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Estimation in Shifted Lindley Distribution

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Abstract

In this article, we propose a shifted version of widely-used Lindley distribution. Some statistical properties such as stochastic ordering, moment generating function, reliability characteristic etc. are studied for this new distribution. For estimating unknown parameters, two types of estimation method viz. method of moments and maximum likelihood method are explored. A simulation study for several choices of parameters is executed. Finally, a real data application illustrates the performance of our proposed distribution.

Key words: Lindley distribution; Stochastic ordering; Parameter estimation; Continuous distribution; Maximum likelihood estimate.

AMS Subject Classifications: 60E05; 62G30; 62E10

1. Introduction

Lifetime distribution tries to capture, mathematically, the length of the life of a system or a device. These distributions have relevance in the fields like environmental sciences, medicine, engineering etc. To analyze lifetime data, gamma, Weibull, Rayleigh etc., distributions are widely used in statistical literature. Chief advantage of these distributions is that they only have more general mathematical closed form compared to the exponential distribution with one additional parameter. Some applied areas such as finance, lifetime analysis and insurance sometimes demand the extended forms of these distributions because there still remain many important problems involving real data in these areas, which do not fit to any of the existent classical statistical models. As a consequence, several classes of generalized distributions have been formed by extending well-known continuous distributions. These generalized distributions entend more flexibility by adding new parameters to the baseline model.

Since last decade, Lindley distribution, proposed by Lindley (1958), has been abruptly acknowledged in different setup by many authors. Pretty recently, in the context of Bayesian statistics as a counter example of fiducial statistics, the Lindley distribution has bagged considerable attention because of its flexibility. Ghitany *et al.*(2008) discussed the various statistical properties of Lindley distribution and showed its applicability over the exponential distribution. They established that in reliability analysis Lindley distribution performs better than exponential model. One of the main reasons to consider the Lindley distribution over the exponential distribution is its time dependent/increasing hazard rate.

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A random variable X is said to have Lindley distribution with parameter θ if its probability density function (PDF) is defined as:

$$f(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x},\ x > 0,\ \theta > 0 \tag{1}$$

and corresponding (cumulative density function) (CDF) is given by

$$F(x;\theta) = 1 - \frac{\theta+1+\theta x}{1+\theta}e^{-\theta x},\ x > 0,\ \theta > 0 \tag{2}$$

Of late a lot of research articles came out on the extension of Lindley distribution. The motivation for all these extension stems on the flexibility of the distribution to accommodate more complex data. Some of the advances in the literature of Lindley distribution are given by Ghitany*et al.* (2011) who has introduced a two-parameter weighted Lindley distribution. Generalized Poisson Lindley distribution has been proposed by Mahmoudi *et al.* (2010). Bakouch *et al.* (2012) came up with extended Lindley (EL) distribution, Adamidis *et al.* (1998) introduced exponential geometric (EG) distribution. Shanker *et al.* (2013) introduced two-parameter Lindley distribution. Following a footstep Ghitany (2013) proposed inferential problems stemmed from power Lindley. Zakerzadeh *et al.* (2012) idealized a new two parameter lifetime distribution: model and properties. Hassan (2014) introduced convolution of Lindley distribution. Ghitany *et al.* (2015) worked on the estimation of the reliability of a stress-strength system from power Lindley distribution. Elbatal *et al.*(2013) proposed a new generalized Lindley distribution. However all these extensions were based on introducing more parameters in the constant part of the base Lindley.

The paper is organized as follows: Section 2 introduces a shifted Lindley distribution and presents its basic properties including the behaviour of the density and some results on stochastic orderings, moments, reliability characteristics. Distribution of the sum of iid random variables has also been discussed. In Section 3, estimation process of parameters is demonstrated at length. Monte Carlo simulation study is carried out in Section 4 followed by a real data analysis in Section 5. This paper concludes with some discussions in Section 6.

2. The Shifted Lindley distribution

The extension, proposed in this article, is completely different. We define.

$$f(x;\theta,\mu) = \frac{\theta^2}{1+\theta(1+\mu)}(1+x)e^{-\theta(x-\mu)}, x > \mu > 0 \tag{3}$$

as a Shifted Lindley distribution with parameters (θ, μ). It will be denoted by $SL(\theta, \mu)$. The CDF of a Shifted Lindley distribution with parameters (θ, μ) is given by

$$F(x;\theta,\mu) = 1 - \frac{1+\theta(1+x)}{1+\theta(1+\mu)}e^{-\theta(x-\mu)}, x > \mu > 0 \tag{4}$$

Note that if we put $\mu = 0$ in equations (3) and in (4), these equations become the PDF and CDF, respectively, of a Lindley distribution with a single parameter θ.

The shape of shifted Lindley distribution depends on its parameters. Figure 1 shows the pdf and cfd of shifted Lindley distribution for some choices of μ and θ. This figure reveals that for smaller θ shifted Lindley pdf exhibits right skewed while for larger θ it looks as an inverted J, more specifically tapering to a standard exponential curve.

