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OCD for D-optimal CRD with Odd Replication

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Abstract

Optimum allocation with an Orthogonal Covariate Design (OCD) with only entries ± 1, can not be accommodated to CRD with at least one of the replications odd, such that treatment-effects and covariate-effects are independently estimable. A CRD with at least one of replications odd, accommodated with an OCD of entries ± 1, can be D-optimal, but it cannot permit the independent estimation of treatment-effects and covariate-effects. Thus, usefulness of covariates to CRD is weakened. The present paper gives (i) the construction of corresponding OCD-component to treatment with replications, 3, 5 and q (odd) where $\mathbf{H}_{q-1}$ exists, separately, for D-optimal CRD and (ii) the construction of OCD for D-optimal CRD among the class of competent design permitting the independent estimation of treatment-effects and covariate-effects, when at least one of the replications is either 3, 5 or q (odd) where $\mathbf{H}_{q-1}$ exists.

Key words: Orthogonal covariate design; Restricted Hadamard matrix; Completely randomised design; Closures of order q to m; Covariate design-component.

1. Introduction

Lopes Troya (1982) initiated the problem of finding optimum covariate designs. In the same spirit, Wierich (1984) , Chadjiconstantinidis and Moyssiadis (1991), Liski *et al.* (2002), Das *et al.* (2003), Rao *et al*. (2003), Dutta *et al*. (2014) and many others have contributed to in the field of covariate design and its set-ups. Our problem for optimum allocation of OCD to CRD is that when one of the replications of treatments of CRD is odd and OCD accommodated to the CRD is with entries ± 1, the CRD cannot be D-optimal, permitting to estimate the treatment-effects and covariate-effects independently. If the CRD with a covariate set-up is D-optimal, it does not permit the independent estimation of treatment-effects and covariate-effects [*cf.* Example No. 5, Dey and Mukerjee (2006)] which can be seen from the relation (1.8). Thus, usefulness of covariates to CRD is weakened. Even when all replications of CRD are even, the optimum allocations of OCD with entries ± 1 for CRD to be D-optimal may not exist. As an example, consider a CRD (v = 3, n_1= n_2= 2, n_3= 4) accommodated with 4 covariates whose respective covariate design-components to the 1st, the 2nd and the 3rd treatments are given by

$$\mathbf{Z}_1=\begin{bmatrix}+1 & +1 & +1 & +1\\ -1 & -1 & -1 & -1\end{bmatrix}, \mathbf{Z}_2=\begin{bmatrix}-1 & +1 & +1 & -1\\ +1 & -1 & -1 & +1\end{bmatrix} \text{ and } \mathbf{Z}_3=\begin{bmatrix}+1 & +1 & -1 & -1\\ +1 & -1 & +1 & +1\\ -1 & +1 & -1 & +1\\ -1 & -1 & +1 & -1\end{bmatrix}$$

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The general FEALM of CRD (v, $n_1, n_2, \ldots, n_v$), assuming every treatment is exposed to the same number of covariates c, is

$$y_{ij} = \mu_i + \sum_{p=1}^{c} z_{ij}^{(p)} \gamma_p + e_{ij};\ i = 1, 2, \ldots, v;\ j = 1, 2, \ldots, n_i;\ \sum n_i = n, \text{ say.} \tag{1}$$

where y_{ij}, μ_i, γ_p, $z_{ij}^{(p)}$ and e_{ij} are the *j*-th observation receiving the *i*-th treatment with common variance σ^2, the *i*-th treatment-effect, the *p*-th covariate- effect, the value of the *p*-th covariate exposed to the *j*-th observation receiving the *i*-th treatment and the random error component of y_{ij} with common variance σ^2 respectively.

Denoting $\boldsymbol{\theta} = (\mu_1, \mu_2, \ldots, \mu_v, \gamma_1, \gamma_2, \ldots, \gamma_c)'$, the information matrix for $\boldsymbol{\theta}$ is given by [*cf.* (2.3.8) of Das *et al.* (2015)]

$$\sigma^{-2}\mathrm{I}(\boldsymbol{\theta}) = \sigma^{-2}\begin{bmatrix} \mathbf{N} & \mathbf{T} \\ \mathbf{T}' & \mathbf{Z}'\mathbf{Z} \end{bmatrix} \tag{2}$$

where $\mathbf{N} = diag(n_1, n_2, \ldots, n_v)$ (3)

$$\mathbf{T} = (\mathbf{T}_1', \mathbf{T}_2', \ldots, \mathbf{T}_v')'\ ;\ \mathbf{T}_i = \mathbf{1}_{n_i}'\mathbf{Z}_i \tag{4}$$

$$\mathbf{Z} = \mathbf{Z}^{(\mathrm{n}\times\mathrm{c})} = (\mathbf{Z}_1', \mathbf{Z}_2', \ldots, \mathbf{Z}_v')' \tag{5}$$

such that no column of Z is zero-column.

$$\mathbf{Z}_i = \mathbf{Z}_i^{(n_i\times c)} = \begin{bmatrix} z_{i1}^{(1)} & z_{i1}^{(2)} & \cdots & z_{i1}^{(c)} \\ z_{i2}^{(1)} & z_{i2}^{(2)} & & z_{i2}^{(c)} \\ \vdots & & \ddots & \vdots \\ z_{in_i}^{(1)} & z_{in_i}^{(2)} & \cdots & z_{in_i}^{(c)} \end{bmatrix} = [\mathbf{Z}_i^{(1)}, \mathbf{Z}_i^{(2)}, \ldots, \mathbf{Z}_i^{(c)}] \tag{6}$$

which will be known as the covariate design (CD)-component to the *i*-th treatment. Then, $\mathbf{Z}'\mathbf{Z} = \sum \mathbf{Z}_i'\mathbf{Z}_i$ and $\det(\mathbf{I}(\boldsymbol{\theta})) = \det(\mathbf{N}) \times \det(\mathbf{C}) = (\prod_{i=1}^{v} n_i) \times \det.(\mathbf{C})$, by (3) to (6)

$$\text{where } \mathbf{C} = \mathbf{Z}'\mathbf{Z} - \sum n_i^{-1}\mathbf{T}_i'\mathbf{T}_i \text{ and } \mathbf{T}_i = \mathbf{1}_{n_i}'\mathbf{Z}_i \tag{7}$$

The maximization of $\det(\mathbf{I}(\boldsymbol{\theta}))$ can be done in two stages: (i) the maximization of $\prod_{i=1}^{v} n_i$ and (ii) that of $\det(\mathbf{C})$, [*cf.* Dey and Mukerjee (2006)] which were generalised and more broadened by Dutta *et al.* (2014). From (2) it can be seen that $\mathbf{Z}'\mathbf{Z}$ is a diagonal matrix iff the covariates are orthogonal to one another and $\mathbf{1}_{n_i}'\mathbf{Z}_i = \mathbf{0}'\ \forall\ i$, *i.e.* $\mathbf{X}'\mathbf{Z} = \mathbf{0}'$ iff the covariates are orthogonal to treatments *i.e.* the covariate-effects are independently estimable iff $\mathbf{Z}'\mathbf{Z}$ is a diagonal matrix and further, the covariate-effects and the treatment-effects are independently estimable iff

$$\mathbf{1}_{n_i}'\mathbf{Z}_i = \mathbf{0}'\ \forall\ i\ ,\ i.e.\ \mathbf{X}'\mathbf{Z} = \mathbf{0}'. \tag{8}$$

Thus, when $\mathbf{1}_{n_i}'\mathbf{Z}_i = \mathbf{0}'$ $\forall$ i, each covariate matrix $\mathbf{Z}_i$ behaves independently in the sense that a covariate design-component (covariate matrix) $\mathbf{Z}_i$ does not affect another covariate design-component $\mathbf{Z}_{i'}$ $(i \neq i')$. When at least one of n_i's is odd and $z_{ij}^{(p)} = \pm 1$, at least one of the relations (i) $\mathbf{T} = \mathbf{O}$, (ii) $\mathbf{Z}'\mathbf{Z} = n\mathbf{I}_c$ and (iii) n is multiple of v, is violated in obtaining c-Orthogonal Covariate Design (OCD) for the CRD $(v, n_1, n_2, \ldots, n_v)$; $\Sigma n_i = n$. Now, we will look for the situations where the relation (i) holds but the relation (ii) does not. Further, by (7) if $\|\mathbf{Z}_i^{(p)}\|$ are at maximum $\forall\, i, p$, then det.($\mathbf{I}(\boldsymbol{\theta})$) obtains its maximum for D-optimal CRD. Now, the focus is on maximization of $\|\mathbf{Z}_i^{(p)}\| \ni \mathbf{T}_i = \mathbf{0}'$ $\forall\, i, p$.

