C_2 =150, C_3 =200, and C_4 =250. The measurable indicators are computed coupled with total expected cost and total expected profit. These measures have guided in deciding the optimal value of '*r*' in order to minimize its expected cost and maximize the expected profit. Different Cost and profit graphs have been plotted by varying the parameters under consideration. These graphs are illustrated and discussed below.

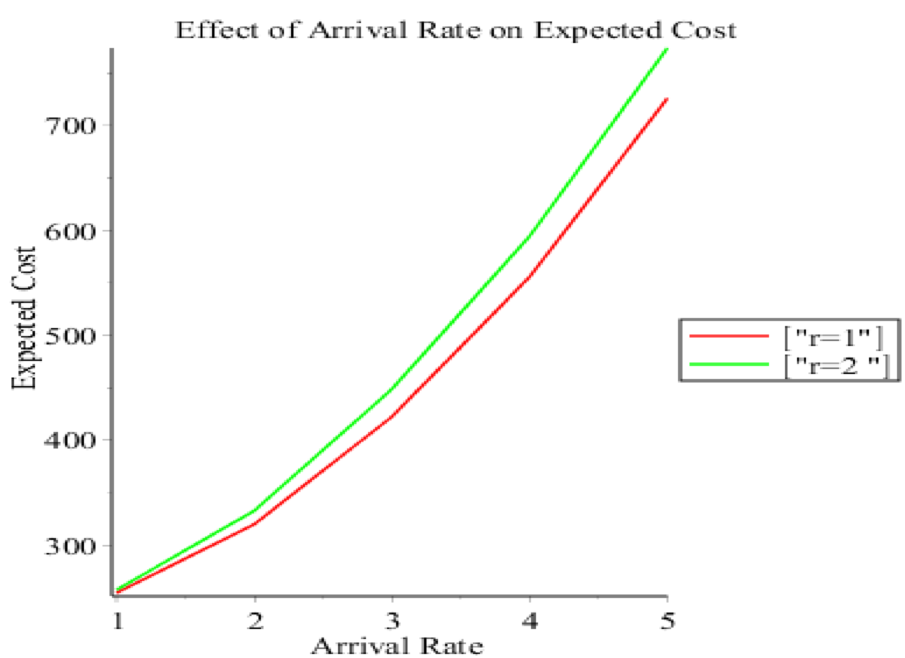

Figure 1(a)

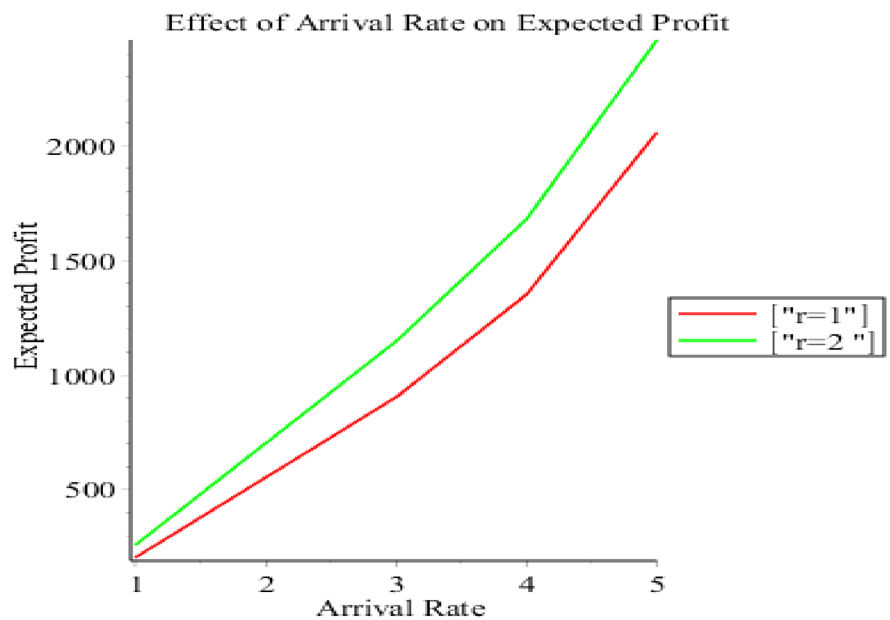

Figure 1(b)

In figures 1(a) and 1(b), we fix $\mu_1 = 3, \mu_2 = 5, = 0.5, \xi = 0.01$, and $q = 0.8$ and display the expected cost and expected profit by varying arrival rates for both the systems r = 1, and r = 2. It is clear from the graph that expected cost for both the systems are almost same and increases as arrival increases. Same trend can be seen for profit as well and if arrival rate becomes same or greater than the slow service rate, it is beneficial to use the faster rate to maximize the profit.