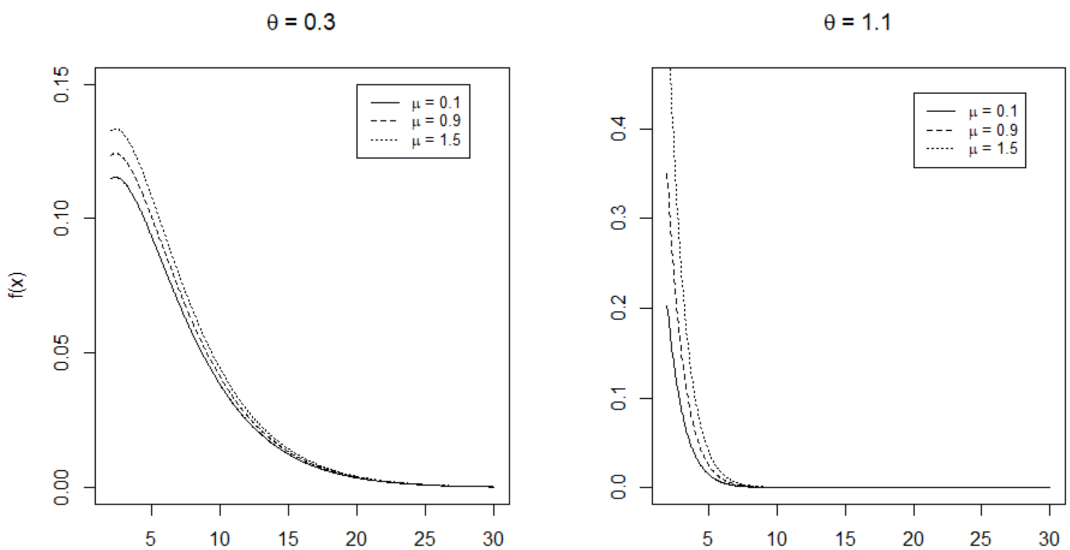

Figure 1: The PDF's of various Shifted Lindley distributions for different values of parameters

2.1. Stochastic orders

One of the main objectives of statistics is the comparison of random quantities. These comparisons are mainly based on the comparison of some measures associated to these random quantities. Stochastic ordering of positive continuous random variables is an important tool for judging such comparative behavior. Suppose X_i is distributed as $SL(\mu_i, \theta_i)$, $i = 1, 2$. Let F_i denote the cumulative distribution of X_i and f_i denote the probability density function of X_i. A random variable X_1 is said to be smaller than a random variable X_2 in the

- Stochastic order $(X_1 \leq_{st} X_2)$ if $F_1(x) \geq F_2(x)$ for all x.
- Hazard rate order $(X_1 \leq_{hr} X_2)$ if $h_1(x) \geq h_2(x)$ for all x.
- Likelihood ratio order $(X_1 \leq_{Lr} X_2)$ if $\frac{f_1(x)}{f_2(x)}$ decreases in x.

In order to establish stochastic ordering of distributions we refer the following result from Shaked *et al.* (1994).

$$X_1 \leq_{LR} X_2 \Longrightarrow X_1 \leq_{hr} X_2 \Longrightarrow X_1 \leq_{st} X_2.$$

Taking a cue from this above-mentioned result, a pair of theorems are proposed regarding the stochastic ordering pattern of $SL(\theta, \mu)$ for different choices of (θ, μ).

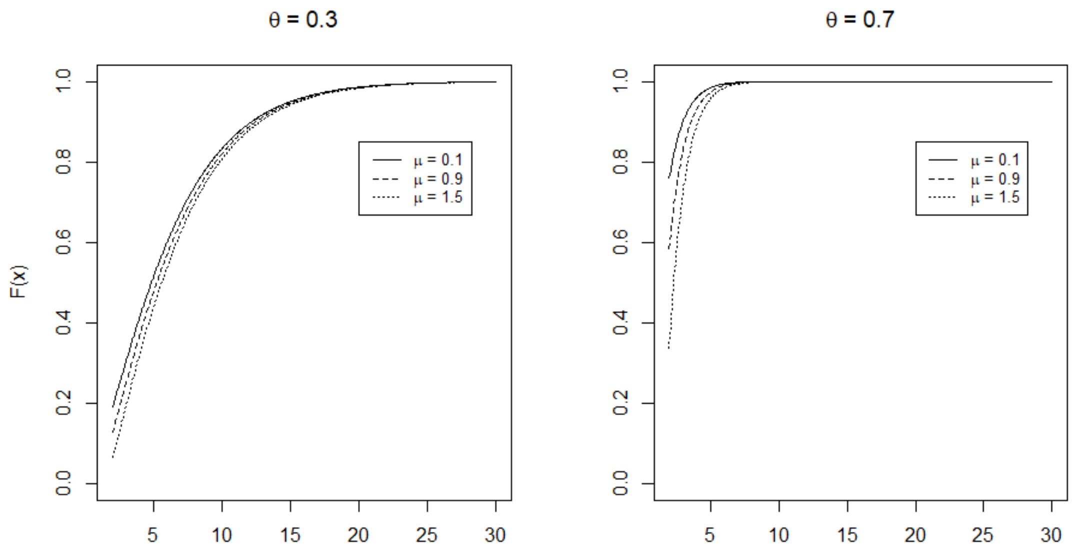

Figure 2: The CDF's of various Shifted Lindley distributions for different values of parameters

Theorem 1: Let $X_1 \sim SL(\theta_1, \mu_1)$ and $X_2 \sim SL(\theta_2, \mu_2)$. If $\mu_1 = \mu_2$ and $\theta_2 < \theta_1$, then $X_1 \leq_{Lr} X_2$ and hence $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$.

Proof:

Assume $\mu_1 = \mu_2$. Then $\frac{\delta}{\delta x} ln \frac{f_1(x)}{f_2(x)} = \theta_2 - \theta_1$. So $\frac{\delta}{\delta x} ln \frac{f_1(x)}{f_2(x)} < 0$ if $\theta_2 < \theta_1$ implying $\frac{f_1(x)}{f_2(x)} \downarrow x$.
This means that $X_1 \leq_{Lr} X_2$ and hence $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$.

Theorem 2: Let $X_1 \sim SL(\theta_1, \mu_1)$ and $X_2 \sim SL(\theta_2, \mu_2)$. If $\theta_1 = \theta_2 = \theta > 0$ and $\mu_1 > \mu_2$; then $X_1 \geq_{st} X_2$.