Definition 1: Given a set S (whether finite or infinite) = $\{\mathbf{x}_i / \mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im})'; x_{ij} \in [-1, 1]; i = 1,2, \ldots\}$, $\mathbf{x}_{\alpha_1}$, $\mathbf{x}_{\alpha_2}$, …, $\mathbf{x}_{\alpha_q}$ are said to be closures of order q to m among $\mathbf{x}_i$'s (simply, closures of order q) if $\|\mathbf{x}_{\alpha_1}\| \geq \|\mathbf{x}_{\alpha_2}\| \geq \ldots \geq \|\mathbf{x}_{\alpha_q}\| \geq \|\mathbf{x}_\beta\|$ $\forall\, \mathbf{x}_\beta \in S$ where $\mathbf{x}_{\alpha_1}, \mathbf{x}_{\alpha_2}, \ldots, \mathbf{x}_{\alpha_q} \in S$ and other vectors are said to be non-closures of order q among $\mathbf{x}_i$'s.

When $q = 1$, $\mathbf{x}_{\alpha_1}$ is said to be closure to m among all other vectors (simply, a closure to m).

Remark 1:

(a) For a given set S, closures of a particular order is not unique. E.g., letting $S = \{(-1, 1, -1, 1)', (-1, -1, 1, 1)', (-1, 1, 1, -1)', (1, -1, 1, -1)', (1, 1, -1, -1)', (1, -1, -1, 1)'\} = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_6\}$, say, any $q(\leq 4)$ out of 6 $\boldsymbol{s}_i$'s are closures of order $q(\leq 4)$.

(b) A vector $\boldsymbol{x} \in S$ may be one of the closures of an order and may not be of other smaller order.

(c) A vector $\mathbf{x} \in S \subseteq R^m$, which is one of the closures of an order, may not be closure of the same order if there exists at least another vector $\mathbf{y}$ ($\in S$) $\ni \|\mathbf{x}\| = \|\mathbf{y}\|$. That is, if there exist some vectors $\mathbf{x}_{\alpha_1}$, $\mathbf{x}_{\alpha_2}$, …, $\mathbf{x}_{\alpha_u} \ni \|\mathbf{x}_{\alpha_1}\| = \|\mathbf{x}_{\alpha_2}\| = \ldots = \|\mathbf{x}_{\alpha_u}\| = \|\boldsymbol{x}\|$, each of them can be treated as one of the closures of the order q to m and at the same time can be treated as one of the non-closures of the order q to m.

(d) By the definition of closure of an order to m, a vector $\boldsymbol{x}_{\boldsymbol{\alpha}}$ can be treated as a closure of any order to m if $\mathbf{x}_{\alpha j} = \pm 1$ $\forall$ j =1, 2, …, m i.e. $\|\mathbf{x}_{\boldsymbol{\alpha}}\| = m$.

Definition 2: An OCD-component $\mathbf{Z}_i = [\mathbf{Z}_i^{(1)}, \mathbf{Z}_i^{(2)}, \ldots, \mathbf{Z}_i^{(c)}]$ is said to be optimal if $\mathbf{Z}_i^{(1)}, \mathbf{Z}_i^{(2)}, \ldots, \mathbf{Z}_i^{(c)}$ are closures of order c to n_i.

Definition 3: An $m \times n$-matrix $\mathbf{A} = [a_{ij}]$; $a_{ij} \in [-1, 1]$, is said to be restricted Hadamard matrix if $\mathbf{A}'\mathbf{A}$ is diagonal and $\mathbf{1}'\mathrm{A} = \mathbf{0}'$.

For finding the number of orthogonal covariate components to n_i observations receiving the i-th treatment (OCD-component to the i-th treatment) in CRD, a lemma is given at below:

Lemma 1: For a restricted Hadamard matrix $\mathbf{A}_{m \times n}$, n is less than or equal to $m - 1$.

Proof of the lemma is given in APPENDIX-A.

Some propositions whose proof are straight forward, are given below, for future use.

Proposition 1: If $n_i \times m$-H_i ; i = 1, 2, …, s, are s restricted Hadamard matrices, then $\mathbf{H}$ = $(\mathbf{H}_1', \mathbf{H}_2', \ldots, \mathbf{H}_s')'$ is again so.

Obviously, $\mathbf{H}'\mathbf{H}$ = diag.$(\sum_i a_{i1}, \sum_i a_{i2}, \ldots, \sum_i a_{im})$ where $\mathbf{H}_i'\mathbf{H}_i$ = diag.$(a_{i1}, a_{i2}, \ldots, a_{im})$.

Proposition 2: If $n \times m$- $\mathbf{H}$ = $(\mathbf{h}_1, \ldots, \mathbf{h}_m)$ is a restricted Hadamard matrix, then $\mathbf{H}^*$ = $(\mathbf{h}_{\alpha_1}, \ldots, \mathbf{h}_{\alpha_s})$; $s < m$, is again so, where $\mathbf{h}_{\alpha_i}$'s are distinct and columns of $\mathbf{H}$.

Obviously, $\mathbf{H}^{*'}\mathbf{H}^*$= diag.$(a_{\alpha_1}, \ldots, a_{\alpha_s})$ where a_{α_i} = $\|\mathbf{h}_{\alpha_i}\|$.

Proposition 3: If $n \times m$- $\mathbf{H}$ = $(\mathbf{h}_1, \ldots, \mathbf{h}_m)$ is a restricted Hadamard matrix, then $\mathbf{H}^*$ = $(\mathbf{h}_1, \ldots, \mathbf{h}_m, \mathbf{0}, \ldots, \mathbf{0})$ of order $n \times (m + s)$ is again so.

Obviously, $\mathbf{H}^{*'}\mathbf{H}^*$= $diag.(a_1, \ldots, a_m, 0, \ldots, 0)$, where $\mathbf{H}'\mathbf{H}$ = $diag.(a_1, \ldots, a_m)$.

2. Construction of OCD-component

As covariate-effects and treatment-effects are to be independently estimable, $\mathbf{T} = (\mathbf{T}_1', \ldots, \mathbf{T}_v')'$ i.e. $(\mathbf{1}_{n_1}'\mathbf{Z}_1, \ldots, \mathbf{1}_{n_v}'\mathbf{Z}_v)'$ i.e. $\mathbf{X}'\mathbf{Z} = \mathbf{0}_{v\times c}$. Each of $\mathbf{Z}_i$'s has its own complete structure in the sense that no entry in $\mathbf{Z}_i$ does not depend on any entry in $\mathbf{Z}_{i'}$ $(i\neq i')$ concerning the independent estimation of covariate-effects and treatment-effects. In other words, none of $\mathbf{Z}_i$'s depends on the remaining others concerning the independent estimation of covariate-effects and treatment-effects. So, in this section, we focus on construction-methods of OCD-components (which are closures of some order) to the β-th treatment of CRD to be D-optimal among all possible OCD-components of the competent design, permitting the independent estimation of covariate-effects and treatment-effects for which n_β= 3, 5 and q (odd) provided Hadamard matrix $\mathbf{H}_{q-1}$ exists. The need of such OCD-components lies when at least one of n_i's is 3, 5 or q odd. Meanwhile, for the remaining n_i's, apart from 3, 5 and q (odd) provided $\mathbf{H}_{q-1}$ exists, their corresponding OCD-components are assumed to be known and existed. The construction-methods are dealt with through induction of solving restrictions imposed by the orthogonal conditions among covariates and that of covariates to the treatments.

2.1. Construction of OCD-component for n_β = 3 in CRD.

Let n_β = 3 be one of the n_i's and $\mathbf{Z}_\beta^{(p)} = (z_{\beta 1}^{(p)}, z_{\beta 2}^{(p)}, z_{\beta 3}^{(p)})'$. Then, by Lemma 1, p equals to 2 at most *i.e.* c = 2.

Case I. Considering c = 1, by the condition of orthogonal of covariate to the β-th treatment,

$$\sum_{j=1}^{3} z_{\beta j}^{(1)} = 0. \tag{9}$$

WOLG, assume $\widehat{z_{\beta 1}^{(1)}} = a_1, \widehat{z_{\beta 2}^{(1)}} = a_2 \ni a_1, a_2 \in [-1, 1]$ for (9) becomes a non-homogeneous linear equation involving one unknown variable so as $\widehat{z_{\beta 3}^{(1)}} = -(a_1 + a_2) \in [-1, 1]$.