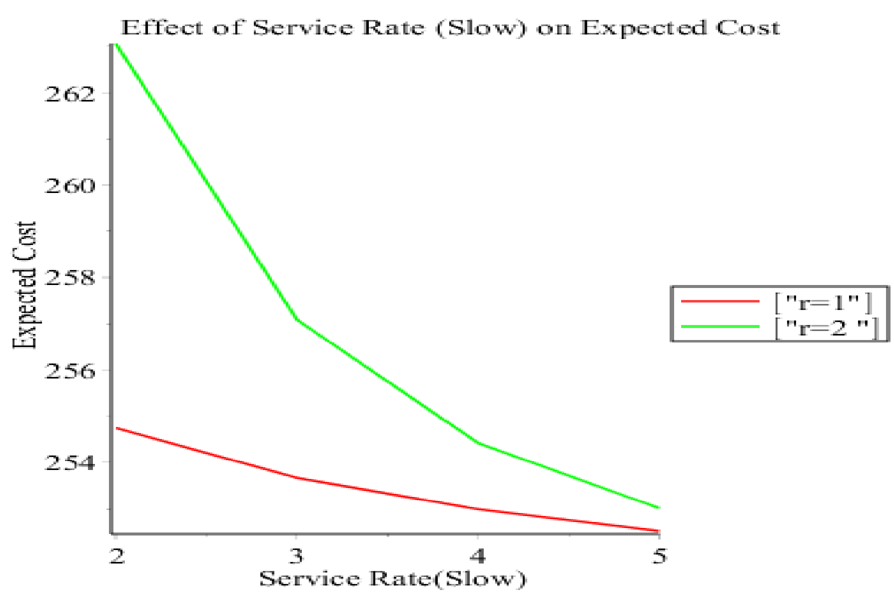

Figure 2(a)

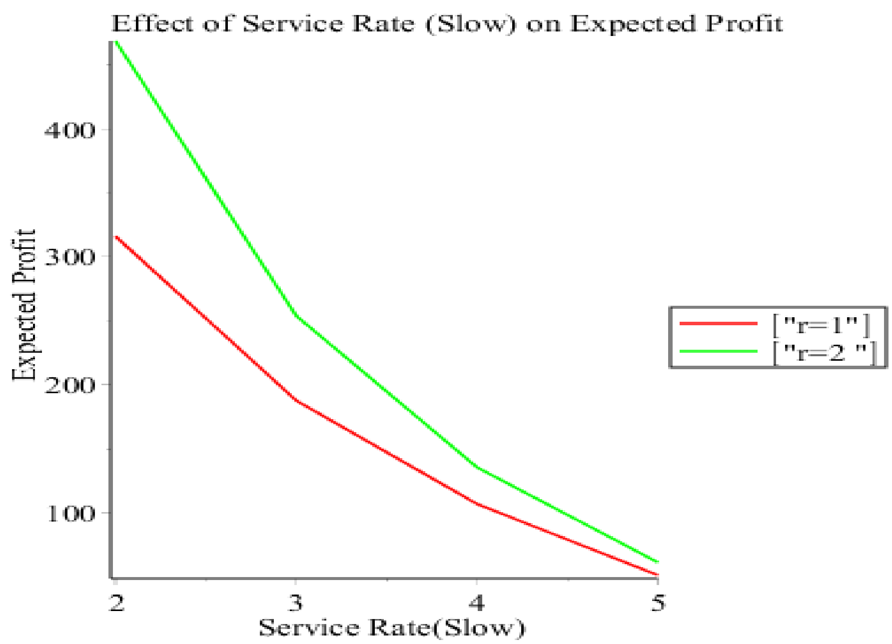

Figure 2(b)

In figures 2(a) and 2(b), we fix $\lambda = 1$, $\mu_2 = 6$, $\beta = 0.5$, $\xi = 0.01$, and $q = 0.8$ and display the expected cost and expected profit by varying slow service rate for both the systems r = 1, and r = 2. It is clear from the graph that expected cost for both the systems decreases as service rate increases. But the decrement is more rigorous for system r = 2 than for r = 1. Same trend can be seen for profit as well. This is because the server remains ideal for rest of the time.