Proof:

The ratio of two pdf's does not involve x. So the technique adopted in checking of likelihood ratio ordering fails. Therefore, we would head to investigate via ratio of two corresponding distribution functions and hence directly infering on stochastic ordering of the distribution.

$$\frac{F_1(x)}{F_2(x)} = \frac{1 - \frac{1+\theta(1+x)}{1+\theta(1+\mu_1)} e^{-\theta(x-\mu_1)}}{1 - \frac{1+\theta(1+x)}{1+\theta(1+\mu_2)} e^{-\theta(x-\mu_2)}} = \frac{1 - \left[1 + \frac{\theta(x-\mu_1)}{1+\theta(1+\mu_1)}\right] e^{-\theta(x-\mu_1)}}{1 - \left[1 + \frac{\theta(x-\mu_2)}{1+\theta(1+\mu_2)}\right] e^{-\theta(x-\mu_2)}}.$$

Assume $\mu_1 > \mu_2$. Then $1 + \theta(1 + \mu_1) > 1 + \theta(1 + \mu_2)$ and $\theta(x - \mu_1) < \theta(x - \mu_2)$

$$\frac{\theta(x-\mu_1)}{1+\theta(1+\mu_1)} < \frac{\theta(x-\mu_2)}{1+\theta(1+\mu_2)}$$
$$1 + \frac{\theta(x-\mu_1)}{1+\theta(1+\mu_1)} < 1 + \frac{\theta(x-\mu_2)}{1+\theta(1+\mu_2)}. \quad (5)$$

Also

$$-e^{-\theta(x-\mu_1)} < -e^{-\theta(x-\mu_2)} \tag{6}$$

Combining (5) and (6) we have

$$1-\left[1+\frac{\theta(x-\mu_1)}{1+\theta(1+\mu_2)}\right]e^{-(x-\mu_1)} < 1-\left[1+\frac{\theta(x-\mu_2)}{1+\theta(1+\mu_2)}\right]e^{-(x-\mu_2)}$$

which results $F_1 < F_2$ upon further simplification. Consequently $X_1 \geq_{st} X_2$. Therefore if $\mu_1 > \mu_2$, $X_1 \geq_{st} X_2$ and vice-versa.

2.2. Moments

In applications, moments are necessary and very important. Through moments, it is possible to study many of the interesting characteristics and features of a distribution. The mean of the distribution can be obtained as:

$$\begin{aligned} \mu_1' = E(X) &= \frac{\theta^2}{1+\theta(1+\mu)}\int_\mu^\infty x(1+x)e^{-\theta(x-\mu)}dx \\ &= \mu+\frac{2}{\theta}-\frac{1+\mu}{1+\theta(1+\mu)} \end{aligned} \tag{7}$$

To find all higher order moment we will use the following result:

Theorem 3: For $k \geq 0$, the recurrence relation for the higher order moments are

$$\mu_{k+1}' = \mu_1'\mu_k' - \frac{d}{d\theta}\mu_1' \tag{8}$$

Proof:

$$\begin{aligned} \frac{d}{d\theta}\mu_k' &= \int_\mu^\infty x^k(1+x)\Big[\frac{\theta^2}{1+\theta(1+\mu)}(-1)(x-\mu)e^{-\theta(x-\mu)} \\ &+ e^{-\theta(x-\mu)}\quad \Big[\frac{2\theta}{1+\theta(1+\mu)}-\frac{\theta^2(1+\mu)}{(1+\theta(1+\mu))^2}\Big]\Big] \\ &= -\frac{\theta^2}{1+\theta(1+\mu)}\int_\mu^\infty x^{k+1}(1+x)\ \ e^{-\theta(x-\mu)}\ dx \\ &+ \mu\frac{\theta^2}{1+\theta(1+\mu)}\int_\mu^\infty x^k(1+x)\ \ e^{-\theta(x-\mu)}\ dx \\ &+ \Big[\frac{2\theta}{1+\theta(1+\mu)}-\frac{\theta^2(1+\mu)}{(1+\theta(1+\mu))^2}\Big]\int_\mu^\infty x^k(1+x)\ \ e^{-\theta(x-\mu)}\ dx \\ &= -\mu_{k+1}' + \ \mu_1'\mu_k' \end{aligned}$$

i.e.

$$\mu_{k+1}' = \mu_1'\mu_k' - \frac{d}{d\theta}\mu_k'$$

Hence the proof.

Putting $k = 1$ we get,

$$\begin{aligned}\mu_2' &= \Big(\mu + \frac{2}{\theta} - \frac{1+\mu}{1+\theta(1+\mu)}\Big)\mu_1' - \frac{d}{d\theta}\mu_1' \\ &= \Big(\mu + \frac{2}{\theta} - \frac{1+\mu}{1+\theta(1+\mu)}\Big)^2 - \Big[-\frac{2}{\theta^2} + \frac{(1+\mu)^2}{(1+\theta(1+\mu))^2}\Big]\end{aligned}$$

and hence

$$\begin{aligned}\mu_2 &= \mu_2' - \mu_1'^2 \\ &= \frac{2}{\theta^2} - \frac{(1+\mu)^2}{(1+\theta(1+\mu))^2}\end{aligned}$$

Putting $\mu = 0$ will imply

$$\mu_2 = \frac{2}{\theta^2} - \frac{1}{(1+\theta)^2}$$

which is the variance of a Lindley distribution with parameter θ. Similarly it can be shown that, for a $SL(\mu, \theta)$ distribution

$$\mu_3 = \frac{4}{\theta^3} - \frac{2(1+\mu)^3}{(1+\theta(1+\mu))^3}$$

and

$$\mu_4 = \frac{24}{\theta^4} - \frac{3(1+\mu)^4}{(1+\theta(1+\mu))^4} - \frac{12(1+\mu)^2}{\theta^2(1+\theta(1+\mu))^2}$$

In reference to the moments of shifted Lindley distribution, next we present heat plot (Figure 3) which unravels the intertwining effect of parameters μ and θ on mean, variance, skewness and kurtosis. Heat plot (or heatmap) is a data visualization technique that shows impact of variables in terms of intensity of color in two dimensions. The variation in color exhibits obvious visual clues about the relationship between two categories. From the matrix layout with color and shading of heat plot furnished here, it comes up that the mean and variance drop as both the parameters increase while the skewness and kurtosis shoot up with the increase of both parameters.