Therefore, $$\widehat{\mathbf{Z}_{\boldsymbol{\beta}}^{(1)}} = (a_1, a_2, -(a_1 + a_2))' \tag{10}$$

Case II. Considering c = 2, by the condition of orthogonal of covariates to the β-th treatment and that among covariates, we have

$$\sum_{j=1}^{3} z_{\beta j}^{(1)} = \sum_{j=1}^{3} z_{\beta j}^{(2)} = 0 \tag{11}$$

$$\sum_{j=1}^{3} z_{\beta j}^{(1)} z_{\beta j}^{(2)} = 0. \tag{12}$$

Using (10) of the Case I: c = 1, for estimating $\mathbf{Z}_{\boldsymbol{\beta}}^{(2)}$, (11) and (12) give

$$\sum_{j=1}^{3} z_{\beta j}^{(2)} = 0 \tag{13}$$

$$a_1 z_{\beta 1}^{(2)} + a_2 z_{\beta 2}^{(2)} - (a_1 + a_2)\, z_{\beta 3}^{(2)} = 0. \tag{14}$$

WOLG, assume $\widehat{z_{\beta 1}^{(2)}} = b_1 \in [-1, 1]$ for (13) and (14) become a system of 2 linearly independent non-homogeneous linear equations involving two unknown variables (independent is due to orthogonal between $\mathbf{Z}_{\beta}^{(1)}$ and $\mathbf{Z}_{\beta}^{(2)}$)

$$b_1 + z_{\beta 2}^{(2)} + z_{\beta 3}^{(2)} = 0 \tag{15}$$

$$a_1 b_1 + a_2 z_{\beta 2}^{(2)} - (a_1 + a_2) z_{\beta 3}^{(2)} = 0 \tag{16}$$

so as $\widehat{z_{\beta 2}^{(2)}} = -(a_2 + 2a_1)b_1/(2a_2 + a_1) \in [-1, 1]$, $\widehat{z_{\beta 3}^{(2)}} = (a_1 - a_2)b_1/(2a_2 + a_1) \in [-1, 1]$, by (15) and (16).

Consequently, $\widehat{\mathbf{Z}_{\beta}^{(2)}} = (b_1, -(a_2 + 2a_1)b_1/(2a_2 + a_1), (a_1 - a_2)b_1/(2a_2 + a_1))'$. (17)

Therefor, $\widehat{\mathbf{Z}_{\boldsymbol{\beta}}} = \begin{bmatrix} a_1 & b_1 \\ a_2 & -(a_2 + 2a_1)b_1/(2a_2 + a_1) \\ -(a_1 + a_2) & (a_1 - a_2)b_1/(2a_2 + a_1) \end{bmatrix}$, by (10) and (17) (18)

It is seen from (18) that infinite number of $\mathbf{Z}_{\boldsymbol{i}}$'s exist for various values of a_1, a_2, and b_1. All these $\mathbf{Z}_{\boldsymbol{i}}$'s are restricted Hadamard matrices of order 3×2. Searching the largest value of $\left\|\mathbf{Z}_{\beta}^{(p)}\right\| \ni z_{\beta j}^{(p)} \in [-1, 1]$ *i.e.* searching the closures of order 2 to 3, the possible forms of $\mathbf{Z}_{\beta}$ are given in the following table.

Table 1:

Sr. No.	1	2	3	4	5	6	7	8
$\mathbf{Z}_{\boldsymbol{\beta}}$	$\begin{matrix} \frac{1}{2} & 1 \\ \frac{1}{2} & -1 \\ -1 & 0 \end{matrix}$	$\begin{matrix} -\frac{1}{2} & 1 \\ -\frac{1}{2} & -1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} \frac{1}{2} & -1 \\ \frac{1}{2} & 1 \\ -1 & 0 \end{matrix}$	$\begin{matrix} -\frac{1}{2} & -1 \\ -\frac{1}{2} & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} \frac{1}{2} & 1 \\ -1 & 0 \\ \frac{1}{2} & -1 \end{matrix}$	$\begin{matrix} -\frac{1}{2} & 1 \\ 1 & 0 \\ -\frac{1}{2} & -1 \end{matrix}$	$\begin{matrix} \frac{1}{2} & -1 \\ -1 & 0 \\ \frac{1}{2} & 1 \end{matrix}$	$\begin{matrix} -\frac{1}{2} & -1 \\ 1 & 0 \\ -\frac{1}{2} & 1 \end{matrix}$
Sr. No.	9	10	11	12	13	14	15	16
$\mathbf{Z}_{\boldsymbol{\beta}}$	$\begin{matrix} -1 & 0 \\ \frac{1}{2} & -1 \\ \frac{1}{2} & 1 \end{matrix}$	$\begin{matrix} 1 & 0 \\ -\frac{1}{2} & -1 \\ -\frac{1}{2} & 1 \end{matrix}$	$\begin{matrix} -1 & 0 \\ \frac{1}{2} & 1 \\ \frac{1}{2} & -1 \end{matrix}$	$\begin{matrix} 1 & 0 \\ -\frac{1}{2} & 1 \\ -\frac{1}{2} & -1 \end{matrix}$	$\begin{matrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \\ 0 & -1 \end{matrix}$	$\begin{matrix} -1 & \frac{1}{2} \\ 1 & \frac{1}{2} \\ 0 & -1 \end{matrix}$	$\begin{matrix} 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \\ 0 & 1 \end{matrix}$	$\begin{matrix} -1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \\ 0 & 1 \end{matrix}$
Sr. No.	17	18	19	20	21	22	23	24
$\mathbf{Z}_{\boldsymbol{\beta}}$	$\begin{matrix} 1 & \frac{1}{2} \\ 0 & -1 \\ -1 & \frac{1}{2} \end{matrix}$	$\begin{matrix} -1 & \frac{1}{2} \\ 0 & -1 \\ 1 & \frac{1}{2} \end{matrix}$	$\begin{matrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ -1 & -\frac{1}{2} \end{matrix}$	$\begin{matrix} -1 & -\frac{1}{2} \\ 0 & 1 \\ 1 & -\frac{1}{2} \end{matrix}$	$\begin{matrix} 0 & -1 \\ -1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{matrix}$	$\begin{matrix} 0 & -1 \\ 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{matrix}$	$\begin{matrix} 0 & 1 \\ -1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{matrix}$	$\begin{matrix} 0 & 1 \\ 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{matrix}$

A lemma to be used later on, is proposed here.

Lemma 2. The $\mathbf{Z}_\beta$'s given in the Table 1 are the possible optimal OCD-components of CRD $(v, n_1, n_2, \ldots, n_v)$ to the β-th treatment with replication 3 *i.e.* $n_\beta = 3$.

Proof of the lemma is given in APPENDIX-B.

2.2. Construction of OCD-component for $n_\beta = 5$ in CRD.

Let $n_\beta = 5$ be one of the n_i's and $\mathbf{Z}_\beta^{(p)} = (z_{\beta 1}^{(p)}, z_{\beta 2}^{(p)}, z_{\beta 3}^{(p)}, z_{\beta 4}^{(p)}, z_{\beta 5}^{(p)})'$.

For c orthogonal covariates, by the condition of orthogonal of the covariates to the β-th treatment and that among themselves, we have $c(c+1)/2$ relations

$$\left.\begin{aligned} \textstyle\sum_j z_{\beta j}^{(p)} &= 0 \;\; \forall\, p = 1, 2, \ldots\ldots c \\ \textstyle\sum z_{\beta j}^{(p)} z_{\beta j}^{(p')} &= 0 \;\; \forall\, p \neq p' = 1, 2, \ldots\ldots c \end{aligned}\right\} \quad (19)$$

By the Lemma 1, there exist at most 4 orthogonal covariates *i.e.* p equals to 4 at most, which will be enumerated in the following. Considering $c = 1, 2, 3, 4$ successively and assuming 4, 3, 2, 1 values of $z_{\beta j}^{(p)}$ as follows (for c = 1) $z_{\beta 1}^{(1)} = a_1$, $z_{\beta 2}^{(1)} = a_2$, $z_{\beta 3}^{(1)} = a_3$, $z_{\beta 4}^{(1)} = a_4$, (for c = 2) $z_{\beta 1}^{(2)} = b_1$, $z_{\beta 2}^{(2)} = b_2$, $z_{\beta 3}^{(2)} = b_3$, (for c = 3) $z_{\beta 1}^{(3)} = c_1$, $z_{\beta 2}^{(3)} = c_2$, (for c = 4) $z_{\beta 1}^{(4)} = d_1$ in (19) correspondingly, ∍ a_1 to a_4, b_1 to b_3, c_1, $c_2, d_1 \in [-1, 1]$, there are q independent non-homogenous linear equations involving q unknown variables for c = q; q = 1, 2, 3, 4 [independent is due to orthogonal among $\mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{p})}{}'s$] which give the solutions $\widehat{z_{\beta 5}^{(1)}}, \widehat{z_{\beta 4}^{(2)}}, \widehat{z_{\beta 5}^{(2)}}, \widehat{z_{\beta 3}^{(3)}}$, $\widehat{z_{\beta 4}^{(3)}}$, $\widehat{z_{\beta 5}^{(3)}}$, $\widehat{z_{\beta 2}^{(4)}}$, $\widehat{z_{\beta 3}^{(4)}}$, $\widehat{z_{\beta 4}^{(4)}}$, $\widehat{z_{\beta 5}^{(4)}}$. These solutions are in terms of a_1 *to* a_4, b_1 *to* b_3, c_1, c_2, and d_1. Thus, the covariate matrix $\mathbf{Z}_{\boldsymbol{\beta}}$ to the β-th treatment is given by

$$\mathbf{Z}_\beta = ((a_1,\ a_2,\ a_3, a_4, \widehat{z_{\beta 5}^{(1)}})', (b_1, b_2, b_3, \widehat{z_{\beta 4}^{(2)}}, \widehat{z_{\beta 5}^{(2)}})', (c_1, c_2, \widehat{z_{\beta 3}^{(3)}}, \widehat{z_{\beta 4}^{(3)}}, \widehat{z_{\beta 5}^{(3)}})', (d_1, \widehat{z_{\beta 2}^{(4)}}, \widehat{z_{\beta 3}^{(4)}}, \widehat{z_{\beta 4}^{(4)}}, \widehat{z_{\beta 5}^{(4)}})') \tag{20}$$

It is seen from (20) that infinite number of $\mathbf{Z}_i$'s exist for various values of a_1 to a_4, b_1 to b_3, c_1, c_2 and d_1. Any OCD-component to a treatment having replication 5, is given by (20). All these $\mathbf{Z}_i$'s are restricted Hadamard matrices of order 5×4. Searching the largest value of $\left\|\mathbf{Z}_\beta^{(p)}\right\| \ni z_{\beta j}^{(p)} \in [-1,1]$ i.e. searching the closures of order 4 to 5, the possible forms of $\mathbf{Z}_\beta$ are given in the following table.

