Statistics and Applications {ISSN 2454-7395 (online)}
Volume 23, No. 1, 2025 (New Series), pp 283–301
https://www.ssca.org.in/journal

Bayesian Estimation of Scale Parameter of Inverted Kumaraswamy Distribution under Various Combinations of Different Priors and Loss Functions

Ableen Kaur, Parmil Kumar and Hemani Sharma
Department of Statistics, University of Jammu, Jammu, J &K - 180006

Received: 28 July 2023; Revised: 17 April 2024; Accepted: 19 June 2024

Abstract

Bayesian estimation, a non-classical method of estimation has emerged as one of the most accepted method in statistical inference. In this paper, the Bayesian estimators of the parameters of Inverted Kumaraswamy distribution under two priors, namely Gamma and Uniform have been obtained considering three different cases: (i) when α is known and β is unknown, (ii) when α is unknown and β is known, and (iii) when α and β both are unknown. The symmetric and asymmetric loss functions *viz.*, Linear exponential (LINEX), Squared error (SE) and Entropy loss (EL) functions have been used for the Bayesian estimation. Lindley's approximation (L-approximation) has been used to obtain approximate Bayes estimators. Their performance was compared using simulated risks. An intensive simulation study is carried out with the help of Matlab and R software to examine the behavior of estimators based on their relative mean square errors.

Key words: Bayesian estimation; Inverted Kumaraswamy distribution; Lindley's approximation; Relative mean square error; Symmetric and asymmetric loss function.

AMS Subject Classifications: 62F15

1. Introduction

In recent literature, several novel distributions have been proposed for describing various real life situations in many applied sciences. Kumaraswamy (1980) obtained a distribution, which is derived from beta distribution after fixing some parameters in beta distribution. But it has a closed-form cumulative distribution function which is invertible and for which the moments do exist. If X follows Kum(α,β), then the Cumulative Distribution Function CDF is given by

$$F(x) = (1 - (1 - x)^{\alpha})^{\beta}; 0 < x < 1, \alpha, \beta > 0$$

The distribution is appropriate to natural phenomena whose outcomes are bounded from both sides, such as the individuals' heights, test scores, temperatures and hydrological daily data of rain fall.

Corresponding Author: Ableen Kaur
Email: ableenkaur23@gmail.com

Abd Al-Fattah *et al.* (2017) derived the Inverted Kumaraswamy (IKum) distribution from Kumaraswamy (Kum) distribution using the transformation $T = \frac{1}{X} - 1$ so if X follows Kum(α,β) where α and β are shape parameters, then the T has a IKum distribution with CDF

$$F(t) = (1 - (1+t)^{-\alpha})^{\beta}; t > 0, \alpha, \beta > 0 \tag{1}$$

and probability density function (pdf)

$$f(t) = \alpha\beta(1+t)^{-(\alpha+1)}(1 - (1+t)^{-\alpha})^{\beta-1}; t > 0, \alpha, \beta > 0 \tag{2}$$

Also the Reliability and Hazard rate functions are given by

$$R(t) = P(T > t) = 1 - F(t) = 1 - (1 - (1+t)^{-\alpha})^{\beta} \tag{3}$$

$$H(t) = \frac{f(t)}{R(t)} = \frac{\alpha\beta(1+t)^{-(\alpha+1)}(1 - (1+t)^{-\alpha})^{\beta-1}}{1 - (1 - (1+t)^{-\alpha})^{\beta}} \tag{4}$$

Abd Al-Fattah *et al.* (2017) found IKum distribution to be a right skewed distribution, which according to Moustafa and Mahmoud (2018), will affect long term reliability predictions, producing optimistic predictions of rare events occurring in the right tail of the distribution compared with other distributions. Also the IKum distribution provides good fit to several data in literature.