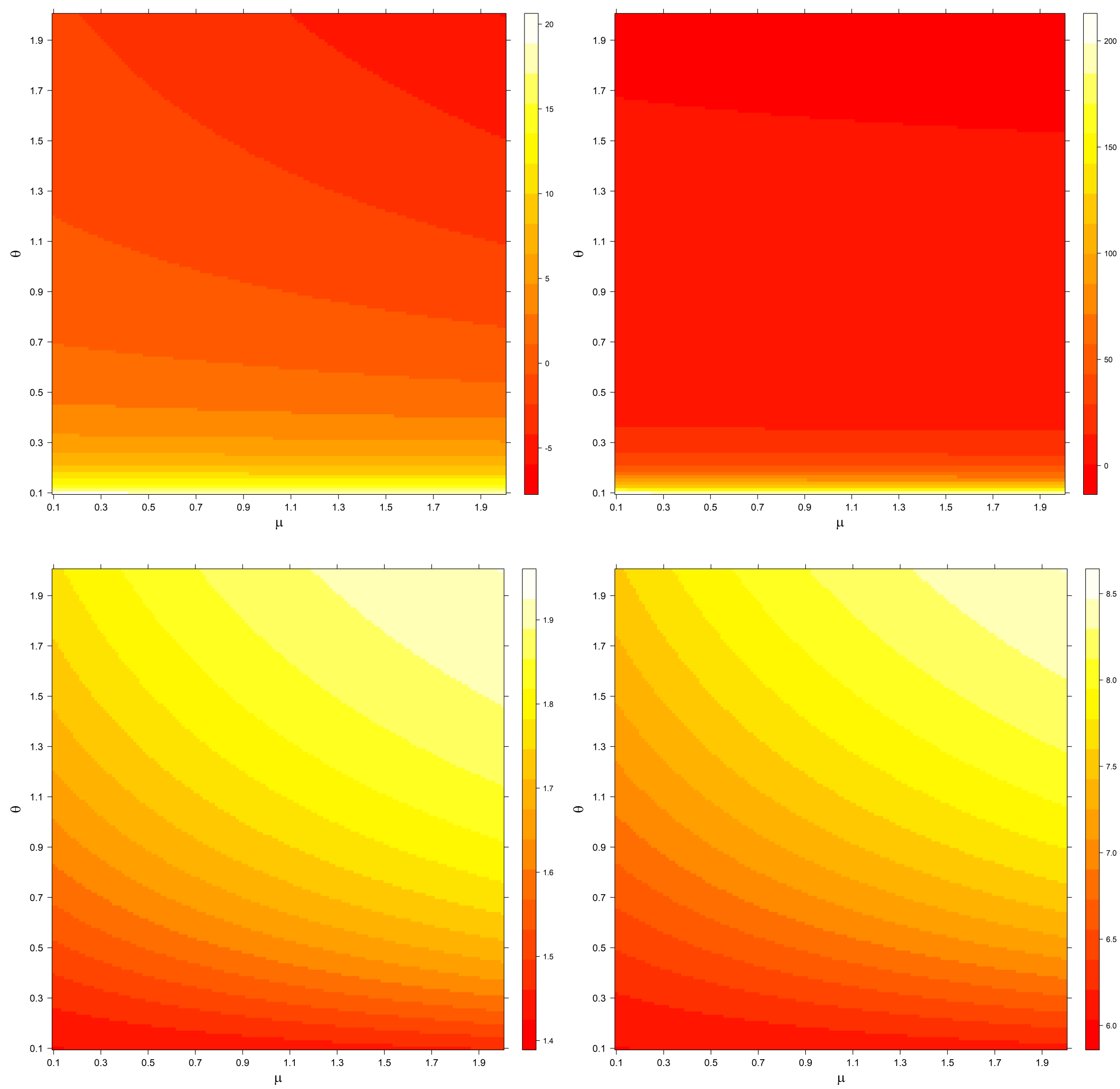

Figure 3: The mean, variance, skewness and kurtosis (from left to right) of the shifted Lindley distributions with respect to the parameters μ and θ.

2.3. Moment generating function (MGF)

In this subsection, we derived the MGF of $SL(\mu, \theta)$ distribution.