Table 2:

Sl. No.	a_1	a_2	a_3	a_4	b_1	b_2	b_3	c_1	c_2	d_1
1	1	-1	1	-1	1	1	-1	1	-1	$\pm 1/4$
2					1	-1	-1	1	1	$\pm 1/4$
3					-1	-1	1	-1	1	$\pm 1/4$
4					-1	1	1	-1	-1	$\pm 1/4$
5					-1	-1	1	1	-1	$\pm 1/4$
6					1	-1	-1	-1	-1	$\pm 1/4$
7					1	1	-1	-1	1	$\pm 1/4$
8					-1	1	1	1	1	$\pm 1/4$
9	1	1	1	1	1	1	-1	1	-1	$\pm 1/4$
10					1	-1	1	1	1	$\pm 1/4$
11					-1	-1	1	-1	1	$\pm 1 / 4$
12					-1	1	-1	-1	-1	$\pm 1 / 4$
13					-1	-1	1	1	-1	$\pm 1 / 4$
14					1	-1	1	-1	-1	$\pm 1 / 4$
15					1	1	-1	-1	1	$\pm 1 / 4$
16					-1	1	-1	1	1	$\pm 1 / 4$
17	1	1	-1	-1	1	-1	1	1	-1	$\pm 1 / 4$
18					1	-1	-1	1	-1	$\pm 1 / 4$
19					-1	1	-1	-1	1	$\pm 1 / 4$
20					-1	1	1	-1	1	$\pm 1 / 4$
21					-1	1	-1	1	-1	$\pm 1 / 4$
22					1	-1	-1	-1	1	$\pm 1 / 4$
23					1	-1	1	-1	1	$\pm 1 / 4$
24					-1	1	1	1	-1	$\pm 1 / 4$
25	-1	1	-1	1	-1	-1	1	-1	1	$\pm 1 / 4$
26					-1	1	1	-1	-1	$\pm 1 / 4$
27					1	1	-1	1	-1	$\pm 1 / 4$
28					1	-1	-1	1	1	$\pm 1 / 4$
29					1	1	-1	-1	1	$\pm 1 / 4$
30					-1	1	1	1	1	$\pm 1 / 4$
31					-1	-1	1	1	-1	$\pm 1 / 4$
32					1	-1	-1	-1	-1	$\pm 1 / 4$

33	-1	1	1	-1	-1	-1	1	-1	1	$\pm 1/4$
34					-1	1	-1	-1	-1	$\pm 1/4$
35					1	1	-1	1	-1	$\pm 1/4$
36					1	-1	1	1	1	$\pm 1/4$
37					1	1	-1	-1	1	$\pm 1/4$
38					-1	1	-1	1	1	$\pm 1/4$
39					-1	-1	1	1	-1	$\pm 1/4$
40					1	-1	1	-1	-1	$\pm 1/4$
41	-1	-1	1	1	-1	1	-1	-1	1	$\pm 1/4$
42					-1	1	1	-1	1	$\pm 1/4$
43					1	-1	1	1	-1	$\pm 1/4$
44					1	-1	-1	1	-1	$\pm 1/4$
45					1	-1	1	-1	1	$\pm 1/4$
46					-1	1	1	1	-1	$\pm 1/4$
47					-1	1	-1	1	-1	$\pm 1/4$
48					1	-1	-1	-1	1	$\pm 1/4$

A lemma follows, for use in sequel.

Lemma 3. The $\mathbf{Z}_\beta$'s given in the Table 2 are the possible optimal OCD-components of CRD (v , $n_1, n_2, \ldots, n_v$) to the β-th treatment having replication 5 i.e. $n_\beta = 5$.

Proof of the lemma is given in Appendix-C.

2.3. Construction of OCD-component for n_β = odd in CRD where $\mathbf{H}_{n_\beta - 1}$ exists

Let n_β (odd) be one of the n_i's . Under the condition that $\mathbf{H}_{n_\beta - 1}$ exists, a more general construction of OCD-component of CRD (v, $n_1, n_2, \ldots, n_v$) to the β-th treatment, follows as a lemma.

Lemma 4. An optimum allocation of OCD-component to the β-th treatment with replication n_β (odd) where $\mathbf{H}_{n_\beta - 1}$ exists, is given by

$$\mathbf{Z}_\beta = \mathbf{Z}_\beta^{n_\beta \times (n_\beta - 1)} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \ldots. & \mathbf{h}_{n_\beta - 2} & \delta / (n_\beta - 1) \mathbf{1}_{n_\beta - 1} \\ 0 & 0 & \ldots. & 0 & -\delta \end{bmatrix}$$

where $\delta = \pm 1$ and $\mathbf{h}_j$'s are given by $\mathbf{H}_{n_\beta - 1} = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n_\beta - 2}, \mathbf{1}_{n_\beta - 1}]$; $j = 1, 2, \ldots, n_\beta$.

Proof of the lemma is given in Appendix-D.

3. Construction of OCD for D-Optimal CRD.

In this section there are 3 sub-sections, focusing on the constructions of c-OCD for D-optimal CRD $(v, n_1, n_2, \ldots, n_v)$, permitting the independent estimation of treatment-effects and covariate-effects among the component CRD's, when at least one of $n_i{}'s$ is equal to 3, 5 or q

odd such that there exists $\mathbf{H}_{q-1}$. By the definition of restricted Hadamard matrix in Definition 3, CD-components should be a restricted Hadamard matrix so as treatment-effects and covariate-effects are independently estimable. Further, to obtain the maximum det.($\mathbf{I}(\boldsymbol{\theta})$), each of OCD-components of the CRD $(v, n_1, n_2, \dots, n_v)$ to the β-th treatment are to be closures of order c to n_β for all β.

3.1. OCD for D-optimal CRD ($n_\beta = 3$)

For a 2 OCD-component of a D-optimal CRD $(v, n_1, n_2, \dots, n_v)$ D^*, permitting the independent estimation of treatment-effects and covariate-effects, let $\mathbf{Z}_i$ be the OCD-component to the i-th treatment of the D-optimal CRD D^*.

Since D^* permits the independent estimation of treatment-effects and covariates-effects, then

$$\mathbf{T}_i^* = \mathbf{0}' \;\forall \text{ i } i.e. \boldsymbol{X}^{*\prime}\mathbf{Z}^* = 0,$$
$$\mathbf{Z}^{*\prime}\mathbf{Z}^* = (\mathbf{Z}_1', \dots, \mathbf{Z}_v')(\mathbf{Z}_1', \dots, \mathbf{Z}_v')' = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \text{ say.}$$

And further, det.$(\mathbf{I}^*(\boldsymbol{\theta}))$of $D^* = (\prod_{\mathrm{i}=1}^{v} n_i)$ det.$(\mathbf{Z}^{*\prime}\mathbf{Z}^* - \mathbf{T}^{*\prime}\mathbf{N}^{*-1}\mathbf{T}^*)$
$$= (\textstyle\prod_{i=1}^{v} n_i)\alpha_1\alpha_2 \geq \text{det.}(\mathbf{I}^{**}(\boldsymbol{\theta}))\text{of } D^{**}$$

where D^{**} belongs to the class of all the competent CRD's $(v, n_1, n_2, \dots, n_v)$, permitting the independent estimation of treatment-effects and covariate-effects and $\mathbf{I}^{**}(\boldsymbol{\theta})$is the information matrix for $\boldsymbol{\theta}$ through D^{**}. Taking $\mathbf{Z}_i (i = 1, 2, \dots, v)$ as the OCD-component to the i-th treatment and $\mathbf{Z}_{(v+1)}$= ((1/2, 1/2, -1)$'$, (1, 1, 0)$'$) as that to the (v + 1)-th treatment of a new CRD $(v, n_i\, ; i = 1, 2, \dots, v\, ;\; n_{v+1} = 3)D$, say.