The inverse distributions, also known as inverted or reciprocal distributions, have been widely applied to a wide variety of scenarios in this context. Lately, many researchers have considered and studied the properties of inverted distributions. For example, Tiao and Cuttman (1965) obtained Inverted Dirichlet distribution and its application to a problem in bayesian inference. Prakash (2012) studied the inverted exponential model and Flaih *et al.* (2012) presented exponentiated inverted Weibull distribution. Iqbal *et al.* (2017) developed a general form of IKum distribution. Fan and Gui (2022) studied the statistical inference of inverted exponentiated Rayleigh distribution based on joint progressively type-II censored data. Aldahlan *et al.* (2022) estimated the parameters of the Beta Inverted Exponential Distribution under Type-II Censored Samples. Sana *et al.* (2023) considered the problem of estimation of unknown parameters based on lower record values for Inverted Kumaraswamy distribution using Lindley's approximation.

To our best knowledge no such Bayesian analysis for Inverted Kumaraswamy distribution under these combinations of priors and loss functions has been done.

The paper is carried out as follows: In Section 2 the likelihood function is obtained, followed by the derivation of posterior distribution of the unknown parameter in all three cases under the considered priors in Section 3. In Section 4 different loss functions are used to compute the estimates of the parameters. Section 5 depicts the simulation study conducted for performance evaluation along with the results in tabular form. The study is concluded in Section 6, followed by references used for literature review.

2. Likelihood function for the inverted Kumaraswamy distribution

Let $X_1, X_2, ..., X_n$ be a random sample of size n taken from the Inverted Kumaraswamy distribution. Then the likelihood function for the given sample observations is

$$L(x;\alpha,\beta) = \alpha^n \beta^n \prod_{i=1}^{n} (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1}$$

$$L(x;\alpha,\beta) = \alpha^n \beta^n \prod_{i=1}^{n} \frac{(1+x_i)^{-(\alpha+1)}}{(1-(1+x_i)^{-\alpha})} e^{\beta \sum_{i=1}^{n} \ln\left(1-(1+x_i)^{-\alpha}\right)} \tag{5}$$

3. Priors and posterior distributions for the unknown parameters of inverted Kumaraswamy distrubution

In Bayesian estimation selection of appropriate prior for the parameters is a crucial step. In this paper, we consider one informative and one non-informative prior. The corresponding posterior distributions were derived for each case.

3.1. CASE I: When β is unknown and α is known

3.1.1. Posterior distribution under gamma prior

$$\pi_1(\beta|a,b) = \frac{e^{-b\beta}\beta^{a-1}b^a}{\Gamma(a)}; \beta, a, b > 0 \tag{6}$$

Using the likelihood function (5) and the prior (6), the posterior distribution for the parameter β becomes

$$\begin{aligned}\pi_1(\beta|x) &= \frac{L(x;\alpha,\beta) * \pi_1(\beta|a,b)}{\int_0^\infty L(x;\alpha,\beta) * \pi_1(\beta|a,b) d\beta} \\ &= \frac{\alpha^n \beta^n \prod_{i=1}^{n} \frac{(1+x_i)^{-(\alpha+1)}}{((1-(1+x_i)^{-\alpha})} e^{\beta \sum_{i=1}^{n} \ln\left(1-(1+x_i)^{-\alpha}\right)} \frac{e^{-b\beta}\beta^{a-1}b^a}{\Gamma(a)}}{\int_0^\infty \alpha^n \beta^n \prod_{i=1}^{n} \frac{(1+x_i)^{-(\alpha+1)}}{((1-(1+x_i)^{-\alpha})} e^{\beta \sum_{i=1}^{n} \ln(1-(1+x_i)^{-\alpha})} \frac{e^{-b\beta}\beta^{a-1}b^a}{\Gamma(a)} d\beta}\end{aligned}$$

$$\pi_1(\beta|x) = \frac{\beta^{(n+a)-1} \exp\left(-\beta R\right) R^{n+a}}{\Gamma(n+a)}, \tag{7}$$

where $R(x,\alpha) = b - \sum_{i=1}^{n} \ln\left(1-(1+x_i)^{-\alpha}\right)$

3.1.2. Posterior distribution under uniform prior

$$\pi_2(\beta|k) = k; \beta, k > 0 \tag{8}$$

Using the likelihood function (5) and the prior (8), the posterior distribution for the parameter β becomes