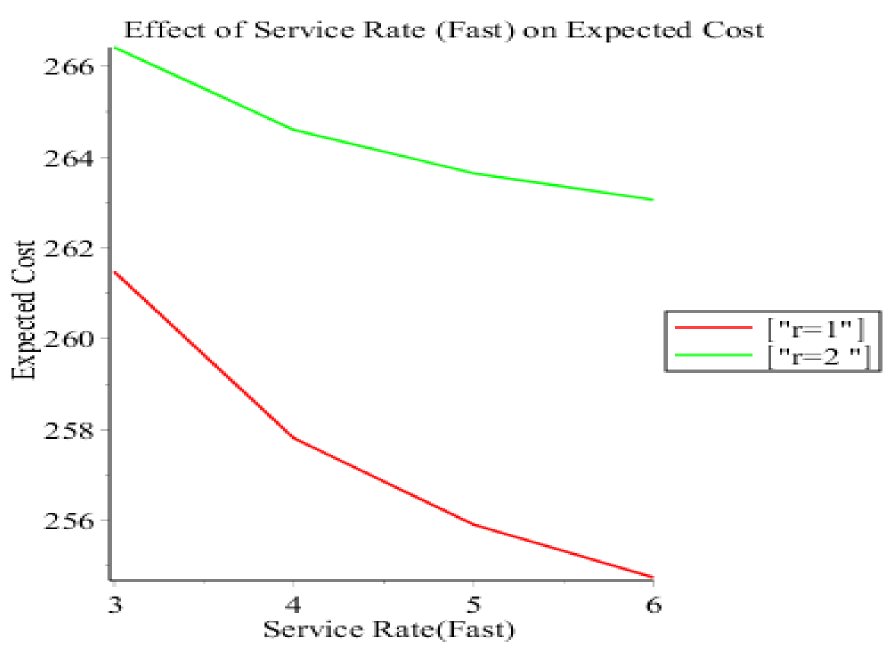

Figure 3(a)

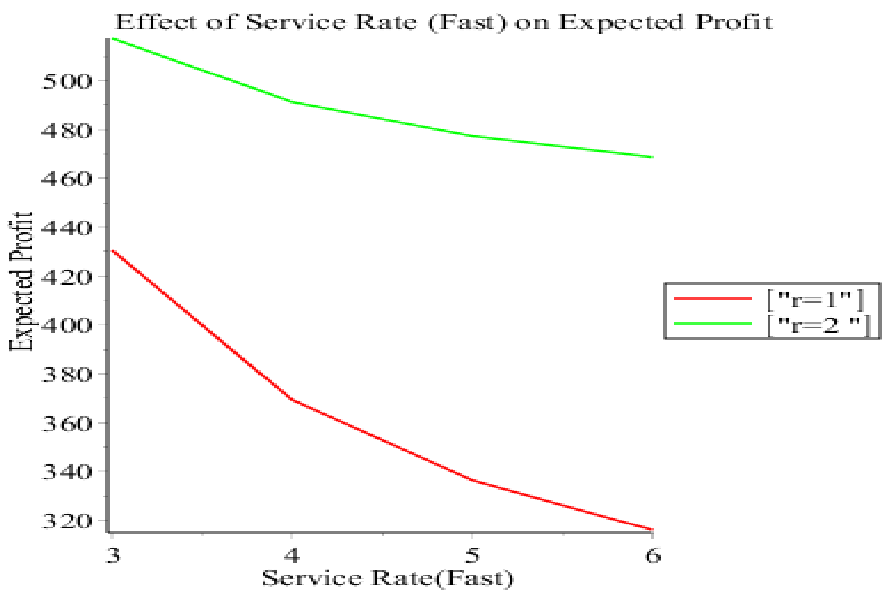

Figure 3(b)

In figures 3(a) and 3(b), we fix $\lambda = 1$, $\mu_1 = 2$, $\beta = 0.5$, $\xi = 0.01$, and $q = 0.8$ and display the expected cost and expected profit by varying fast service rate for both the systems r = 1, and r = 2. It is clear from the graph that expected cost for both the systems decreases as service rate increases. Same trend can be seen for profit as well. Varying fast service rate rarely affects the expected cost and slightly affects the expected profit.

