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Estimation of distinct elements of a covariance matrix : MINQUE and MINQE

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Abstract

The method of MINQUE (Minimum Norm Quadratic Unbiased Estimation), was originally proposed by [C.R. Rao(1970)] for estimating heteroscedastic covariance matrix of a linear model and later generalized in a series of papers [C.R. Rao (1971, 1972)] for variance and covariance components. In some situations it may produce negative values for estimating non-negative variance components. For this reason P.S.R.S. Rao and Chaubey (1978) proposed some modifications, including MINQE which does not impose the unbiased condition. Chaubey (1980) showed how the method of MINQUE can be adapted for estimating the distinct elements of an intra-class covariance matrix. This extension is straight forward when no a priori guess is incorporated in the estimation process. However, for the case when a priori guess about distinct elements is used, in general, we may need to consider a different minimization problem, especially for the covariance components model. In this paper we consider the method of MINQE and MINQUE for estimating distinct elements of a variance covariance matrix in the case of a variance components model, providing the solution. It is shown that for the particular case of the intra-class correlation model, the original form for the solution holds with slight modification.

Key words : Covariance matrix; Minimum norm quadratic unbiased estimation; Minimum norm quadratic estimation.

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1 Introduction

Rao, C.R. (1970) introduced the principle of MINQUE (minimum norm quadratic unbiased estimation) for estimation of variances in a heteroscedatic linear model. Subsequently, he extended this method to variance and covariance components models [Rao, C.R. (1971, 1972)]. This method may produce negative estimates for non-negative variance components and hence various alternatives have been suggested in the literature. One alternative which parallels the development of MINQUE was proposed by Rao, P.S.R.S. and Chaubey (1978). These methods also provide a choice of incorporating *a priori* information about the variances and covariances. Here, we consider estimation of the distinct elements of a covariance matrix in the context of a covariance components model. We will highlight the fact that as shown by Rao, P.S.R.S. and Chaubey (1978) these methods can be easily adapted to this problem for the case of linear model, however, they require some care while taking into consideration of *a priori* values. In what follows we will refer to the methods of MINQUE and MINQE as weighted-MINQUE and weighted-MINQE.

First we consider the variance components model

$$Y = X\beta + U_1\xi_1 + ... + U_k\xi_k, \tag{1}$$

where Y is an $n-$vector of observations, X is an $(n \times q)$ design matrix of full column rank, β is $q-$vector of regression parameters, U_i is a given $(n \times c_i)$ matrix and ξ_i represents a c_i-vector representing a hypothetical random variable, such that

$$E(\xi_i) = 0,\ D(\xi_i) = \sigma_i^2 I_{c_i \times c_i},\ \mathrm{Cov}(\xi_i, \xi_j) = 0,\ \text{for } i \neq j; i, j = 1, 2, ..., k.$$

Letting $U = (U_1|...|U_k)$ and $\xi = (\xi_1|...|\xi_k)'$, the model in Eq. (1) can be compactly written as

$$Y = X\beta + U\xi. \tag{2}$$

Note that from Eq. (1) we have

$$E(Y) = X\beta \quad D(Y) = \sigma_1^2 V_1 + ... + \sigma_k^2 V_k, \tag{3}$$

where $V_i = U_i U_i'$. The parameters $\sigma_1^2, ..., \sigma_k^2$ are called the variance components.

The principle of MINQUE considers a quadratic form $Y'AY$ as an estimator of the linear function

$$\theta = \sum_{i=1}^{k} p_i \sigma_i^2,$$

where the matrix A is determined by minimizing

$$tr(AVAV),$$

where $V = UU' = \sum_{i=1}^{k} U_iU_i' = \sum_{i=1}^{k} V_i$ subject to the conditions

$$\begin{array}{rcl} (i)\ \text{Invariance}: \quad AX &=& 0 \\ (ii)\text{Unbiasedness}: \ tr(AV_i) &=& p_i, i = 1, 2, ..., k. \end{array}$$

This principle is motivated by the choice of A such that the difference $\xi'(U'AU - \Delta)\xi$ is small, where

$$\Delta = Bdiag\left(\frac{p_1}{c_1}I_{c_1}, ..., \frac{p_k}{c_k}I_{c_k}\right),$$

thereby choosing to minimize the norm $\|U'AU - \Delta\|$, where

$$\|B\|^2 = tr(B^2).$$

The solution to this problem is given in the following theorem.