$$\begin{aligned}\pi_2(\beta|x) &= \frac{L(x;\alpha,\beta)*\pi_2(\beta|k)}{\int_0^\infty L(x;\alpha,\beta)*\pi_2(\beta|k)d\beta}\\ &= \frac{\alpha^n\beta^n\prod_{i=1}^n\frac{(1+x_i)^{-(\alpha+1)}}{((1-(1+x_i)^{-\alpha})}e^{\beta\sum_{i=1}^n\ln\left(1-(1+x_i)^{-\alpha}\right)}k}{\int_0^\infty\alpha^n\beta^n\prod_{i=1}^n\frac{(1+x_i)^{-(\alpha+1)}}{((1-(1+x_i)^{-\alpha})}e^{\beta\sum_{i=1}^n\ln\left(1-(1+x_i)^{-\alpha}\right)}kd\beta}\end{aligned}$$

$$\pi_2(\beta|x) = \frac{\beta^n\exp\left(-\beta T\right)T^{n+1}}{\Gamma(n+1)}, \tag{9}$$

where $T(x,\alpha) = -\sum_{i=1}^n \ln\left(1-(1+x_i)^{-\alpha}\right)$

3.2. CASE II: When β is known and α is unknown

3.2.1. Posterior distribution under gamma prior

$$\pi_1(\alpha|a,b) = \frac{e^{-b\beta}\beta^{a-1}b^a}{\Gamma(a)}; \alpha, a, b > 0 \tag{10}$$

Using the likelihood function (5) and the prior (10), the posterior distribution for the parameter β becomes

$$\begin{aligned}\pi_1(\alpha|x) &= \frac{L(x;\alpha,\beta)*\pi_1(\alpha|a)}{\int_0^\infty L(x;\alpha,\beta)*\pi_1(\alpha|a)d\alpha}\\ &= \frac{\alpha^n\beta^n\prod_{i=1}^n(1+x_i)^{-(\alpha+1)}(1-(1+x_i)^{-\alpha})^{\beta-1}\frac{e^{-b\beta}\beta^{a-1}b^a}{\Gamma(a)}}{\int_0^\infty\alpha^n\beta^n\prod_{i=1}^n(1+x_i)^{-(\alpha+1)}(1-(1+x_i)^{-\alpha})^{\beta-1}\frac{e^{-b\beta}\beta^{a-1}b^a}{\Gamma(a)}d\alpha}\\ &= \frac{\alpha^{n+a-1}e^{-b\alpha}\prod_{i=1}^n(1+x_i)^{-(\alpha+1)}(1-(1+x_i)^{-\alpha})^{\beta-1}}{\int_0^\infty\alpha^{n+a-1}e^{-b\alpha}\prod_{i=1}^n(1+x_i)^{-(\alpha+1)}(1-(1+x_i)^{-\alpha})^{\beta-1}d\alpha}\\ &= K_1^{-1}\alpha^{n+a-1}e^{-b\alpha}\prod_{i=1}^n(1+x_i)^{-(\alpha+1)}\left(1-(1+x_i)^{-\alpha}\right)^{\beta-1}\end{aligned}$$

where,

$$K_1 = \int_0^\infty \alpha^{n+a-1}e^{-b\alpha}\prod_{i=1}^n(1+x_i)^{-(\alpha+1)}\left(1-(1+x_i)^{-\alpha}\right)^{\beta-1}d\alpha \tag{11}$$

3.2.2. Posterior distribution under uniform prior

$$\pi_2(\alpha|a) = k; k > 0 \tag{12}$$

$$\begin{aligned}
\pi_2(\alpha|x) &= \frac{L(x;\alpha,\beta) * \pi_2(\alpha|a)}{\int_0^\infty L(x;\alpha,\beta) * \pi_2(\alpha|a) d\alpha} \\
&= \frac{\alpha^n \beta^n \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} k}{\int_0^\infty \alpha^n \beta^n \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} k d\alpha} \\
&= \frac{\alpha^n \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1}}{\int_0^\infty \alpha^n \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} d\alpha} \\
&= K_2^{-1} \alpha^n \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} \Big(1-(1+x_i)^{-\alpha}\Big)^{\beta-1}
\end{aligned}$$

where,

$$K_2 = \int_0^\infty \alpha^n \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} \Big(1-(1+x_i)^{-\alpha}\Big)^{\beta-1} d\alpha \tag{13}$$