Theorem 4: If $X \sim SL(\mu, \theta)$, then the moment generating function $M_X(t)$ has the following form:

$$M_X(t) \quad = \quad \theta^2 \frac{[1 + (\theta - t)(1 + \mu)]}{(\theta - t)^2 [1 + \theta(1 + \mu)]} e^{t\mu}, \; |t| < \theta \tag{9}$$

Proof:

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \frac{\theta^2}{1+\theta(1+\mu)} \int_{\mu}^{\infty} e^{tX}(1+x)e^{-\theta(x-\mu)}dx \\
&= \frac{\theta^2}{1+\theta(1+\mu)} \int_{\mu}^{\infty} (1+x)e^{-\theta(x-\mu)+tx}dx \\
&= \frac{\theta^2}{1+\theta(1+\mu)} \int_{\mu}^{\infty} (1+x)e^{-(\theta-t)(x-\mu)}e^{t\mu}dx \\
&= \frac{\theta^2}{1+\theta(1+\mu)} e^{t\mu} \int_{\mu}^{\infty} (1+x)e^{-(\theta-t)(x-\mu)}dx \\
&= \frac{\theta^2}{1+\theta(1+\mu)} e^{t\mu} \frac{1+(\theta-t)(1+\mu)}{(\theta-t)^2} \\
&= \theta^2 \frac{[1+(\theta-t)(1+\mu)]}{(\theta-t)^2[1+\theta(1+\mu)]} e^{t\mu}
\end{aligned}$$

So, when $\mu = 0$

$$\begin{aligned}
M_X(t) &= \frac{\theta^2[1+(\theta-t)]}{(\theta-t)^2(1+\theta)} \\
&= \frac{1}{1+\theta} \left[\frac{\theta^2}{(\theta-t)^2} + \frac{\theta^2}{(\theta-t)}\right]
\end{aligned}$$

which coincides the MGF of a Lindley distribution with parameter θ.

In the same way the characteristic function of the shifted Lindley distribution becomes as follows.

$$\phi_X(t) = M_X(it) = \frac{\theta^2}{1+\theta(1+\mu)} \left[\frac{1}{(\theta-it)^2} + \frac{(1+\mu)}{(\theta-it)}\right] e^{it\mu} \tag{10}$$

where $i = \sqrt{-1}$ is the unit imaginary number.

2.4. Quantile function

Let X denotes a random variable with the probability distribution function Eq. (4). The quantile function , say $Q(p)$, defined by $F(Q(p)) = p$ is the root of the equation

$$\frac{1+\theta(1+Q(p))}{1+\theta(1+\mu)} e^{-\theta(Q(p)-\mu)} = 1-p \tag{11}$$

for $0 < p < 1$. On further simplification of the equation (11),

$$\begin{aligned}
1+\theta(1+Q(p))e^{-\theta(Q(p)-\mu)} &= [1+\theta(1+\mu)](1-p) \\
(1+Q(p))e^{-\theta Q(p)} &= \left[\{1+\theta(1+\mu)\}\frac{1-p}{\theta} - \frac{1}{\theta}\right] e^{-\theta\mu}
\end{aligned}$$

Substituting $Z(p) = 1 + Q(p)$ in the above

$$\begin{aligned} Z(p)e^{-\theta Z(p)}e^{\theta} &= \left[\{1+\theta(1+\mu)\}\frac{1-p}{\theta} - \frac{1}{\theta}\right] e^{-\theta\mu} \\ (-\theta Z(p))e^{-\theta Z(p)} &= -\theta e^{-\theta}\left[\{1+\theta(1+\mu)\}\frac{1-p}{\theta} - \frac{1}{\theta}\right] e^{-\theta\mu} \end{aligned}$$

So, the solution for $Z(p)$ is

$$Z(p) = -\frac{1}{\theta}W\Big(- \theta e^{-\theta}\left[\{1+\theta(1+\mu)\}\frac{1-p}{\theta} - \frac{1}{\theta}\right] e^{-\theta\mu}\Big)$$

for $0 < p < 1$, where $W(.)$ is the Lambert W function (see Corless *et al.* (1996)). Inverting the above equation in $Z(p)$

$$Q(p) = -\frac{1}{\theta}W\Big(\left[1 - \{1+\theta(1+\mu)\}(1-p)\right] e^{-\theta(\mu+1)}\Big) - 1,$$

for $0 < p < 1$.

2.5. Distribution of the sum of iid RVs

Theorem 5: If $X_1, X_2, \cdots, X_n$ are IID RVs from $SL(\mu, \theta)$, then the pdf of $Z = X_1 + X_2 + \cdots + X_n$ is

$$g(z; n, \theta, \mu) = \sum_{k=0}^{n} p_{k,n}(\theta, \mu) f_{SG}(z; 2n-k, \mu, \theta) \tag{12}$$

where $p_{k,n}(\theta,\mu) = \binom{n}{k}\frac{(\theta(1+\mu))^k}{(1+\theta(1+\mu))^n}$ and $f_{SG}(z;k,\mu,\theta) = \frac{\theta[\theta(x-\mu)]^{k-1}}{\Gamma k}e^{-\theta(x-\mu)}$, the pdf of a shifted gamma (SG) distribution (or Pearson type III) with parameters (k, μ, θ).

Proof: Recall that, if $X_1 \sim SL(\mu, \theta)$ then the pdf of X_1 is

$$f_{X_1}(x;\theta,\mu) = \frac{\theta^2}{1+\theta(1+\mu)}(1+x)e^{-\theta(x-\mu)} = \frac{(1+\mu)\theta}{1+\theta(1+\mu)}f_{SG(x;1,\theta,\mu)} + \frac{1}{1+\theta(1+\mu)}f_{SG(x;2,\theta,\mu)}$$

Next let us have the following lemma to expedite the proof.