Theorem 1.1 [Rao, C.R. (1972)]*Let A be a symmetric matrix and V be a symmetric and non-singular matrix then* $\min tr(AVAV)$ *subject to the conditions (1) and (ii) is attained at*

$$A_{IU} = \sum_{j=1}^{k} \lambda_j RV_jR, \tag{4}$$

where

$$\begin{array}{rcl} R &=& V^{-1}Q_V = Q_V'V^{-1} \\ Q_V &=& I - P_V, \\ P_V &=& X(X'V^{-1}X)^{-1}X'V^{-1} \end{array}$$

where $p = (p_1, p_2, ..., p_k)'$ *and* $\lambda = (\lambda_1, ..., \lambda_k)'$ *is obtained by solving*

$$S\lambda = p,$$

where the $(i, j)^{th}$ *element of* S *is given by* $S_{ij} = tr(RV_iRV_j)$.

Rao, P.S.R.S. and Chaubey (1978) showed that by removing the condition of unbiasedness, we necessarily obtain non-negative estimators. This results in minimizing

$$tr(AVAV) - 2tr(U'AU\Delta) \tag{5}$$

whose solution is given by the following theorem.

Theorem 1.2 [Rao, P.S.R.S. and Chaubey (1978)]*Let A be a symmetric matrix and V be a symmetric and non-singular matrix then* $\min tr(AVAV) - 2tr(U'AU\Delta)$ *subject to the conditions (1) is attained at*

$$A_I = \sum_{j=1}^{k} \frac{p_j}{c_j} RV_j R, \tag{6}$$

where R is as defined before.

Suppose that $\Lambda_1^2, ..., \Lambda_k^2$ represent *a priori* weights reflecting the relative magnitudes of the variance components $\sigma_1^2, ..., \sigma_k^2$, then we can write the model 2 in terms of transformed hypothetical variables $\eta_i = \Lambda_i \xi_i, i = 1, ..., k$ as

$$Y = X\beta + U^* \eta, \tag{7}$$

where $U^* = (U_1^*|...|U_k^*), U_i^* = \Lambda_i U_i$ and $\eta = (\eta_1|...|\eta_k)'$. The weighted MINQUE and MINQE are then obtained by minimization of

$$tr(\Lambda^{\frac{1}{2}}(U'AU - \Delta)\Lambda^{\frac{1}{2}})^2,$$

where $\Lambda = diag(\Lambda_1^2 I_{c_1}, ..., \Lambda V^{*-1}{}_k^2 I_{c_k}$, subject to conditions (i) and (ii) and condition (i), respectively. The solutions for A for getting the weighted-MINQUE is thus provided by

$$A_{wIU} = \sum_{i=1}^{k} \lambda_j \ R_* V_j R_*, \tag{8}$$

where $R_* = Q_{V^{*-1}} V^{*-1}$ and $\lambda = (\lambda_1, ...\lambda_k)'$ is obtained by solving

$$S_* \lambda = p,$$

where $p = (p_1, p_2, ..., p_k)'$ and the $(i, j)^{th}$ element of S_* is given by $S_{*ij} = tr(R_* V_i R_* V_j)$.

And that for the weighted-MINQE is given by

$$A_{wI} = \sum_{i=1}^{k} \frac{p_j \Lambda_j^2}{c_j} \ R_* V_{j*} R_*, \tag{9}$$

where $V_{j*} = \Lambda_j^2 V_j$.

2 Covariance components model

Rao, C.R. (1971b, 1972) extended the principle of MINQUE to more general models involving hypothetical variables with unknown covariances. He considered the model in Eq. (1), however the hypothetical variables $\xi_1, ..., \xi_k$ were assumed to have a common covariance matrix, *i.e*

$$D(\xi_i) = \Sigma, i = 1, 2, ..., k.$$

In this formulation all the elements of the covariance matrix were assumed to be unknown. Chaubey (1977) considered a special case of the above model with $k = 1$ and showed that the MINQUE and MINQE estimators can be easily adapted to this case when some elements of the covariance matrix may be considered equal by decomposing the covariance matrix Σ as

$$\Sigma = \alpha_1 T_1 + ... + \alpha_d T_d, \tag{10}$$

where $\alpha_i, i = 1, 2, ..., d$ denote the $d-$distinct elements of Σ. It can be noted that the matrices $T_i, i = 1, 2, ..., d$, contain zero's and one's as their elements. Chaubey and Rao, P.S.R.S. (1978) showed that the method of weighted-MINQUE and weighted-MINQE also parallels along the same lines for the special case mentioned above. In general, however, this may not hold.