3.3. CASE III: When α and β both are unknown

3.3.1. Posterior distribution under gamma prior

Suppose the parameters are independent and follow Gamma distribution,

$$\begin{aligned}
\pi(\alpha|a_1,b_1) &\propto \alpha^{a_1-1} e^{-b_1\alpha}; \alpha > 0, a_1, b_1 > 0 \\
\pi(\beta|a_2,b_2) &\propto \beta^{a_2-1} e^{-b_2\beta}; \beta > 0, a_2, b_2 > 0
\end{aligned}$$

where, $a_1 \& b_1$ and $a_2 \& b_2$, are non-negative hyperparameters and are known. The joint prior distribution for α and β is given by

$$\pi_{11}(\alpha,\beta|a_1,b_1,a_2,b_2) \propto \alpha^{a_1-1}\beta^{a_2-1}e^{-b_1\alpha_1-b_2\beta}$$

The joint posterior density function of parameters α and β is obtained as

$$\pi_{11}(\alpha,\beta|x) = \frac{L(x;\alpha,\beta) * \pi_{11}(\alpha,\beta|a_1,b_1,a_2,b_2)}{\int_0^\infty\int_0^\infty L(x;\alpha,\beta) * \pi_{11}(\alpha,\beta|a_1,b_1,a_2,b_2) d\alpha d\beta}$$

the above equation cannot be obtained in closed form so in order to find the Bayes estimator of the parameters we have used Lindley approximation method. The joint posterior density function can be written as

$$\pi_{11}(\alpha,\beta|x) \propto \alpha^{n+a_1-1}\beta^{n+a_2-1}\prod_{i=1}^n (1+x_i)^{-(\alpha+1)}\Big(1-(1+x_i)^{-\alpha}\Big)^{\beta-1} e^{-b_1\alpha_1-b_2\beta} \tag{14}$$

3.3.2. Posterior distribution under uniform prior

Suppose the parameters are independent and follow Uniform distribution,

$$\begin{aligned}
\pi(\alpha|k_1) &= k_1; \alpha > 0, k_1 > 0 \\
\pi(\beta|k_2) &= k_2; \beta > 0, k_2 > 0
\end{aligned}$$

The joint prior distribution for α and β is given by

$$\pi_{12}(\alpha,\beta|k_1,k_2)=k_1k_2$$

The joint posterior density function of parameters α and β is obtained as

$$\pi_{12}(\alpha,\beta|x)=\frac{L(x;\alpha,\beta)*\pi_{12}(\alpha,\beta|k_1,k_2)}{\int_0^\infty\int_0^\infty L(x;\alpha,\beta)*\pi_{12}(\alpha,\beta|k_1,k_2)d\alpha d\beta}$$

the above equation cannot be obtained in closed form so in order to find the Bayes estimator of the parameters we have used Lindley approximation method. The joint posterior density function can be written as

$$\pi_{11}(\alpha,\beta|x)\propto\alpha^n\beta^n\prod_{i=1}^{n}(1+x_i)^{-(\alpha+1)}\Big(1-(1+x_i)^{-\alpha}\Big)^{\beta-1} \tag{15}$$

4. Bayesian estimation under different loss functions

This section presents the Bayes estimates of the unknown parameter obtained under three loss functions *viz.*, Linear exponential, Squared error and Entropy loss functions.

4.1. Case I: When β is unknown and α is known

4.1.1. Bayesian estimation by using gamma prior under different loss functions

- **Bayes estimator under LINEX loss function**

The LINEX loss function is given by

$$L(\hat\beta,\beta)=\exp(q_1(\hat\beta-\beta))-h(\hat\beta-\beta)-1;q_1,h\neq0 \tag{16}$$