$$\text{Then, } \mathbf{Z} = \left[\mathbf{Z}_1',\; \dots, \mathbf{Z}_v',\; \begin{pmatrix} 1/2 & 1 \\ 1/2 & 1 \\ -1 & 0 \end{pmatrix}'\right]' \tag{21}$$

$$\mathbf{T}_i = \mathbf{0}' \;\forall\, i = 1, 2, \dots, v{+}1 \text{ i. e. } \mathbf{X}'\mathbf{Z} = \mathbf{0} \tag{22}$$

$$\mathbf{Z}'\mathbf{Z} = (\mathbf{Z}_1', \dots, \mathbf{Z}_v', \mathbf{Z}_{v+1}')(\mathbf{Z}_1', \dots, \mathbf{Z}_v', \mathbf{Z}_{v+1}')'$$

$$= \begin{pmatrix} \alpha_1 + 3/2 & 0 \\ 0 & \alpha_2 + 2 \end{pmatrix}, \text{ from Proposition 1, by (21) and (22).} \tag{23}$$

$$\text{Then, det.}\big(\mathbf{I}(\boldsymbol{\theta})\big)\text{of } D = (\textstyle\prod_{i=1}^{v} n_i)(\alpha_1 + 3/2)(\alpha_2 + 2), \text{ by (23)} \tag{24}$$

which is the maximum of determinant of information matrix of a CRD (v+1, $n_1, n_2, \dots, n_v$, n_{v+1}=3) among all the competent CRD's accommodated with various possible OCD-components to the (v+1)-th treatment. Thus, a theorem is immediate.

Theorem 1: Existence of 2-OCD of D-optimal CRD (v,n_1,n_2, ...,n_v), permitting the independent estimation of treatment-effects and covariate-effects implies that of 2-OCD of D-optimal CRD (v+1,n_1,.....n_v,$n_{v+1} = 3$), maintaining the same estimation status.

Proof: Suppose the proposed OCD for the new resultant CRD is not an optimal allocation. Then, there exists an optimal OCD $\mathbf{Z}^{***}$ ∍ det.($\mathbf{Z}^{***\prime}\mathbf{Z}^{***}$) > ($\alpha_1$+3/2) ($\alpha_2 + 2$), by (24) *i.e.* by (18) ∃ some $a_1, a_2, b_1 \in [-1,1]$ ∍ $[\alpha_1+a_1^2+a_2^2+(a_1+a_2)^2][\alpha_2+b_1^2+\{(a_2+2a_1)b_1/(2a_2+a_1)\}^2+\{(a_1-a_2)b_1/(2a_2+a_1)\}^2] > (\alpha_1+3/2)(\alpha_2+2)$

i.e. $[a_1^2+a_2^2+(a_1+a_2)^2]\,[b_1^2+\{(a_2+2a_1)b_1/(2a_2+a_1)\}^2+\{(a_1-a_2)b_1/(2a_2+a_1)\}^2]>3$
since $(\delta_1+x_1)(\delta_2+x_2) > (\delta_1+y_1)(\delta_2+y_2)$ iff $x_1x_2> y_1y_2\ \forall\ \delta_1, \delta_2, x_1, x_2, y_1, y_2>0$

i.e. $\{2a_1^2+2a_2^2+2a_2a_1\}b_1^2\{4a_2^2+a_1^2+4a_1a_2+a_2^2+4a_1^2+4a_1a_2+a_2^2+a_1^2-2a_1a_2\}/(2a_2+a_1)^2 > 3$ which is impossible as it is same to (B1) and (B1) is impossible.

Remark 2: By applying an induction-method for constructing OCD-components to treatments with replication number 3, the construction of 2-OCD of D-optimal CRD with replication number n_i (permitting independent estimation of treatment-effects and covariate-effects), out of which some are equal to 3, is always possible.

Note 1: By Lemma 1, $n_i \geq 3\ \forall\ i = 1, 2, \ldots, v$, otherwise 2-OCD of D-optimal CRD $(v, n_1, n_2, \ldots, n_v)$, permitting independent estimation of treatment-effects and covariate-effects, never exists.

Hadamard matrix $\mathbf{H}_{n_i}$(if exists for $n_i \geq 3\ \forall\ i$) can be rewritten as $\mathrm{H}_{n_i} = \left[\mathbf{h}_1^{(i)}, \ldots, \mathbf{h}_{n_i-1}^{(i)},\ \mathbf{1}_{n_i}\right]$. The vectors $\mathbf{h}_1^{(i)}, \ldots, \mathbf{h}_{n_i-1}^{(i)}$ are closures of order n_i -1 to n_i. Therefore, a 2-OCD of D-optimal CRD $(v, n_1, n_2, \ldots, n_{v-1})$ exists, permitting independent estimation of treatment-effects and covariate-effects. Applying Theorem 1, a corollary follows.

Corollary 1: Existence of 2-OCD of D-optimal CRD $(v, n_1, \ldots, n_{v-1}, n_v=3)$ permitting the independent estimation of treatment-effects and covariate-effects, is always possible, provided $\mathbf{H}_{n_i}$ exists and $n_i \geq 3\ \forall\ i= 1, 2, \ldots, v$-1.

Using the OCD-component in Sr. No. 17, Table 1, an example of Theorem 1 is given below:

Example 1: Given a 2-OCD $\mathbf{Z}^* = (\mathbf{Z}_1', \mathbf{Z}_2')'$ of D-optimal CRD ($v = 2, n_1 = n_2 = 4$) permitting independent estimation of treatment-effects and covariate-effects, taking the OCD-component in Sr. No. 17, Table 1 as OCD-component of the 3rd treatment of a CRD ($v = 2, n_1 = n_2 = 4, n_3 = 3$), the det. of the information matrix i.e. $|\mathrm{I}(\boldsymbol{\theta})|$ attains maximum value $(4^2\times3)(8+2)(8+3/2)$ and the resultant CRD maintains the same estimation status where the OCD is given by $\mathbf{Z} = (\mathbf{Z}_1', \mathbf{Z}_2', \mathbf{Z}_3')'$; $\mathbf{Z}_1 = \mathbf{Z}_2 = ((-1, -1, 1, 1)', (-1, 1, -1, 1)')$, $\mathbf{Z}_3 = ((1, 0, -1)', (1/2, -1, 1/2)')$.

3.2. OCD for D-optimal CRD (n_β = 5).

For a 4-OCD of a CRD $(v, n_1, n_2, \ldots, n_v)$ D^*, permitting the independent estimation of treatment-effects and covariate-effects, let $\mathbf{Z}_i$ be the OCD-component to the i-th treatment of the CRD D^*. Since D^* permits the independent estimation of treatment-effects and covariates-effects, then

$$\mathbf{T}_i^* = \mathbf{0}'\ \ \forall\, \mathrm{i}\ \text{ i.e. } \mathbf{X}^{*\prime}\mathbf{Z}^* = 0,$$

$$\mathbf{Z}^{*\prime}\mathbf{Z}^* = \sum \mathbf{Z}_i'\mathbf{Z}_i = \text{diag.}(\alpha_1, \ldots, \alpha_4), \text{ say}$$

and further, det. $(\mathbf{I}^*(\theta))$ of D^* = det.$(\mathbf{I}^*(\boldsymbol{\theta}))$ of $D^* = (\prod_{i=1}^{v} n_i)$ det.$(\mathbf{Z}^{*\prime}\mathbf{Z}^* - \mathbf{T}^{*\prime}\mathbf{N}^{*-1}\mathbf{T}^*)$
$= (\prod_{i=1}^{v} n_i)(\prod_{i=1}^{4} \alpha_i) \geq$ det.$(\mathbf{I}^{**}(\boldsymbol{\theta}))$ of D^{**},

where D^{**} belongs to the class of all competent CRD's (v, n_1, n_2, ..., n_v) accommodated with 4-OCD permitting the independent estimation of treatment-effects and covariate-effects and $\mathbf{I}^{**}(\boldsymbol{\theta})$is the information matrix for $\boldsymbol{\theta}$ through D^{**}.

Taking $\mathbf{Z}_i$ (i = 1, 2, …..,v) as the OCD-component to the i-th treatment and $\mathbf{Z}_{v+1}$ = ((1, –1, 1, –1, 0)$'$, (1, 1, -1, -1, 0)$'$, (1, -1, -1, 1, 0)$'$, (1/4, 1/4, 1/4, 1/4, -1)$'$) as that to the (v+1)-th treatment of a new CRD (v+1, n_1, n_2, …, n_v, n_{v+1}= 5) D, say.

Then, for the design D, $\mathbf{Z} = (\mathbf{Z}_1', ..., \mathbf{Z}_v', \mathbf{Z}_{v+1}')'$ (25)

$\mathbf{T}_i = \mathbf{0}' \ \forall$ i = 1, 2, ..., v+1 i.e. $\mathbf{X}'\mathbf{Z} = \mathbf{0}$ (26)

Now, $\mathbf{Z}'\mathbf{Z}$ = diag. (α_1, α_2, α_3, α_4) + $\mathbf{Z}_{v+1}'\mathbf{Z}_{v+1}$, from Proposition 1

= diag. (α_1+4, α_2+4, α_3+4, α_4+5/4), by (25) and (26). (27)

Then, det.($\mathbf{I}(\theta)$) of D = ($\prod_{i=1}^{v+1} n_i$)(α_1+4)(α_2+4)(α_3+4)(α_4+5/4), by (27) (28)

which is the maximum of determinant of information matrix of the CRD (v+1, n_1, n_2, …, n_v, n_{v+1}= 5) among all the competent CRD's accommodated with various possible OCD-components to the (v+1)-th treatment. Thus, the following theorem is immediate.