MINQUE and MINQE

Using the decomposition of the covariance matrix in Eq. (10) we can write the variance-covariance matrix of $\epsilon = U\xi$ as

$$D(\epsilon) = \sum_{j=1}^{d} \alpha_i T_i^*, \tag{11}$$

where

$$T_i^* = \sum_{j=1}^{d} U_j T_i U_j'.$$

The invariance condition for estimation of $\theta = \sum_i p_i \alpha_i$ by a quadratic form $Y'AY$ remains the same as before, namely $AX = 0$, however, the unbiasedness condition now becomes,

$$tr(AT_I^*) = p_i, i = 1, ..., d.$$

Denoting by s_i, the number of one's in T_i, $\xi_j' T_i \xi_j$ is a natural unbiased estimator (NUE) of $s_i \alpha_i$ for all j, hence we can write the NUE of θ as

$\xi' B \xi$, where

$$B = Bdiag(\frac{1}{k}\sum_i \frac{p_i}{s_i}T_i, ..., \frac{1}{k}\sum_i \frac{p_i}{s_i}T_i).$$

Following the motivation from Rao, C.R. (1970,1971a), we are thus led to minimize the Euclidean norm $\|U'AU - B\|$. It can be shown as in Chaubey (1977) that $tr(U'AUB) = tr(B^2)$ and hence for obtaining the MINQUE of θ we are led to minimize $tr(AVAV)$ subject to

$$\begin{aligned} (i) \text{ Invariance}: \ AX &= 0 & (12)\\ (ii) \text{ Unbiasedness}: \ tr(AT_i^*) &= p_i, \ i = 1, 2, ..., d. & (13) \end{aligned}$$

The solution for A is thus obtained from Theorem 1.1 by substituting T_i^* for V_i, *i.e.*

$$A_{IU}^{(d)} = \sum_{j=1}^{k} \lambda_j R T_j^* R, \tag{14}$$

where $\lambda = (\lambda_1, ..., \lambda_k)'$ is obtained by solving

$$W\lambda = p,$$

where the $(i,j)^{th}$ element of W is given by $W_{ij} = tr(RT_i^*RT_j^*)$ and $p = (p_1, p_2, ..., p_k)'$.

Without the unbiased condition, the above motivation leads us to minimize

$$tr(AVAV) - 2tr(U'AUB)$$

subject to the condition $AX = 0$. The solution is therefore given by Theorem 1.2 as

$$A_I^{(d)} = \frac{1}{k}\sum_{i=1}^{d} \frac{p_i}{s_i} R T_i^* R. \tag{15}$$

Weighted MINQUE and MINQE

Given *a priori* estimates of α_i as δ_i, or otherwise, let Λ_0 denote an *a priori* estimate of Σ. This matrix may be used to transform the variable ξ into $\eta = \Lambda_*^{-\frac{1}{2}}\xi$, where Λ_* is a block-diagonal matrix of k blocks given by

$$\Lambda_* = Bdiag((\Lambda_0, ..., \Lambda_0) \tag{16}$$

into the model

$$Y = X\beta + U_*\eta,$$

where $U_* = U\Lambda_*^{\frac{1}{2}}$.

The matrix A must now be determined by minimizing

$$tr(U_*'AU_* - \Lambda_*^{\frac{1}{2}}B\Lambda_*^{\frac{1}{2}})^2 = tr(AV*AV*) + tr(B\Lambda_*)^2 - 2tr(B\Lambda_*U'AU\Lambda_*)$$

subject to the invariance and unbiasedness conditions. The problem would be simplified as in the case of estimation of variance components or in the case of estimation of distinct elements for the special case, $k = 1, U_1 = I$, if $tr(B\Lambda_*)^2 = tr(B\Lambda_*U'AU\Lambda_*)$. However, in general this may not hold and we therefore have to solve the general problem of minimizing

$$tr(AV_*AV_*) - 2tr(AD),$$

where $D = U\Lambda_*B\Lambda_*U'$ subject to the unbiasedness and invariance conditions. The solution to this general problem is provided in the following theorem.