By using LINEX loss function as given in (16), the risk function is given by

$$R(\hat\beta,\beta)=E[L(\hat\beta,\beta)]=\int_0^\infty\exp(q_1(\hat\beta-\beta)-h(\hat\beta-\beta)-1).\pi_1(\beta|x)d\beta$$

$$=\int_0^\infty\Big[\exp\big(q_1\hat\beta\big).\exp\big(-q_1\beta\big)-h\hat\beta+h\beta-1\Big].\frac{\beta^{(n+a)-1}\exp\big(-\beta R\big)R^{n+a}}{\Gamma(n+a)}d\beta$$

$$=\exp(q_1\hat\beta)\frac{R^{n+a}}{[R+q_1]^{n+a}}+\frac{h}{R}(n+a)-(h\hat\beta+1)$$

Now solving $\frac{\partial R(\hat\beta,\beta)}{\partial\hat\beta}=0$, we get

$$\exp\big(q_1\hat\beta\big)=\frac{h}{q_1}\left(\frac{R+q_1}{R}\right)^{n+a}$$

Taking log on both sides, we obtain the Bayes estimator as

$$\hat\beta_{LINEX}=\frac{1}{q_1}\left[\ln\left(\frac{h}{q_1}\right)+(n+a)\ln\left(\frac{R+q_1}{R}\right)\right] \tag{17}$$

• Bayes estimator under squared error loss function

The squared error loss function is given by

$$L(\hat{\beta}, \beta) = (\hat{\beta} - \beta)^2 \tag{18}$$

By using Squared error loss function as given in (18), the risk function is given by

$$R(\hat{\beta}, \beta) = E[L(\hat{\beta}, \beta)] = \int_0^\infty \left[(\hat{\beta} - \beta)^2\right].\pi_1(\beta|x)d\beta$$

$$= \int_0^\infty \left[(\hat{\beta} - \beta)^2\right].\frac{\beta^{(n+a)-1}\exp(-\beta R)R^{n+a}}{\Gamma(n+a)}d\beta$$

$$= \hat{\beta}^2 + \frac{(n+a+1)(n+a)}{R^2} - 2\hat{\beta}\frac{(n+a)}{R}$$

Now solving $\frac{\partial R(\hat{\beta},\beta)}{\partial \hat{\beta}} = 0$, we get

$$\implies 2\hat{\beta} - 2\frac{(n+a)}{R} = 0$$

$$\implies \hat{\beta}_{SELF} = \frac{(n+a)}{R} \tag{19}$$

• Bayes estimator under entropy loss function

The entropy loss function is given by

$$L(\hat{\beta}, \beta) = b[\Delta - \ln(\Delta) - 1]; b > 0 \tag{20}$$

Assuming $b = 1$,$\Delta = \frac{\hat{\beta}}{\beta}$, we have

$$L(\hat{\beta}, \beta) = \left[\left(\frac{\hat{\beta}}{\beta}\right) - \ln\left(\frac{\hat{\beta}}{\beta}\right) - 1\right] \tag{21}$$

By using Entropy loss function as given in (21), the risk function is given by

$$R(\hat{\beta}, \beta) = E[L(\hat{\beta}, \beta)] = \int_0^\infty \left[\left(\frac{\hat{\beta}}{\beta}\right) - \ln\left(\frac{\hat{\beta}}{\beta}\right) - 1\right].\pi_1(\beta|x)d\beta$$

$$= \int_0^\infty \left[\left(\frac{\hat{\beta}}{\beta}\right) - \ln\left(\frac{\hat{\beta}}{\beta}\right) - 1\right].\frac{\beta^{(n+a)-1}\exp(-\beta R)R^{n+a}}{\Gamma(n+a)}d\beta$$

$$= \hat{\beta}.\frac{R}{n+a-1} - \ln(\hat{\beta}) + \frac{\psi(n+a)}{\Gamma n+a} - 1$$

Now solving $\frac{\partial R(\hat{\beta},\beta)}{\partial \hat{\beta}} = 0$, we get

$$\implies \frac{R}{n+a-1} - \frac{1}{\hat{\beta}} = 0$$

$$\implies \hat{\beta}_{ELF} = \frac{n+a-1}{R} \tag{22}$$

4.1.2. Bayesian estimation by using uniform prior under various loss functions

• Bayes estimator under LINEX loss function

The LINEX loss function is given by

$$L(\hat{\beta},\beta) = \exp(q_1(\hat{\beta}-\beta)) - h(\hat{\beta}-\beta) - 1, q_1, h \neq 0 \tag{23}$$