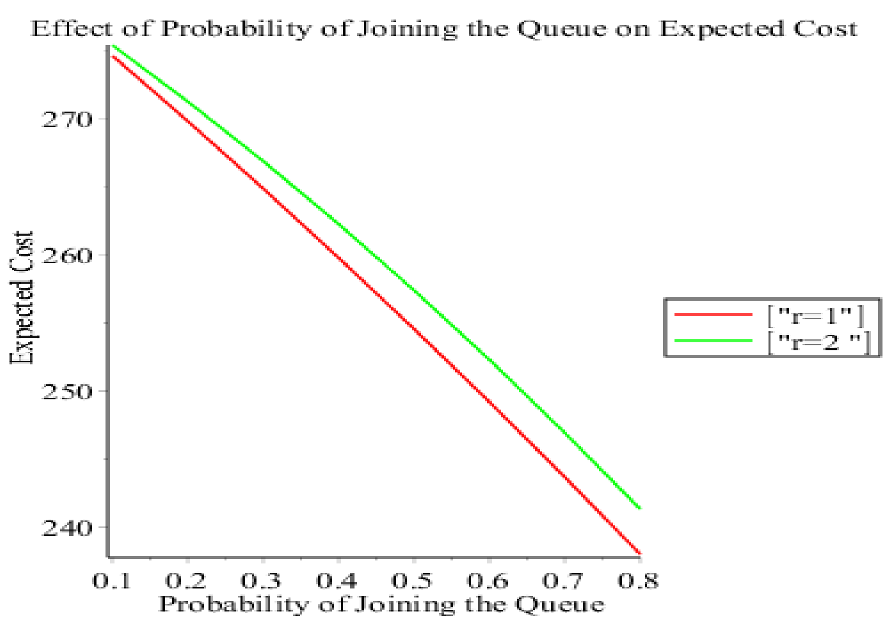

Figure 4(a)

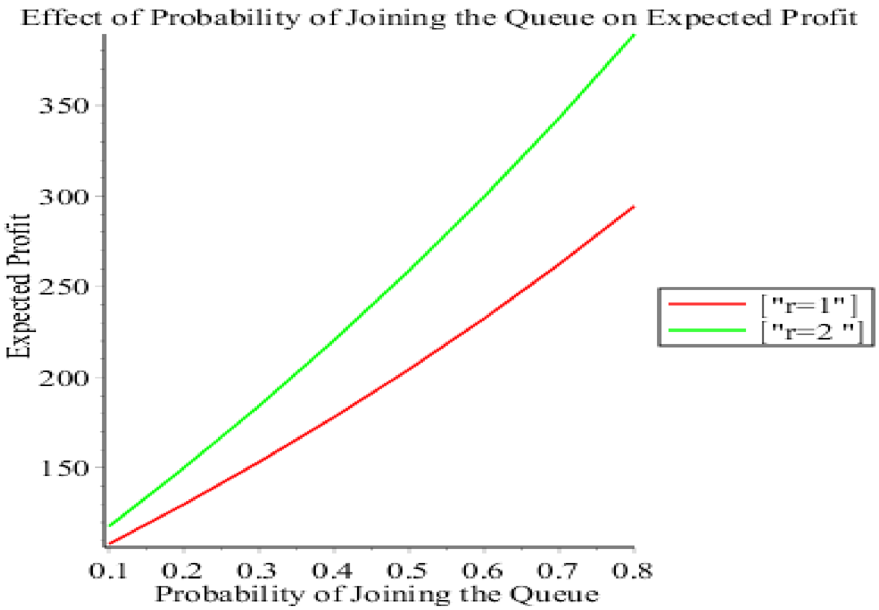

Figure 4(b)

In figures 4(a) and 4(b), we fix $\lambda = 1$, $\mu_1 = 3$, $\mu_2 = 5$, $\xi = 0.01$, and $q = 0.8$ and display the expected cost and expected profit by varying joining probability for both the systems *r* = 1, and *r* = 2. It is clear from the graph that expected cost for both the systems decreases as joining probability increases, whereas expected profit increases as joining probability increases. Thus, profit could be maximized by encouraging the customers to join the system.