Theorem 2: Existence of 4-OCD of D-optimal CRD (v, n_1, n_2, ..., n_v) implies that of 4-OCD of D-optimal CRD (v+1, n_1, n_2, ..., n_v, n_{v+1}= 5) maintaining the same estimation status.

Proof: Suppose the proposed OCD for the new resultant CRD D is not an optimal allocation. Then, there exists an OCD $\mathbf{Z}^{***} \ni$ det.($\mathbf{Z}^{***\prime}\mathbf{Z}^{***}$) > ($\alpha_1$+4)($\alpha_2$+4)($\alpha_3$+4)(α_4+5/4), by (28) *i.e.* by (20), $\exists$ some a_1 to a_4, b_1 to $b_3, c_1, c_2,\ d_1 \in [-1,1] \ni$

$$[\alpha_1+\textstyle\sum_j \left(\widehat{z_{\beta j}^{***(1)}}\right)^2][\alpha_2+\sum_j \left(\widehat{z_{\beta j}^{***(2)}}\right)^2][\alpha_3+\sum_j \left(\widehat{z_{\beta j}^{***(3)}}\right)^2][\alpha_4+\sum_j \left(\widehat{z_{\beta j}^{***(4)}}\right)^2] > (\alpha_1+4)(\alpha_2+4)(\alpha_3+4)(\alpha_4+5/4)$$

i.e. $[\sum_j \left(\widehat{z_{\beta j}^{***(1)}}\right)^2][\sum_j \left(\widehat{z_{\beta j}^{***(2)}}\right)^2][\sum_j \left(\widehat{z_{\beta j}^{***(3)}}\right)^2][\sum_j \left(\widehat{z_{\beta j}^{***(4)}}\right)^2] > 80$

since $\prod_{i=1}^{q}(\delta_i + x_i) > \prod_{i=1}^{q}(\delta_i + y_i)$ iff $\prod_{i=1}^{q} x_i > \prod_{i=1}^{q} y_i;\ \forall\ \delta_i's,\ x_i's,\ y_i's > 0$

$$\text{i.e.}[\textstyle\sum_{k=1}^{4} a_k^2 + \left(\widehat{z_{\beta 5}^{***(1)}}\right)^2][\sum_{l=1}^{3} b_l^2 + \left(\widehat{z_{\beta 4}^{***(2)}}\right)^2 + \left(\widehat{z_{\beta 5}^{***(2)}}\right)^2][c_1^2 + c_2^2 + \left(\widehat{z_{\beta 3}^{***(3)}}\right)^2 + \left(\widehat{z_{\beta 4}^{***(3)}}\right)^2 + \left(\widehat{z_{\beta 5}^{***(3)}}\right)^2][d_1^2 + \left(\widehat{z_{\beta 2}^{***(4)}}\right)^2 + \left(\widehat{z_{\beta 3}^{***(4)}}\right)^2 + \left(\widehat{z_{\beta 4}^{***(4)}}\right)^2 + \left(\widehat{z_{\beta 5}^{***(4)}}\right)^2] > 80 \quad (29)$$

which is impossible as it is same to (C1) and (C1) is impossible. Hence proved.

Remark 3: By applying an induction-method for constructing OCD-components of treatments with replication number 5, the construction of 4-OCD of D-optimal CRD with replication number n_i's (permitting independent estimation of treatment-effects and covariate-effects), out of which some are equal to 5, is always possible.

Note 2: By Lemma 1, $n_i \geq 5\ \forall\ i$ = 1, 2, …,v, otherwise 4-OCD of D-optimal CRD (v, n_1 , ..., n_v), permitting independent estimation of treatment-effects and covariate-effects, never exists.

A Hadamard matrix $\mathbf{H}_{n_i}$(if exists for $n_i \geq 5\ \forall\ i$) can be rewritten as $\mathbf{H}_{n_i} = \left[\mathbf{h}_1^{(i)}, \ldots, \mathbf{h}_{n_i-1}^{(i)},\ \mathbf{1}_{n_i}\right]$. The vectors $\mathbf{h}_1^{(i)}, \ldots, \mathbf{h}_{n_i-1}^{(i)}$ are closures of order n_i-1 to n_i. Therefore, a 4-OCD of D-optimal CRD (v, n_1, ... ,n_{v-1}) exists, permitting independent estimation of treatment-effects and covariate-effects. Applying Theorem 2, a corollary follows.

Corollary 2: Existence of 4 covariate-OCD of D-optimal CRD (v, n_1, …., n_{v-1}, n_v = 5), permitting the independent estimation of treatment-effects and covariate-effects, is always possible, provided $\mathbf{H}_{n_i}$ exists and $n_i \geq 5\ \forall\ i$ = 1, 2, …, v-1.

Using the OCD-component in Sr. No. 1, Table 2, an example of Theorem 2 is given below:

Example 2: Given a 4-OCD $\mathbf{Z}^*= (\mathbf{Z}_1', \mathbf{Z}_2')'$ of D-optimal CRD (v = 2, $n_1 = n_2$ = 8) permitting independent estimation of treatment-effects and covariate-effects, taking the OCD-component in Sr. No. 1, Table 2 as OCD-component of the 3rd treatment of a CRD (v = 3, $n_1 = n_2$ = 8, n_3 = 5), the det. of the information matrix i.e. $|\mathbf{I(\theta)}|$ attains maximum value ($8^2 \times$ 5)$(16+4)^3$(16+5/4) and the resultant CRD maintains the same estimation status where the OCD is given by $\mathbf{Z} = (\mathbf{Z}_1', \mathbf{Z}_2', \mathbf{Z}_3')'$; $\mathbf{Z}_1 = \mathbf{Z}_2$ = ((1, 1, -1, -1, -1, 1, -1, 1)$'$, (1, 1, 1, -1, -1, -1, 1, -1)$'$, (1, -1, 1, 1, -1, -1, -1, 1)$'$, (1, 1, -1, 1, 1, -1, -1, -1)$'$), $\mathbf{Z}_3$ = ((1, -1, 1, -1, 0)$'$, (1, 1, -1, -1, 0)$'$ (1, -1, -1, 1, 0)$'$, (1/4, 1/4, 1/4, 1/4, -1)$'$)

3.3. OCD for D-optimal CRD ($n_\beta = q$: $\mathbf{H}_{q-1}$ ∃)

In this sub-section, under more general case of n_β odd such that $\mathbf{H}_{n_\beta-1}$ exists, the existence of OCD for D-optimal CRD would be claimed, assuming that OCD-components of the other treatments (perhaps, with even replications) are known. Using Hadamard matrices, construction of OCD for D-optimal CRD (v, n_1, n_2, ..., n_v), permitting the independent estimation of treatment-effects and covariate-effects, is proposed.

Theorem 3: Optimum allocation of c-OCD of D-optimal CRD $(v, n_1, n_2, \ldots, n_v)$ permitting the independent estimation of treatment-effects and covariate-effects, is always possible if for some n_α's ∃ $\mathbf{H}_{n_\alpha}$ and for remaining n_β's ($\neq n_\alpha$), ∃ $\mathbf{H}_{n_\beta-1}$ where c = min. (n_α-1, n_β-2 ∀ α, β).

Proof of the theorem is given in Appendix-E.

Remark 4: As $\mathbf{H}_{n_\alpha}$ exists, n_α is even. Of course, $\mathbf{h}_1^{(\alpha)}, \ldots\ldots, \mathbf{h}_{n_\alpha-1}^{(\alpha)}$ given in (E1), are closures of order n_α-1 to n_α and consequently, are optimal OCD-components of the α-th treatment.

Remark 5: As $\mathbf{H}_{n_\beta-1}$ exists, n_β odd. The gravity of Theorem 3 is that when some of the replications of the CRD are odd, OCD of D-optimal CRD can be constructed, permitting independent estimation of treatment-effects and covariate-effects.

An example of Theorem 3 is given below.

Example 3: For a CRD (v =2, n_1 = 4, n_2= 3) to be D-optimal, as $\mathbf{H}_4$ and $\mathbf{H}_2$ exist, c = min.(n_1-1, n_2-2) = 1. Then, $\mathbf{Z}^{(1)} = (+1, +1, -1, -1)'$, $\mathbf{Z}^{(2)} = (+1, -1, 0)'$ using the Hadamard matrices of n = 4 and 2 in the Table 2, Hedayat and Wallis (1978). The determinant of $\mathbf{I}(\boldsymbol{\theta})$ of the CRD with covariate design $\mathbf{Z} = (\mathbf{Z}_1', \mathbf{Z}_2')'$ obtains its maximum 2×4(6-1) i.e. 40 over all possible OCD accommodated to the CRD.