By using LINEX loss function as given in (23), the risk function is given by

$$R(\hat{\beta},\beta) = E[L(\hat{\beta},\beta)] = \int_0^\infty \exp(q_1(\hat{\beta}-\beta) - h(\hat{\beta}-\beta) - 1).\pi_2(\beta|x)d\beta$$

$$= \int_0^\infty \left[\exp(q_1\hat{\beta}).\exp(-q_1\beta) - h\hat{\beta} + h\beta - 1\right].\frac{\beta^n \exp(-\beta T)^{n+1}}{\Gamma(n+1)}d\beta$$

$$= \exp(q_1\hat{\beta})[\left(\frac{T}{(T+q_1)}\right)]^{n+1} - h\frac{(n+2)}{T} - (h\hat{\beta}+1)$$

Now solving $\frac{\partial R(\hat{\beta},\beta)}{\partial\hat{\beta}} = 0$, we get

$$\exp(q_1\hat{\beta}) = \frac{h}{q_1}\left(\frac{T+q_1}{T}\right)^{n+1}$$

Taking log on both sides, we obtain the Bayes estimator as

$$\hat{\beta}_{LINEX} = \frac{1}{q_1}\left[ln\left(\frac{h}{q_1}\right) + (n+1)log\left(\frac{T+q_1}{T}\right)\right] \tag{24}$$

• Bayes estimator under squared error loss function

The squared error loss function is given by

$$L(\hat{\beta},\beta) = (\hat{\beta}-\beta)^2 \tag{25}$$

By using Squared error loss function as given in (25), the risk function is given by

$$R(\hat{\beta},\beta) = E[L(\hat{\beta},\beta)] = \int_0^\infty \left[(\hat{\beta}-\beta)^2\right].\pi_2(\beta|x)d\beta$$

$$= \int_0^\infty \left[(\hat{\beta}-\beta)^2\right].\frac{\beta^n \exp(-\beta T)^{n+1}}{\Gamma(n+1)}d\beta$$

$$= \hat{\beta}^2 + \frac{(n+2)(n+1)}{T^2} - 2\hat{\beta}\frac{(n+1)}{T}$$

Now solving $\frac{\partial R(\hat{\beta},\beta)}{\partial\hat{\beta}} = 0$, we get

$$\implies 2\hat{\beta} - 2\frac{(n+1)}{T} = 0$$

$$\implies \hat{\beta}_{SELF} = \frac{(n+1)}{T} \tag{26}$$

• Bayes estimator under entropy loss function

The entropy loss function is given by

$$L(\hat{\beta}, \beta) = b[\Delta - \ln(\Delta) - 1]; b > 0 \tag{27}$$

Assuming $b = 1$,$\Delta = \frac{\hat{\alpha}}{\alpha}$, we have

$$L(\hat{\beta}, \beta) = \left[\left(\frac{\hat{\alpha}}{\alpha}\right) - \ln\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right] \tag{28}$$

By using Entropy loss function as given in (28), the risk function is given by

$$R(\hat{\beta}, \beta) = E[L(\hat{\beta}, \beta)] = \int_0^\infty \left[\left(\frac{\hat{\beta}}{\beta}\right) - \ln\left(\frac{\hat{\beta}}{\beta}\right) - 1\right].\pi_2(\beta|x)d\beta$$

$$= \int_0^\infty \left[\left(\frac{\hat{\beta}}{\beta}\right) - \ln\left(\frac{\hat{\beta}}{\beta}\right) - 1\right].\frac{\beta^n \exp\left(-\beta T\right)^{n+1}}{\Gamma(n+1)}d\beta$$

$$= \hat{\beta}.\frac{T}{n} - \ln(\hat{\beta}) + \frac{\psi(n+1)}{\Gamma n + 1} - 1$$