Theorem 2.1 *Let A be a symmetric matrix and V_* be a symmetric and non-singular matrix then $\min tr(AV_*AV_*) - 2tr(AD)$ subject to the conditions (12) and (13) is attained at*

$$A_{wIU}^{(d)} = \sum_{j=1}^{k} \lambda_j RT_j^* R + RDR, \tag{17}$$

where $\lambda = (\lambda_1, ..., \lambda_k)'$ is obtained by solving

$$W_*\lambda = p - q,$$

where the $(i,j)^{th}$ element of W_* is given by $W_{*ij} = tr(RT_i^*RT_j^*)$ and $q_i = tr(RDRT_i^*)$.

Using the standard Lagrangian multiplier method we can show that the solution must satisfy

$$V_*AV_* = D + \sum_{j=1}^{d} \lambda_j T_j^* + (1/2)(XM + M'X'),$$

where $(\lambda_1, ..., \lambda_d)$ are constants representing Lagrangian multipliers for the unbiasedness constraints and M is a matrix representing those for the invariance constratint. Since $AX = 0$ is implies $A = Q_{V_*}'AQ_{V_*}$, we get

$$A = Q_{V_*}'V_*^{-1}DV_*^{-1}Q_{V_*} + \sum_j \lambda_j Q_{V_*}'V_*^{-1}T_j^*V_*^{-1}Q_{V_*}.$$

We now use the unbiasedness conditions in Eq. (13) that gives the following condition,

$$W_*\lambda = p - q,$$

which completes the proof.

The weighted MINQE is obtained from Theorem 1.2 and we have the corresponding matrix of the quadratic form as given by

$$A_{wI}^{(d)} = RDR = \frac{1}{k}\sum_{i=1}^{d}\frac{p_i}{s_i}RT_i^0R, \tag{18}$$

where

$$T_i^0 = \sum_{j=1}^{k} U_j\Lambda_0 T_i \Lambda_0 U_j'.$$

In the unweighted case $\Lambda_0 = I$, then

$$tr(B\Lambda_*)^2 = tr(B\Lambda_* U' AU\Lambda_*).$$

This was shown to hold in the special case of $k = 1, U_1 = I$, in Rao, P.S.R.S and Chaubey (1978) for the unweighted case. Chaubey (1977) provided the details for estimating the elements of a intraclass covariance matrix in general linear model. In the next section we show that this condition also holds for the weighted case, which simplifies computations.

3 Intraclass covariance regression model

The regression model with intraclass covariance matrix is given by

$$Y = X\beta + \epsilon,$$

where the covariance matrix of ϵ has all diagonal elements= α_1 and off-diagonal elements= α_2. That is the covariance matrix is given by

$$D(\epsilon) = \alpha_1 T_1 + \alpha_2 T_2,$$

where $T_1 = I_{n\times n}, T_2 = J_{n\times n} - I$, where J is a matrix of all one's. This is a special case of (10), where we note that $T_i^* = T_i, i = 1, 2$, hence

$$\begin{aligned} V_* &= \Lambda_0 = \Lambda_* \\ &= (\delta_1 - \delta_2)I + \delta_2 J \end{aligned} \tag{19}$$

and

$$D = V_* B V_*, \tag{20}$$

where δ_1 and δ_2 are known values such that V_* is non-singular. We can show that in this case

$$tr(B\Lambda_*)^2 = tr(A\Lambda_* B\Lambda_*).$$

By direct calculations, we have

$$B\Lambda_* = a_1 I + a_2 J, \tag{21}$$

where

$$\begin{aligned} a_1 &= (\delta_1 - \delta_2)\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) && (22) \\ a_2 &= (\delta_1 - \delta_2)\frac{p_2}{s_2}\delta_2\left(\frac{p_1}{s_1} + (n-1)\frac{p_2}{s_2}\right) && (23) \end{aligned}$$

Thus, we have

$$(B\Lambda_*)^2 = a_1 I + (na_2^2 + 2a_1a_2)J \tag{24}$$

and therefore after some lengthy calculations we get,

$$tr(B\Lambda_*)^2 = b_1\delta_1^2 + b_2\delta_2^2 + b_{12}\delta_1\delta_2, \tag{25}$$

where,

$$\begin{aligned} b_1 &= \frac{p_1^2}{n} + \frac{p_2^2}{n^2 - n} && (26) \\ b_2 &= \frac{p_1^2(n-1)}{n} + \frac{2p_1p_2}{n}(n-2) + \frac{p_2^2}{n(n-1)}(n^2 - 3n + 3) && (27) \\ b_{12} &= 2\left[\frac{p_2^2(n-2)}{n(n-1)} + \frac{2p_1p_2}{n}\right] && (28) \end{aligned}$$