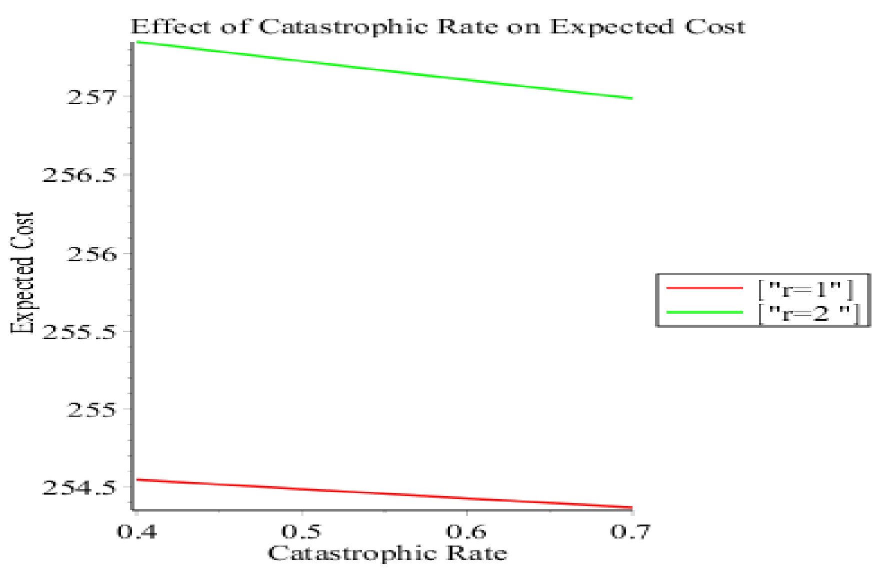

Figure 5(a)

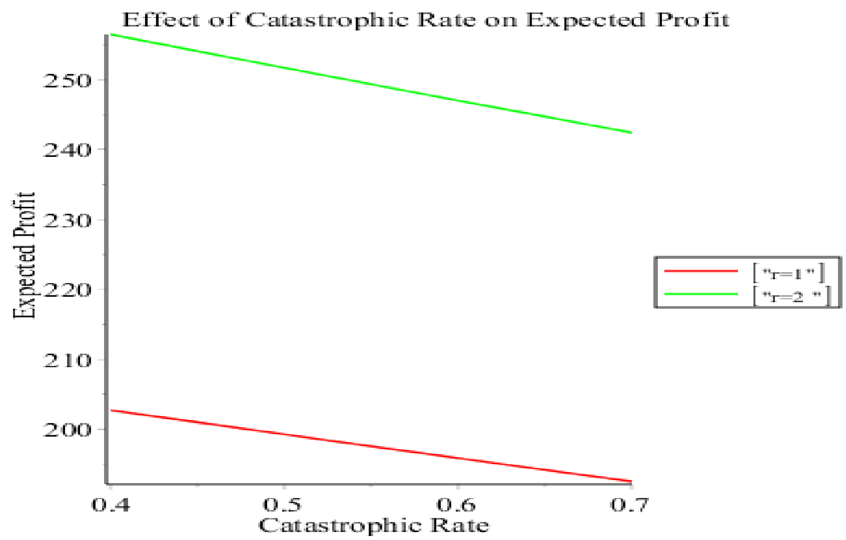

Figure 5(b)

In figures 5(a) and 5(b), we fix $\lambda = 1$, $\mu_1 = 3$, $\mu_2 = 5$, $\beta = 0.5$, and $q = 0.8$ and display the expected cost and expected profit by varying catastrophic rate for both the systems r = 1, and r = 2. It is clear from the graph that expected cost for both the systems decreases as catastrophic rate increases. Same trend can be seen for profit as well. Increasing catastrophic rate barely affects the cost and profit function.