Now solving $\frac{\partial R(\hat{\beta}, \beta)}{\partial \hat{\beta}} = 0$, we get

$$\implies \frac{T}{n} - \frac{1}{\hat{\beta}} = 0$$

$$\implies \hat{\beta}_{ELF} = \frac{n}{T} \tag{29}$$

4.2. Case II: When β is known and α is unknown

4.2.1. Bayesian estimation by using gamma prior under different loss functions

• Bayes estimator under LINEX loss function

The bayes estimator of α under LINEX loss function is given by

$$\hat{\alpha}_{LINEX} = -\frac{1}{h} \ln E[e^{-h\alpha}|x] \tag{30}$$

where,

$$E[e^{-h\alpha}|x] = \frac{\int_0^\infty \alpha^{n+a-1} e^{-\alpha(b+h)} \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} d\alpha}{\int_0^\infty \alpha^{n+a-1} e^{-b\alpha} \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} d\alpha}$$

• **Bayes estimator under squared error loss function**

The bayes estimator of α under SELF is given by

$$\hat{\alpha}_{SELF} = E[\alpha|x] \tag{31}$$

where,

$$E[\alpha|x] = \frac{\int_0^\infty \alpha^{n+a} e^{-b\alpha} \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} d\alpha}{\int_0^\infty \alpha^{n+a-1} e^{-b\alpha} \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} d\alpha}$$

• **Bayes estimator under Entropy loss function**

The bayes estimator of α under ELF is given by

$$\hat{\alpha}_{ELF} = (E[\alpha^{-1}|x])^{-1} \tag{32}$$

where,

$$E[\alpha^{-1}|x] = \frac{\int_0^\infty \alpha^{n+a-1} e^{-b\alpha+\alpha^{-1}} \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} d\alpha}{\int_0^\infty \alpha^{n+a-1} e^{-b\alpha} \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} d\alpha}$$

4.2.2. Bayesian estimation by using Uniform prior under different loss functions

• **Bayes estimator under LINEX loss function**

The bayes estimator of α under LINEX loss function is given by

$$\hat{\alpha}_{LINEX} = -\frac{1}{h} \ln E[e^{-h\alpha}|x] \tag{33}$$

where,

$$E[e^{-h\alpha}|x] = \frac{\int_0^\infty \alpha^n e^{-h\alpha} \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} a\alpha}{\int_0^\infty \alpha^n \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} d\alpha}$$

• **Bayes estimator under squared error loss function**

The bayes estimator of α under SELF is given by

$$\hat{\alpha}_{SELF} = E[\alpha|x] \tag{34}$$

where,

$$E[\alpha|x] = \frac{\int_0^\infty \alpha^{n+1} \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} a\alpha}{\int_0^\infty \alpha^n \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} d\alpha}$$

• **Bayes estimator under entropy loss function**

The bayes estimator of α under ELF is given by

$$\hat{\alpha}_{ELF} = (E[\alpha^{-1}|x])^{-1} \tag{35}$$

where,

$$E[\alpha^{-1}|x] = \frac{\int_0^\infty \alpha^{n-1} \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} a\alpha}{\int_0^\infty \alpha^n \prod_{i=1}^n (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta-1} d\alpha}$$

4.3. Case III: When α and β both are unknown

In previous section, we obtained the mathematical expression for the Bayes estimates of the parameters. We notice that these estimators are in the form of ratio of two integrals. Thus, Lindley's approximation method is a good alternative to solve such types of problems see Lindley (1980). Therefore, we briefly discuss about this approximation technique and apply it to evaluate the Bayesian estimates by considering the function I(x), defined as follows;

$$I(x) = E[\alpha, \beta|x] = \frac{\int u(\alpha, \beta) e^{L(\alpha,\beta)+G(\alpha,\beta)} d(\alpha, \beta)}{\int e^{L(\alpha,\beta)+G(\alpha,\beta)} d(\alpha, \beta)} \tag{36}$$

where,

$u(\alpha, \beta)$ is the function of α and β only; $L(\alpha, \beta)$ is the log likelihood function; $G(\alpha, \beta)$ is the log of joint prior density.