Now towards computing $tr(A\Lambda_* B\Lambda_*)$ we find that

$$\Lambda_* B\Lambda_* = f_1 I + f_2 J, \tag{29}$$

where

$$\begin{aligned} f_1 &= a_1(\delta_1 - \delta_2) && (30) \\ f_2 &= a_2(\delta_1 - \delta_2) + a_1\delta_2 + na_2\delta_2. && (31) \end{aligned}$$

Noting that for unbiasedness to hold, we have

$$\begin{aligned} tr(AT_1) &= tr(A) = p_1 \\ tr(AJ) &= tr(AT_2 + A) = p_1 + p_2 \end{aligned}$$

and direct calculation yields

$$tr(AV_*BV_*) = d_1\delta_1^2 + d_2\delta_2^2 + d_{12}\delta_1\delta_2, \tag{32}$$

where

$$d_1 = \frac{p_1^2}{n} + \frac{p_2^2}{n(n-1)} = a_1 \tag{33}$$

$$d_2 = \frac{p_1^2}{n}(n-1) + \frac{2p_1p_2}{n}(n-2) + \frac{p-2^2}{n(n-1)}(n^2-3n+3) = a_2 \tag{34}$$

$$d_{12} = \frac{2p_2^2}{n(n-1)}(n-2) + \frac{4p_1p_2}{n} = a_{12} \tag{35}$$

This shows that

$$tr(B\Lambda_*)^2 = tr(A\Lambda_*B\Lambda_*),$$

and therefore the weighted MINQUE is obtained using Theorem 1.1, replacing V_i by T_i and V by V_* instead of using Theorem 2.1. However, the weighted-MINQE of α_i is simply given by (using equation (18))

$$\widehat{\alpha}_{i(wI)}^{(d)} = \frac{Y'R'T_iRY}{s_i},$$

where, $s_1 = n, s_2 = n(n-1)$.

4 A numerical example

The following data are used to illustrate the above method for estimating the parameters of a linear model with intraclass covariance matrix. These data are taken from Wiorkowski (1975) about studying the biological activity of adenosine triphosphate (ATP) in red blood cells, measured in the parents and male progeny of 14 randomly selected families. The objective of this study was to find if the ATP levels in children conformed to a simple genetic model which would be equivalent to the observed trial being controlled by a large number of statistically independent loci, each making a small additive contribution to the final observed level of ATP. The model used is given by

$$y_{ij} = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_{ij},$$

for $j = 1, ..., n_i,\ i = 1, 2, ..., k$, where y_{ij} denotes the ATP level for the $j-$th progeny in the $i-$th family, x_{i1} denotes the ATP level of the father in the $i-$th family and x_{i2} denotes that for the mother. The observations in a family are supposed to have a covariance structure of intraclass type, *i.e.*

$$\Sigma = \sigma^2[(1-\rho)I + \rho J],$$

where ρ denotes the intra-class correlation. It is of interest to find if $\rho = 0$ and $\beta_1 = \beta_2 = 0.5$. The data is provided in Table 1 below.

Table 1: ATP Levels for 14 Families

Family	Father	Mother	Male Progeny				
1	3.72	4.43	4.16	4.81			
2	4.54	3.79	4.72				
3	5.05	4.66	4.98	5.03	5.16		
4	4.10	5.42	5.30	4.48	4.85		
5	4.26	4.39	4.87	3.99	4.19	4.28	5.15
6	4.09	5.29	4.74	4.10			
7	4.83	4.99	4.53	4.77	4.77		
8	4.24	4.38	3.72	4.12			
9	5.43	4.73	4.65	4.62			
10	5.23	5.34	5.83	6.03			
11	4.56	5.29	4.86	5.58	5.99		
12	5.16	4.71	5.44	4.34	5.43		
13	3.77	5.13	4.70	5.00	4.63		
14	4.15	4.18	4.82	4.14			

We used SAS-IML procedures for matrix computations. The estimators of regression parameters are those as obtained from estimated weighted least squares method. The formulae for covariances are obtained from Chaubey (1980b). The estimates of parameters using the unweighted MINQUE and MINQE are given in Tables 2 and 3. The unweighted estimates of variance and covariance is used in computing weighted MINQUE and MINQE which are summarized in Tables 4 and 5.