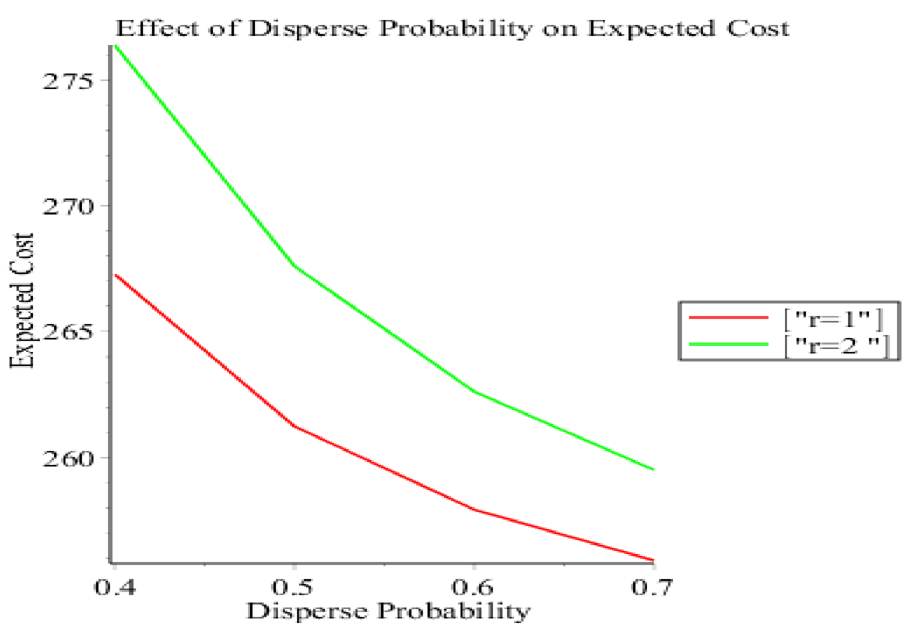

Figure 6(a)

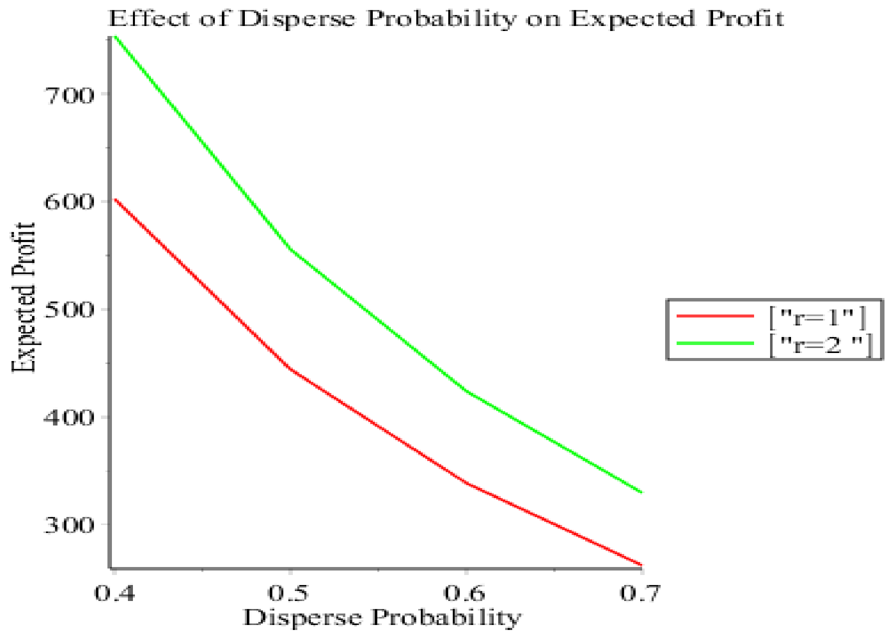

Figure 6(b)

In figures 6(a) and 6(b), we fix $\lambda = 1$, $\mu_1 = 3$, $\mu_2 = 5$, $\beta = 0.5$, and $\xi = 0.01$ and display the expected cost and expected profit by varying disperse probability for both the systems *r* = 1, and *r* = 2. It is clear from the graph that expected cost for both the systems decreases as probability of leaving the system increases. Same trend can be seen for profit as well. Intuitively, increment in feedback probability will increase the cost.