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APPENDIX-A

Proof of Lemma 1:

Suppose c be the maximum number of columns in the restricted Hadamard matrix **A** with m rows. Let $\boldsymbol{a}_{(\boldsymbol{p})} = (a_{1p}, a_{2p}, \dots, a_{mp})'$ be the p-th column of **A** By the definition of restricted Hadamard matrix,

$$\sum_i a_{ip} = 0 \ \forall\ p = 1, 2, \dots, c \text{ and} \tag{A1}$$

$$\sum_i a_{ip} a_{iq} = 0 \ \forall\ p \neq q = 1, 2, \dots, c. \tag{A2}$$

In (A1) and (A2) there are $c(c+1)/2$ equations involving mc unknown variables. When c = 1, there exists only 1 linear equation involving m unknown variables which has non-zero solutions belong to [–1, 1]. When $c = \beta$; β = 2, 3, …, m–1 there exist $\beta(\beta+1)/2$ linearly independent equations (independent is due to (A2)) involving βm unknown variables. Using the solutions for c = 1, 2, …, β–1, $\exists\ \beta(\beta+1)/2 - (\beta-1)\beta/2$, *i.e.* β linearly independent homogeneous linear equations involving m unknown variables, which have non-zero solutions ∈ [–1, 1]. If possible, when $c = m$, there exist $m(m+1)/2$ linearly independent equations involving m^2 unknown variables. Using the solutions for c = 1, 2, …, m-1, there are $m(m+1)/2 - (m-1)m/2$ *i.e.* m linearly independent homogeneous linear equations involving m unknown variables which has no non-zero solution, because any system of linearly independent homogeneous linear equations $\mathbf{B}_{n\times n}\mathbf{x}=\mathbf{0}$ has only zero solution, which is not of our interest. When $c = m + \gamma$ for any $\gamma = 1, 2, \ldots,$

$$\text{rank } (\mathbf{A}) \leq m \text{ since } c > m. \tag{A3}$$

By (A2), all the column vectors of $\boldsymbol{A}$ are orthogonal and consequently, linearly independent which contradicts to (A3). Thus, the case of $c = m + \gamma$ never arises. Hence proved.

APPENDIX-B

Proof of Lemma 2:

Suppose $\mathbf{Z}_\beta$ given in the Table 1 is not an optimal OCD-component of CRD to the β-th treatment with replication 3. Then, there exists an optimal OCD-component $\mathbf{Z}_\beta^*$ to the β-th treatment ∍ det.($\mathbf{Z}_{\boldsymbol{\beta}}^{*\prime}\mathbf{Z}_{\boldsymbol{\beta}}^*$) > det.($\mathbf{Z}_{\boldsymbol{\beta}}'\mathbf{Z}_{\boldsymbol{\beta}}$)

$$i.e. \ \text{det.} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} > 2 \times 3/2, \ \text{ by (18), where } A = a_1^2 + a_2^2 + (a_1 + a_2)^2 \text{ and}$$

$$B = b_1^2 + [(a_2 + 2a_1)b_1/(2a_2 + a_1)]^2 + [(a_1 - a_2)b_1/(2a_2 + a_1)]^2$$

$$i.e. \ 4(a_1^2 + a_2^2 + a_1a_2)^2\, b_1^2 > [2a_2 + a_1]^2. \tag{B1}$$

In order to get closure of order 2 to 3 for det. ($\mathbf{Z}_{\boldsymbol{\beta}}^{*\prime}\mathbf{Z}_{\boldsymbol{\beta}}^*$) to be at maximum, putting the possible largest magnitude of a_1 and a_2 , 1 and -1 (or -1 and 1) respectively in (B1) (but not 1 and 1 (or –1 and –1) because $\widehat{z_{\beta 3}^{(1)}} = -(a_1 + a_2) \in [-1,\ 1]$), we have $4b_1^2 > 1$ *i.e.* $-1/2 > b_1 > 1/2$. So, (B1) is not possible, contradicting to the possible forms of $\mathbf{Z}_{\boldsymbol{\beta}}$ given in Sl. No. 13 and 15 (or Sl. Nos. 14 and 16), Table 1, where $a_1 = 1, a_2 = -1$ (or $a_1 = -1, a_2 = 1$), $b_1 = \pm 1/2$. Hence $\mathbf{Z}_{\boldsymbol{\beta}}$'s given in the Table 1 are the optimal OCD-components of CRD to the β-th treatment with replication 3 *i.e.* the closures of order 2 to 3.

APPENDIX-C

Proof of Lemma 3:

Suppose $\boldsymbol{Z}_{\boldsymbol{\beta}}$ given in the Table 2 is not optimal OCD-component of CRD($v, n_1, n_2, \ldots, n_v$) to the β-th treatment with replication 5. Then, ∃ an optimal OCD-component $\mathbf{Z}_{\boldsymbol{\beta}}^*$ to the β-th treatment ∍ det.($\mathbf{Z}_{\boldsymbol{\beta}}^{*\prime}\mathbf{Z}_{\boldsymbol{\beta}}^*$) > det.($\mathbf{Z}_{\boldsymbol{\beta}}'\mathbf{Z}_{\boldsymbol{\beta}}$)

$$i.e. \ [\textstyle\sum_j \left(\widehat{z_{\beta j}^{*(1)}}\right)^2][\sum_j \left(\widehat{z_{\beta j}^{*(2)}}\right)^2][\sum_j \left(\widehat{z_{\beta j}^{*(3)}}\right)^2][\sum_j \left(\widehat{z_{\beta j}^{*(4)}}\right)^2] > 4^3 \times 5/4, \text{ by (20)}$$

$$i.e. \ [\textstyle\sum_{k=1}^4 a_k^2 + \left(\widehat{z_{\beta 5}^{*(1)}}\right)^2][\sum_{l=1}^3 b_l^2 + \left(\widehat{z_{\beta 4}^{*(2)}}\right)^2 + \left(\widehat{z_{\beta 5}^{*(2)}}\right)^2][c_1^2 + c_2^2 + \left(\widehat{z_{\beta 3}^{*(3)}}\right)^2 + \left(\widehat{z_{\beta 4}^{*(3)}}\right)^2 + \left(\widehat{z_{\beta 5}^{*(3)}}\right)^2][d_1^2 + \sum_{u=2}^5 \left(\widehat{z_{\beta u}^{*(4)}}\right)^2] > 80. \tag{C1}$$

In order to get closures of order 4 to 5 for det.($\mathbf{Z}_{\boldsymbol{\beta}}^{*\prime}\mathbf{Z}_{\boldsymbol{\beta}}^*$) to be at maximum, putting the largest possible magnitude of a_1 to a_4 , b_1 to b_3, c_1, c_2 and d_1 satisfying (19), say, $a_1 = a_2$

= 1, $a_3 = a_4 = -1$, $b_1 = 1$, $b_2 = -1$, $b_3 = c_1 = 1$, $c_2 = -1$, $d_1 = 1/4$. Obviously, $\widehat{z_{\beta 5}^{*(1)}} = 0$, $\widehat{z_{\beta 4}^{*(2)}} = -1$, $\widehat{z_{\beta 5}^{*(2)}} = 0$, $\widehat{z_{\beta 3}^{*(3)}} = -1$, $\widehat{z_{\beta 4}^{*(3)}} = 1$, $\widehat{z_{\beta 5}^{*(3)}} = 0$, $\widehat{z_{\beta 1}^{*(4)}} = \cdots = \widehat{z_{\beta 4}^{*(4)}} = 1/4$, $\widehat{z_{\beta 5}^{*(4)}} = -1$. Then, (C1) gives 5/4 > 5/4. So, (C1) is not possible, contradicting to our assumption for existence of $\mathbf{Z}_{\boldsymbol{\beta}}^{*}$ ∍ det.($\mathbf{Z}_{\boldsymbol{\beta}}^{*\prime}\mathbf{Z}_{\boldsymbol{\beta}}^{*}$) > det.($\mathbf{Z}_{\boldsymbol{\beta}}^{\prime}\mathbf{Z}_{\boldsymbol{\beta}}$). Hence $\mathbf{Z}_{\boldsymbol{\beta}}$'s given in the Table 2 are the optimal OCD-components of CRD to the β-th treatment with replication 5 *i.e.* the closures of order 4 to 5.