Table 2. Unweighted MINQUE and Estimated Variance Covariance Matrix of Estimators

$$\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 0.2174170 \\ 0.0292862 \\ 0.3929127 \\ 0.4084862 \\ 0.5343059 \end{bmatrix}$$

$$\widehat{\mathrm{Cov}} \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 1.3861233 & -0.115021 & -0.179056 & 0 & 0 \\ -0.115021 & 0.0272305 & -0.001588 & 0 & 0 \\ -0.179056 & -0.001588 & 0.0387455 & 0 & 0 \\ 0 & 0 & 0 & 0.0031423 & 0.0012038 \\ 0 & 0 & 0 & 0.0012038 & 0.0024708 \end{bmatrix}$$

Table 3. Unweighted MINQE and Estimated Variance Covariance Matrix of Estimates

$$\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 0.1951973 \\ 0.0000387 \\ 0.347224 \\ 0.4061869 \\ 0.5460642 \end{bmatrix}$$

$$\widehat{\mathrm{Cov}} \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 1.0471448 & -0.08654 & -0.135713 & 0 & 0 \\ -0.08654 & 0.0203289 & -0.001025 & 0 & 0 \\ -0.135713 & -0.001025 & 0.0291933 & 0 & 0 \\ 0 & 0 & 0 & 0.0023442 & 0.0002286 \\ 0 & 0 & 0 & 0.0002286 & 0.0014362 \end{bmatrix}$$

Table 4. Weighted MINQUE and Estimated Variance Covariance Matrix of Estimates

$$\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 0.2171987 \\ 0.0278557 \\ 0.3905864 \\ 0.4083858 \\ 0.5348881 \end{bmatrix}$$

$$\widehat{\mathrm{Cov}}\begin{bmatrix}\hat{\alpha}_1\\ \hat{\alpha}_2\\ \hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix}=\begin{bmatrix}1.3747989 & -0.114033 & -0.177639 & 0 & 0\\ -0.114033 & 0.0269847 & -0.001562 & 0 & 0\\ -0.177639 & -0.001562 & 0.0384249 & 0 & 0\\ 0 & 0 & 0 & 0.0031094 & 0.0011334\\ 0 & 0 & 0 & 0.0011334 & 0.0024038\end{bmatrix}$$

Table 5. Weighted MINQE and Estimated Variance Covariance Matrix of Estimates

$$\begin{bmatrix}\hat{\alpha}_1\\ \hat{\alpha}_2\\ \hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix}=\begin{bmatrix}0.1951973\\ 0.0003872\\ 0.3477227\\ 0.4061869\\ 0.5460941\end{bmatrix}$$

$$\widehat{\mathrm{Cov}}\begin{bmatrix}\hat{\alpha}_1\\ \hat{\alpha}_2\\ \hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix}=\begin{bmatrix}1.0471463 & -0.08654 & -0.35713 & 0 & 0\\ -0.08654 & 0.20329 & -0.001025 & 0 & 0\\ -0.35713 & -0.001025 & 0.0291933 & 0 & 0\\ 0 & 0 & 0 & 0.0023441 & 0.0002283\\ 0 & 0 & 0 & 0.0002283 & 0.0014355\end{bmatrix}$$

It is seen that *a priori* weights do not affect MINQE as much as they affect MINQUE. This may not be true in general, however in the case of intraclass covariance model, we can write the weighted MINQE of α_i as

$$\hat{\alpha}^{(d)}_{i(wI)}=\frac{e'V_*^{-1}T_i^0V_*^{-1}e}{s_i},$$

where $e = Q_{V_*^{-1}}Y$ are the weighted least square residuals. In the present case $T_i^0 = \Lambda_0 T_i \lambda_0$ and $\Lambda_0 = V_*$ we have

$$\hat{\alpha}^{(d)}_{i(wI)}=\frac{e'T_ie}{s_i},$$

which may not be very sensitive to the choice of weights, if the weighted least squared residuals are not very different from the ordinary least squared residuals, which is true for the present data.

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