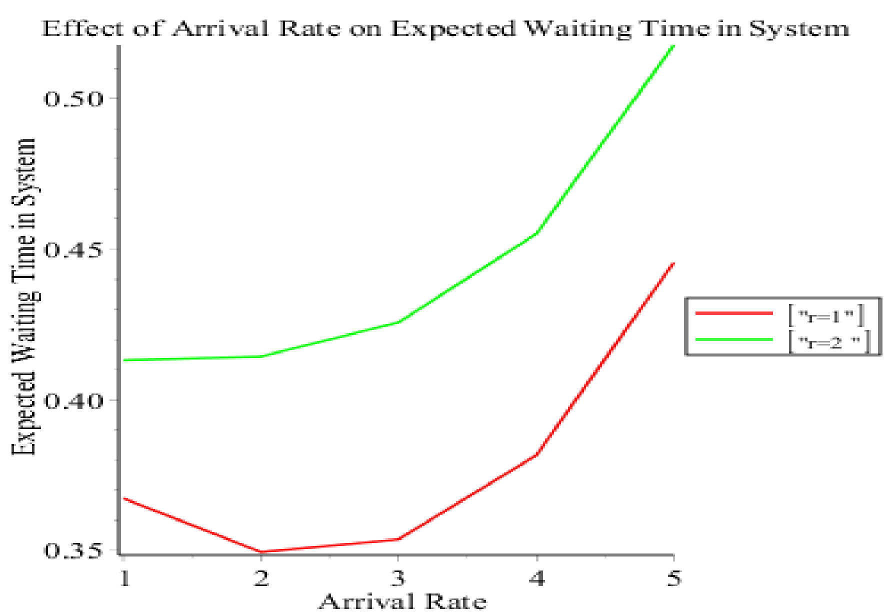

Figure 7(a)

In figure 7(a), we fix $\mu_1 = 3$, $\mu_2 = 5$, $\beta = 0.5$, $\xi = 0.01$, and $q = 0.8$ and display the expected waiting time by varying arrival rates for both the systems *r* = 1, and *r* = 2. It is clear from the graph that expected waiting time for both the systems increases as arrival increases.

Figure 7(b)

In figure 7(b), we fix $\lambda = 1$, $\mu_2 = 6$, $\beta = 0.5$, $\xi = 0.01$, and $q = 0.8$ and display the expected waiting time by varying slow service rate for both the systems *r* = 1, and *r* = 2. It is clear from the graph that expected waiting time for both the systems decreases as service rate increases. But the decrement is more rigorous for system *r* = 2 than for *r* = 1.

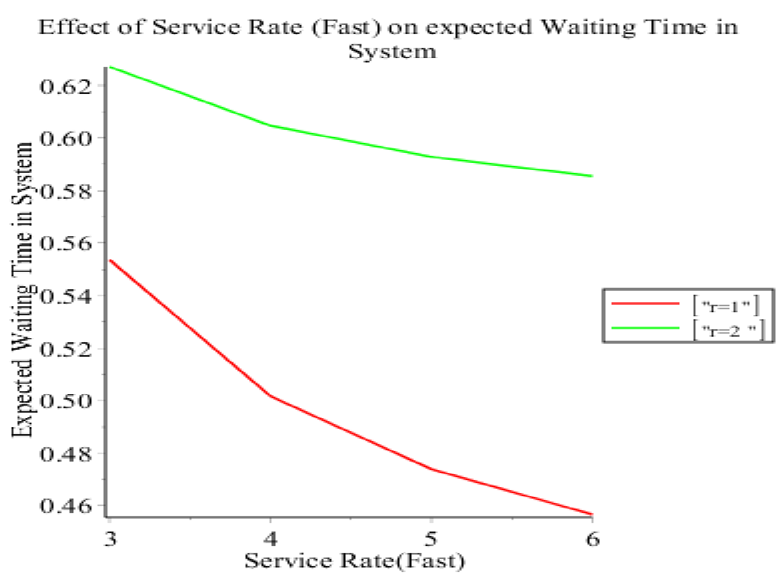

Figure 7(c)

In figure 7(c), we fix $\lambda = 1$, $\mu_1 = 2$, $\beta = 0.5$, $\xi = 0.01$, and $q = 0.8$ and display the expected waiting time by varying fast service rate for both the systems $r = 1$, and $r = 2$. It is clear from the graph that expected waiting time for both the systems decreases as service rate increases.