Lemma 6: If $X \sim SG(k, \mu, \theta)$ then moment generating function of X is given by

$$M_X(t) = (1 - t/\theta)^{-k}\, e^{t\mu}.$$

Proof:

$$\begin{aligned}
M_X(t) &= \frac{\theta^k}{\Gamma k}\int_{\mu}^{\infty}(x-\mu)^{k-1} \quad e^{tx-\theta x+\theta\mu}dx \\
&= \frac{\theta^k}{\Gamma k}\int_{0}^{\infty} z^{k-1} \quad e^{-\theta z+tz+t\mu}dz \qquad \text{put}(x-\mu)=z \\
&= \frac{\theta^k}{\Gamma k} \quad e^{t\mu}\int_{0}^{\infty} z^{k-1} \quad e^{-(\theta-t)z}dz \\
&= \frac{\theta^k}{\Gamma k} \quad e^{t\mu}\frac{\Gamma k}{(\theta-t)^k} = \frac{\theta^k}{(\theta-t)^k} \quad e^{t\mu} \\
&= \frac{1}{(1-t/\theta)^k} \quad e^{t\mu} = (1-t/\theta)^{-k} \quad e^{t\mu}
\end{aligned}$$

The moment generating function (MGF) for X_1 for $|t| < \theta$ is

$$M_{X_1(t)} = E(e^{tX_1}) = \left(\frac{\theta(1+\mu)}{1+\theta(1+\mu)}(1-t/\theta)^{-1} + \frac{1}{1+\theta(1+\mu)}(1-t/\theta)^{-2}\right)e^{(t\mu)}$$

Hence the MGF of Z for $|t| < \theta$ is

$$\begin{aligned}
M_z(t) = E(e^{tZ}) &= \left(\frac{\theta(1+\mu)}{1+\theta(1+\mu)}(1-t/\theta)^{-1} + \frac{1}{1+\theta(1+\mu)}(1-t/\theta)^{-2}\right)^n e^{(nt\mu)} \\
&= \left[\frac{1}{(1+\theta(1+\mu))}\right]^n\Big[\theta(1+\mu)(1-t/\theta)^{-1}+(1-t/\theta)^{-2}\Big]^n e^{(nt\mu)} \\
&= \left[\frac{1}{(1+\theta(1+\mu))}\right]^n(1-t/\theta)^{-2n}\Big[\theta(1+\mu)(1-t/\theta)+1\Big]^n e^{(nt\mu)} \\
&= \left[\frac{1}{(1+\theta(1+\mu))}\right]^n(1-t/\theta)^{-2n}\sum_{k=0}^{n}\binom{n}{k}\Big[\theta(1+\mu)(1-t/\theta)\Big]^k e^{(nt\mu)} \\
&= \left[\frac{1}{(1+\theta(1+\mu))}\right]^n\sum_{k=0}^{n}\binom{n}{k}(\theta(1+\mu))^k(1-t/\theta)^{-(2n-k)}e^{(tn\mu)}
\end{aligned}$$

Using the Lemma stated above, Theorem 5 follows.

2.6. Reliability characteristics of shifted Lindley distribution

In present section, we consider shifted Lindley distribution as a lifetime model and study different reliability characteristics. The reliability function of the $SL(\mu,\theta)$ distribution is given by:

$$R(t) = P(X > t) = 1 - F(t) \tag{13}$$

The mean time to system failure (MTSF) is same as: We know that the hazard function $h(x)$ can be computed as

$$h(t) = \frac{f(t;\theta,\mu)}{1-F(t;\theta,\mu)}$$

which implies

$$h(t) = \theta^2(1+t) \tag{14}$$

The cumulative hazard function $H(x)$ is defined as

$$H(x) = -\log(1 - F(x;\theta,\mu)) = -\log\Big(\frac{\theta+1+\theta x}{1+\theta}e^{-\theta x}\Big) = -\log(R(x))$$

and the failure rate average (fra) is defined by $FRA(x) = H(x)/x$, where $x > \mu$. The conditional survival of t is:

$$R(x|t) = \frac{R(x+t)}{R(t)};\ \theta, R(.) > 0;\ t, x > \mu,\ \mu > 0$$

2.7. Rényi entropy

Entropy is used to measure the randomness of systems, and it is widely used in areas like physics, molecular imaging of tumors and sparse kernel density estimation. If X has the probability distribution function $f(.)$, Rényi entropy is defined by

$$I_\delta(x) = \frac{1}{1-\delta}\log\Big(\int_0^\infty f^\delta(x)\ dx\Big),\ \delta > 0,\ \delta \neq 1.$$

Using equation (3), it is observed that

$$\begin{aligned} f^\delta(x) &= \frac{\theta^{2\delta}}{[1+\theta(1+\mu)]^\delta}(1+x)^\delta e^{-\theta\delta(x-\mu)} \\ &= \frac{\theta^{2\delta}}{[1+\theta(1+\mu)]^\delta}\sum_{i=0}^{\delta}\binom{\delta}{i}x^i e^{-\theta\delta(x-\mu)} \end{aligned}$$

After some algebra, the Rényi entropy of X is reduces to

$$I_\delta(x) = \frac{1}{1-\delta}\log(\sum_{i=0}^{\delta} e_i)$$

where, $e_i = \frac{1}{(\theta\delta)^{i+1}}\Gamma(i+1,\theta\delta\mu)$, $\Gamma(.)$ is the incomplete Gamma function.

3. Estimation

Here, we consider two estimation methods: the methods of moments and maximum likelihood estimation. We provide expressions for the associated Fisher information matrix. Suppose $X_1, X_2, \cdots, X_n$ is a random sample from equation (3). For the moments estimation, let $m_1 = (1/n)\sum_{i=1}^n X_i$ and $m_2 = (1/n)\sum_{i=1}^n (X_i - m_1)^2$. By equating the theoretical moments of equation (3) with the sample moments, the following equations are obtained.

$$m_1 = \mu + \frac{2}{\theta} - \frac{1+\mu}{1+\theta(1+\mu)} \tag{15}$$

$$m_2 = \frac{2}{\theta^2} - \frac{(1+\mu)^2}{[1+\theta(1+\mu)]^2} \tag{16}$$

Solving (15), (16) we can estimate the parameters μ and θ.