APPENDIX-D

Proof of Lemma 4:

By the definition of Hadamard matrix,

$\| \mathbf{h}_j \| = n_\beta$-1 $\forall j$ =1,2, …, n_β-2 and $\| \delta/(n_\beta - 1)\mathbf{1}_{\boldsymbol{n_\beta - 1}} \| = 1/(n_\beta\text{-}1)$

$\Rightarrow \|\mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{j})}\| = n_\beta$-1 $\forall$ j and $\|\mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})}\| = n_\beta/(n_\beta - 1)$, since every $\mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{j})}$ has entries "± 1" except one entry "0" which cannot be non-zero entry as $\mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{j})\prime}\mathbf{1}_{(\boldsymbol{n_\beta - 1})} = 0$

$\Rightarrow \boldsymbol{Z}_{\boldsymbol{\beta}}^{(\mathbf{1})}, \ldots, \mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 2})}$ are closures of order n_β-2 to n_β, since $\|\mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{j})}\| > \|\mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})}\|$ ∀ j (D1)

and $\mathbf{Z}_{\boldsymbol{\beta}}^{(\mathbf{1})}, \ldots, \mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})}$ are closures of order n_β-1 to n_β among all possible OC-components to the β-th treatment. (D2)

Suppose (D2) is not true *i.e.* $\mathbf{Z}_{\boldsymbol{\beta}}$ is not an optimal OCD-component of CRD to the β-th treatment. Then there exists a $(n_\beta-1)$-th orthogonal covariate $\overline{\mathbf{Z}}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})}$ such that

$$\|\overline{\mathbf{Z}}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})}\| > \|\mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})}\| = n_\beta/(n_\beta\text{-}1) \quad \text{(D3)}$$

$$\text{and } \overline{\mathrm{Z}}_\beta^{(j)\prime}\overline{\mathrm{Z}}_\beta^{(n_\beta - 1)} = 0 \;\forall j, \text{ by the orthogonal of covariates.} \quad \text{(D4)}$$

Then, (D4) gives $\overline{\mathbf{Z}}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})} = (gd, gd, \ldots, gd, -g(n_\beta-1)d)'$; $g = 1$ or -1; d a constant, otherwise $\mathbf{X}'\mathbf{Z} \neq 0$.

As every component of $\overline{\mathbf{Z}}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})} \in [-1, 1]$, $|-g(n_\beta\text{-}1)d|$ can take maximum 1 for $\|\overline{\mathbf{Z}}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})}\|$ to be maximized (otherwise, no alternative). So, $d = 1/|g(n_\beta\text{-}1)| = 1/(n_\beta\text{-}1)$ as $g = \pm 1$.

Consequently, $\widehat{\mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})}} = (g/(n_\beta\text{-}1), \ldots, g/(n_\beta-1), -g)'$

$$\Rightarrow \|\widehat{\mathbf{Z}_{\boldsymbol{\beta}}^{(\boldsymbol{n_\beta - 1})}}\| = (n_\beta-1)/(n_\beta - 1)^2 + 1 = n_\beta/(n_\beta\text{-}1). \quad \text{(D5)}$$

Now, (D3) and (D5) gives $n_\beta/(n_\beta-1) > n_\beta/(n_\beta-1)$ which is absurd, contradicting to our assumption.

APPENDIX-E

Proof of Theorem 3:

WOLG, suppose for $n_1, n_2 \ldots\ldots, n_s \ \exists\ \mathbf{H}_{n_\alpha} \forall\ \alpha = 1, 2, \ldots, s$ and for $n_{s+1}, \ldots, n_v \ \exists\ \mathbf{H}_{n_\beta - 1}$ $\forall\ \beta = s+1, \ldots, s+s_1 = v$. Then, $\mathbf{H}_{n_\alpha}$ and $\mathbf{H}_{n_\beta-1}$ can be rewritten as follows.

$$\mathbf{H}_{n_\alpha} = [\mathbf{h}_1^{(\alpha)}, \ldots, \mathbf{h}_{n_\alpha - 1}^{(\alpha)}, \ \mathbf{1}_{n_\alpha}\ ] \ \text{ and } \mathbf{H}_{n_\beta - 1} = [\mathbf{h}_1^{(\beta)}, \ldots, \mathbf{h}_{n_\beta - 2}^{(\beta)}, \mathbf{1}_{n_\beta - 1}\ ]. \tag{E1}$$

$$\text{Consider } \mathbf{Z} = [\mathbf{Z}_1', \ \ldots, \mathbf{Z}_s', \mathbf{Z}_{s+1}', \ldots, \mathbf{Z}_v']' \tag{E2}$$

as OCD of the proposed CRD D where $\mathbf{Z}_\alpha$ and $\mathbf{Z}_\beta$ are the OCD-component to the α-th treatment and the β-th treatment of D respectively and given by $\mathbf{Z}_\alpha = [\mathbf{h}^{(\alpha)\mathbf{1}}, \mathbf{h}^{(\alpha)\mathbf{2}}, \ldots, \mathbf{h}^{(\alpha)\boldsymbol{c}}\ ]$ and $\mathbf{Z}_\beta = [\mathbf{h}^{(\beta)\mathbf{1}}, \mathbf{h}^{(\beta)\mathbf{2}}, \ldots, \mathbf{h}^{(\beta)\mathbf{c}}\ ]$ respectively where $\mathbf{h}^{(\alpha)\boldsymbol{l}}$'s $(l = 1, 2, \ldots, c)$ are any c out of the first $n_\alpha-1$ columns of $\mathbf{H}_{n_\alpha}$ given in (E1) and $\mathbf{h}^{(\beta)\boldsymbol{l}}$'s $(l = 1, 2, \ldots\ldots, c)$ are any c out of the first $n_\beta-2$ columns of $\mathbf{H}_{n_\beta - 1}^* = \begin{bmatrix} \mathbf{h}_1^{(\beta)}, \ldots., \mathbf{h}_{(n_\beta-2)}^{(\beta)}, (\delta/(n_\beta - 1))\mathbf{1}_{(n_\beta-1)} \\ 0, \ \ldots\ldots, \ 0, \qquad -\delta \end{bmatrix}$ and $\delta = \pm 1$.

Then, $\mathbf{Z}'\mathbf{Z} = \sum_{\alpha=1}^{s} \mathbf{Z}_\alpha' \mathbf{Z}_\alpha + \sum_{\beta=s+1}^{v} \mathbf{Z}_\beta' \mathbf{Z}_\beta$

$= (\sum_{\alpha=1}^{s} n_\alpha)\mathbf{I}_c + \left(\sum_{\beta=s+1}^{v}(n_\beta - 1)\right)\mathbf{I}_\mathbf{c}$, from Proposition 1 and Proposition 2, by the properties of Hadamard matrices $\mathbf{H}_{n_\alpha}$ and $\mathbf{H}_{n_\beta - 1}$; $c \leq n_\alpha-1$ and $n_\beta-2$,

$= a\mathrm{I}_C$; $a = n - s_1$ since $(\sum_{\alpha=1}^{s} n_\alpha) + \sum_{\beta=s+1}^{v}(n_\beta - 1) = n - s_1$.

Further, $\mathbf{T}_\alpha = \mathbf{0}' \ \forall\ \alpha$ and $\mathbf{T}_\beta = \mathbf{0}' \ \forall\ \beta$ which ensure the independent estimation of treatment-effects and covariate-effects. Now, det.$(\mathbf{I}(\boldsymbol{\theta}))$ of $D = (\prod_{i=1}^{v} n_i)\text{det.}\,(a\mathbf{I}_c) = (\prod_{i=1}^{v} n_i)a^c$ which is the maximum among the determinants of all information matrices of competent CRD's accommodated with any OCD.

Suppose there exist an optimal OCD $\mathbf{Z}^* = [\mathbf{Z}_1^{*\prime}, \ldots, \mathbf{Z}_s^{*\prime}, \mathbf{Z}_{s+1}^{*\prime}, \ldots, \mathbf{Z}_v^{*\prime}]'$ of the CRD D such that $\text{det.}(\mathbf{I}^*(\boldsymbol{\theta}) \text{ accommodating with } \mathbf{Z}^*) > \text{det.}\,(\mathbf{I}(\boldsymbol{\theta}) \text{ accommodating with } \mathbf{Z})$

$$\textit{i.e. } (\textstyle\prod_{i=1}^{v} n_i)\text{det.}\,[\text{diag.}\,(a_1, \ldots, a_c)] > (\prod_{i=1}^{v} n_i)a^c \tag{E3}$$

since $\mathbf{Z}^{*\prime}\mathbf{Z}^*$ is a diagonal matrix diag. $(a_1, \ldots, a_c)$, say,

$$\textit{i.e. } \textstyle\prod_{p=1}^{c} a_p > a^c. \tag{E4}$$

As all the entries of covariate $\in [-1, 1]$, $n_\beta(\beta = s+1, s+2, \ldots, v)$ are odd (since $\mathbf{H}_{n_\beta - 1}\exists$) and $\mathbf{1}_{n_\beta}'\mathbf{X}_\beta = \mathbf{0}'$, each of c covariates to the α-th treatment i.e. $\mathbf{Z}_\alpha^{(1)}, \ldots, \mathbf{Z}_\alpha^{(c)}$ contains n_α entries "± 1" and that to the β-th treatment contains n_β-1 entries "± 1" and another entry "0". So, by the notation of a_p's in (E3), $a_p \leq \sum_{\alpha=1}^{s} n_\alpha + \sum_{\beta=s+1}^{v}(n_\beta - 1)\ \forall$ p

$= n - s_1$ which contradicts (E4). Hence proved.

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