6. Conclusions

We have presented a detailed study of a queueing system with various parameters. We come across many situations where customer's impatience, dissatisfaction or sudden occurrence of any calamity may cause customer loss and affect the system profit as well. We have incorporated balking, catastrophes, feedback and state dependent service rate altogether to make the model more applicable in real life situations. Many practical congestion situations that we normally encounter such as manufacturing system, call center, communication and telecommunication systems, and health sector may remodel their systems to improve the output by using the results so obtained as tools. Using matrix-geometric solution method, we have analysed the steady-state behaviour of the system and evaluated various performance indicators for the same. An expected cost and profit analysis for the system has been presented and discussed with different set of parameters. From the graphs, it is clear that the optimal value for 'r' is 1. Also, we conclude that server can opt for a faster rate if the arrival rate dominates the initial service rate.

7. Future Considerations

Many real life congestion problems which have special structural properties can be easily solved using matrix-geometric technique even if the dimensions are of higher order. The work can be further extended for markovian and non-markovian queueing networks by considering different parameters along with their transient solutions.

Acknowledgments

We thank the Editors and the reviewers for their constructive comments that helped to improve this paper.

References

Abou-El-Ata, M. O. (1991). The state-dependent queue: M/M/1/N with reneging and general balk functions. *Microelectronics Reliability*, **31 (5)**, 1001-1007.

Ancker Jr, C. J., and Gafarian, A.V. (1963). Some queuing problems with balking and reneging-I. *Operations Research*, **11 (1)**, 88-100.

Bouchentouf, A.A., Cherfaoui, M. and Boualem, M. (2019). Performance and economic analysis of a single server feedback queueing model with vacation and impatient customers. *Opsearch*, **56 (1)**, 300-323.

Bura, G. S., and Bura, R.N. (2015). Time dependent analysis of a queueing system incorporating the effect of environment, catastrophe and restoration. *Journal of Reliability and Statistical Studies*, **8 (2)**, 29-40.

Choudhury, G. and Paul, M. (2005). A two phase queueing system with Bernoulli feedback. *International Journal of Information and Management Sciences*, **16 (1)**, 35-52.

D'Avignon, G. and Disney, R. (1976). Single-server queues with state-dependent feedback. *INFOR: Information Systems and Operational Research*, **14 (1)**, 71-85.

Doshi, B. T. and Jagerman, D. L. (1986). An M/G/1 queue with class dependent balking (reneging). In *Proc. of the international seminar on Teletraffic analysis and computer performance evaluation*, 225-243.

Haight, F. A. (1957). Queueing with balking. *Biometrika*, **44(3/4),** 360-369.

Kumar, B. K., Madheswari, S. P. and Lakshmi, S. A. (2013). An M/G/1 Bernoulli feedback retrial queueing system with negative customers. *Operational Research*, **13 (2),** 187-210.

Kumar, B.K., Anantha Lakshmi, S. R., Anbarasu, S. and Pavai Madheswari, S. (2014). Transient and steady-state analysis of queueing systems with catastrophes and impatient customers. *International Journal of Mathematics in Operational Research*, **6 (5)**, 523-549.

Santhakumaran, A. and Thangaraj, V. (2000). A single server queue with impatient and feedback customers. *International Journal of Information and Management Sciences*, **11 (3)**, 71-80.

Takacs, L. (1963). A single-server queue with feedback. *Bell System Technical Journal*, **42 (2)**, 505-519.

Thangaraj, V. and Vanitha, S. (2009). On the analysis of M/M/1 feedback queue with catastrophes using continued fractions. *International Journal of Pure and Applied Mathematics*, **53 (1)**, 131-151.

Varalakshmi, M., Chandrasekaran, V. M. and Saravanarajan, M. C. (2018). A single server queue with immediate feedback, working vacation and server breakdown. *International Journal of Engineering and Technology*, **7 (4.10)**, 476-479.