3.1. Maximum likelihood (ML) estimation of parameters

The likelihood function for a random sample $X_1, X_2, ..., X_n$ which is taken from $SL(\mu, \theta)$ distribution is:

$$L(\boldsymbol{X}, \mu, \theta) = \frac{\theta^{2n}}{(1+\theta(1+\mu))^n}\Big[\prod_{i=1}^n (1+x_i)\Big] e^{-\theta\sum_{i=1}^n (x_i-\mu)} \tag{17}$$

It is to be noted that mle of μ is

$$\hat{\mu}_{mle} = \min_i X_i = X_{(1)} \tag{18}$$

Differentiating the log-likelihood w.r.t. θ, we get the following equation:

$$\frac{2n}{\theta} - \frac{n(1+\hat{\mu}_{mle})}{1+\theta(1+\hat{\mu}_{mle})} - \sum_{i=1}^n (X_i - \hat{\mu}_{mle}) = 0 \tag{19}$$

which needs to be solved using some iterative procedure.

4. Simulation Study

It may be noted that it is not possible to generate samples from shifted Lindley distribution using the inversion of the CDF. It has already been established that an $SL(\mu, \theta)$ distribution can be viewed as a mixture of two shifted gamma distributions. This property is devised to generate random sample from simulation study. In R, function *rgamma3* of package *FAdist* generates samples from a shifted gamma distribution. Using the aforesaid function a convenient sampling scheme for data generation can be framed as follows.

To estimate the parameters μ and θ, we have generated 10,000 samples from the shifted Lindley distribution. We have considered four different combinations of the parameter to study their influence. Then using the sample moments and equations (15) and (16), we obtain the moment estimates of μ and θ. We have replicated these processes 50, 100 and 500 times and computed standard error of corresponding estimates. The ML estimates are obtained by using the equations (18) and (19) and respective standard errors have been computed using the above techniques.

Algorithm
1. Select values of θ and μ
2. Calculate weight $w = \frac{(1+\mu)\theta}{1+\theta(1+\mu)}$
3. Generate U from $U(0,1)$
4. If $U < w$, generate a sample from $f_{SG(x;1,\theta,\mu)}$ else from $f_{SG(x;2,\theta,\mu)}$ using $rgamma3(1, shape, scale, thres)$

The simulation study is carried out with $N = 10,000$ sample size for $(\mu, \theta) = (0.5, 0.3)$, $(1.5, 1.1)$, $(0.5, 1.1), (1.5, 0.3)$ and replication $n = (50, 100, 500)$. The following measures are calculated to asses the simulation results:
$\hat{\theta}$ and $\hat{\mu}$, estimates obtained through both of the case along with the corresponding standard error of estimates (SE), $Bias_\mu = \sum_{i=1}^{n} \frac{\widehat{\mu}_j - \mu}{n}$, magnitude of relative error $= MRE_\mu = \sum_{i=1}^{n} \frac{\widehat{\mu}_j/\mu}{n}$, mean square error $MSE_\mu = \sum_{i=1}^{n} \frac{(\widehat{\mu}_j - \mu)^2}{n}$, $Bias_\theta = \sum_{i=1}^{n} \frac{\widehat{\theta}_j - \theta}{n}$, $MRE_\theta = \sum_{i=1}^{n} \frac{\widehat{\theta}_j/\theta}{n}$, $MSE_\theta = \sum_{i=1}^{n} \frac{(\widehat{\theta}_j - \theta)^2}{n}$. Results against the parameter θ are shown in Table 3 and that of parameter μ in Table 4. Both of the tables are placed in Appendix at the end of this article.
From Table 3 and Table 4, it may be observed that moment estimators for both are also performing well in terms of small biases. As expected the MRE values are found close to 1, whereas the MSE values are tending close to 0. This study also reveals that moment and ML estimators are equally efficient.

5. Real Data Analysis

The proposed distribution is fitted for a data set available in Duffy *et al.*(1993). The data consists of measurements on strength of the sintered silicon nitride after four-point bend system is applied. On four point bend specimen, the support span of test fixture was 40.373 mm and the inner load span of 19.622 mm. All specimens are subjected to pure four-point bending. Number of complete specimens in the data set is found to be 27. We apply Lindley and shifted Lindley in order to fit this data. Subject to the fitting of shifted Lindley distribution on the data we figure out estimates of the parameters θ and μ by both moment and maximum likelihood method. Estimates alongwith standard errors (SE) are given in Table 1. For the shifted Lindley distribution,it can be seen that both the methods are producing different estimates.

Table 1: Parameter estimates for the four point bend data

Distribution	$\widehat{\mu}_{ML}$ (SE)	$\widehat{\mu}_{MOM}$ (SE)	$\widehat{\theta}_{ML}$ (SE)	$\widehat{\theta}_{MOM}$ (SE)
Lindley	0	0	0.0027	0.0027
	(-)	(-)	(0.0001)	(0.0001)
shifted Lindley	613.9	654.873	0.0096	0.014
	(0.0006)	(0.0531)	(0.0001)	(0.0001)

For further comparison between two distributions fitted to the data, we also report some model selection criteria— Akaike information criterion (AIC), Bayesian information Crite-

Table 2: Model selection criteria for the four point bend data

Distribution	KS(p-value)	AIC	BIC	$CAIC$	$HQIC$
Lindley	2.4(0.823)	391.7628	393.0586	391.9109	392.1481
Shifted Lindley	0.632(0.001)	315.8982	318.49	316.36	316.669

rion (BIC), Corrected AIC (CAIC) and Hannan and Quinn information criterion (HQIC). The definitions used for these selection tools are as: AIC= $-2ln\,L(\theta)+2k$, CAIC= $-2ln\,L(\theta)+2k\frac{n}{n-k-1}$; BIC= $-2ln\,L(\theta)+k\,ln(n)$; and HQIC= $-2ln\,L(\theta)+2k\,ln\{ln(n)\}$, where $lnL(\theta)$ denotes log likelihood, n being the number of observations and k being the number of parameters of the distribution. These are reported in Table 2. Considering all the model selection criteria, reported in Table 2, we found that shifted Lindley fits the data well compared to Lindley distribution. The Kalmogorov-Smirnoff (KS) statistic for shifted Lindley is found to be 0.632 with a p-value of 0.001 confirming the claim of better fit.

6. Conclusion

In this study we have proposed a new distribution called shifted Lindley distribution. Some mathematical properties along with estimation issues are addressed. The hazard rate function of shifted Lindley distribution shows that the subject distribution can be used to model reliability data as well. We derived the moment and maximum likelihood estimates of the parameters along with the biases, mean square error and mean relative errors. A real data application of the shifted Lindley distribution projects that it could provide a meaningful fit than a set of usual statistical distributions, while being considered specially in life time data analysis. A further extension of shifted Lindley might be thought in the context of power Lindley distribution, thereby a comparative study on relative quality of statistical models for a given set of data can be delved into.

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ANNEXURE

Table 3: Estimate, SE, Bias, MSE and MRE of (θ) based on the Moment and ML estimation methods

Replication	μ	θ	**Moment Estimator**(θ)					**ML Estimator** (θ)				
			$\hat{\theta}$	SE	Bias	MSE	MRE	$\hat{\theta}$	SE	Bias	MSE	MRE
50	0.5	0.3	.3001	.0005	0.00011	1.24E-05	1.00037	0.29992	0.00034	-8.00E-05	5.65E-06	0.99973
	1.5	1.1	1.09760	0.00171	-0.00239	0.00015	0.99782	1.09855	0.00109	-0.00145	6.08E-05	0.99868
	0.5	1.1	1.10084	0.001191	0.000847	7.03E-05	1.00077	1.09918	0.00088	-0.00082	3.83E-05	0.99925
	1.5	.3	0.30036	0.00047	0.00036	1.09E-05	1.0012	0.30014	0.00036	0.00014	6.48E-06	1.00046
100	0.5	0.3	0.29996	0.000291	-3.77E-05	8.34E-06	0.99987	0.30024	0.000234	0.000237	5.47E-06	1.0008
	1.5	1.1	1.09861	0.00131	-0.001391	0.00017	0.99873	1.09871	0.00088	-0.00128	7.98E-05	0.99882
	0.5	1.1	1.09958	0.00133	-0.00041	0.00017	0.99962	1.09997	0.000847	-2.45E-05	7.11E-05	0.99997
	1.5	0.3	0.30003	0.00033	2.82E-05	1.09E-05	1.00009	0.30017	0.000268	0.00017	7.16E-06	1.00058
500	0.5	0.3	0.29992	0.000126	-7.15E-05	8.02E-06	0.99976	0.29995	9.49E-05	-5.04E-05	4.50E-06	0.99983
	1.5	1.1	1.1008	0.00058	0.00079	0.00017	1.00072	1.10043	0.00041	0.00043	8.56E-05	1.00039
	0.5	1.1	1.09996	0.000543	-3.65E-05	0.00015	0.99997	1.09991	0.00041	-9.22E-05	8.02E-05	0.99991
	1.5	0.3	0.30032	0.00013	0.00032	9.13E-05	9.13E-06	0.30006	9.76E-05	6.59E-05	4.76E-06	1.00022

Table 4: Estimate, SE, Bias, MSE and MRE of (μ) based on the Moment and ML estimation methods

Replication	μ	θ	**Moment Estimator(μ)**					**ML Estimator (μ)**				
			$\hat{\mu}$	SE	Bias	MSE	MRE	$\hat{\mu}$	SE	Bias	MSE	MRE
50	0.5	0.3	0.50645	0.01143	0.00645	0.00644	1.01290	0.50128	0.00015	0.00128	2.80E-06	1.00256
	1.5	1.1	1.49876	0.001575	-0.00123	0.000123	0.99917	1.50013	1.46E-05	0.00012	2.73E-08	1.00008
	0.5	1.1	0.50267	0.00156	0.00267	0.00013	1.00534	0.50015	2.33E-05	0.000147	4.82E-08	1.00029
	1.5	0.3	1.50655	0.00911	0.00655	0.00411	1.00436	1.50083	0.00013	0.00083	1.50E-06	1.00055
100	0.5	0.3	0.48804	0.00851	-0.01195	0.00732	0.97608	0.50117	0.000125	0.00117	2.93E-06	1.00234
	1.5	1.1	1.49986	0.00117	-0.00014	0.00014	0.99990	1.50012	1.23E-05	0.00012	3.06E-08	1.00008
	0.5	1.1	0.50103	0.00125	0.00103	0.00016	1.00207	0.50014	1.32E-05	0.00014	3.67E-08	1.00028
	1.5	0.3	1.49556	0.006429	-0.00444	0.00411	0.99704	1.50085	8.22E-05	0.00085	1.39E-06	1.00057
500	0.5	0.3	0.49843	0.003470	-0.00156	0.00601	0.99687	0.50118	5.14E-05	0.00118	2.72E-06	1.00237
	1.5	1.1	1.50048	0.00051	0.00048	0.00013	1.00032	1.50013	5.88E-06	0.00012	3.37E-08	1.00008
	0.5	1.1	0.50009	0.00058	9.00E-05	0.000169	1.00018	0.500123	6.30E-06	0.00014	3.89E-08	1.00027
	1.5	0.3	1.50738	0.002759	0.00738	0.00385	1.00492	1.50074	3.12E-05	0.00074	1.03E-06	1.00049