According to Lindley (1980), if ML estimates of the parameters are available and n is sufficiently large then the above ratio of the integral can be approximated as:

$$I(x) = u(\hat{\alpha}, \hat{\beta}) + \tfrac{1}{2}[(\hat{u}_{\beta\beta} + 2\hat{u}_{\beta}\hat{p}_{\beta})\hat{\sigma}_{\beta\beta} + (\hat{u}_{\alpha\beta} + 2\hat{u}_{\alpha}\hat{p}_{\beta})\hat{\sigma}_{\alpha\beta} + (\hat{u}_{\beta\alpha} + 2\hat{u}_{\beta}\hat{p}_{\alpha})\hat{\sigma}_{\beta\alpha} + (\hat{u}_{\alpha\alpha} + 2\hat{u}_{\alpha}\hat{p}_{\alpha})\hat{\sigma}_{\alpha\alpha}] + \tfrac{1}{2}[(\hat{u}_{\beta}\hat{\sigma}_{\beta\beta} + \hat{u}_{\alpha}\hat{\sigma}_{\beta\alpha})(\hat{L}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta} + \hat{L}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha} + \hat{L}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta} + \hat{L}_{\alpha\alpha\beta}\hat{\sigma}_{\alpha\alpha}) + (\hat{u}_{\beta}\hat{\sigma}_{\alpha\beta} + \hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha}) \times (\hat{L}_{\alpha\beta\beta}\hat{\sigma}_{\beta\beta} + \hat{L}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha} + \hat{L}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta} + \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})]$$

where, $\hat{\alpha}$ and $\hat{\beta}$ are the MLE of α and β respectively. The expressions for the MLE of the parameters of Inverted Kumaraswamy distribution have been derived by Al-Fattah et. al. (2017) and Sana et.al. (2023)

$$\hat{u}_{\alpha} = \frac{\partial u(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}}, \hat{u}_{\beta} = \frac{\partial u(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}}, \hat{u}_{\alpha\beta} = \frac{\partial^2 u(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}\partial \hat{\beta}}, \hat{u}_{\beta\alpha} = \frac{\partial^2 u(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}\partial \hat{\alpha}},$$

$$\hat{u}_{\alpha\alpha} = \frac{\partial^2 u(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}^2}, \hat{u}_{\beta\beta} = \frac{\partial^2 u(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}^2}, \hat{p}_{\alpha} = \frac{\partial G(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}}, \hat{p}_{\beta} = \frac{\partial G(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}},$$

$$\hat{L}_{\alpha\alpha} = \frac{\partial^2 L(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}^2}, \hat{L}_{\beta\beta} = \frac{\partial^2 L(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}^2}, \hat{L}_{\alpha\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}^3}, \hat{L}_{\alpha\alpha\beta} = \frac{\partial^3 L(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}\partial \hat{\alpha}\partial \hat{\beta}},$$

$$\hat{L}_{\beta\beta\alpha} = \frac{\partial^3 L(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}\partial \hat{\beta}\partial \hat{\alpha}}, \hat{L}_{\beta\alpha\beta} = \frac{\partial^3 L(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}\partial \hat{\alpha}\partial \hat{\beta}}, \hat{L}_{\alpha\beta\beta} = \frac{\partial^3 L(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}\partial \hat{\beta}\partial \hat{\beta}}, \hat{L}_{\beta\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}\partial \hat{\alpha}\partial \hat{\alpha}}$$

4.3.1. Bayesian estimation by using gamma prior under different loss functions

- **Bayes estimator under squared error loss function**

After substitution, the equation (14) reduces like Lindleys integral, therefore, for the

Bayes estimates of the parameter α under squared error loss function are,

$$u(\alpha,\beta)=\alpha$$
$$L(\alpha,\beta)=n\ln\alpha+n\ln\beta-(\alpha+1)\sum\ln(1+x_i)+(\beta-1)\sum\ln(1-(1+x_i)^{-\alpha})$$
$$G(\alpha,\beta)=(a_1-1)\ln\alpha+(a_2-1)\ln\beta-b_1\alpha-b_2\beta$$

It may be verified that,

$$u_\alpha=1, u_{\alpha\alpha}=u_{\alpha\beta}=u_{\beta\alpha}=u_{\beta\beta}=0$$
$$p_\alpha=\frac{a_1-1}{\alpha}-b_1, p_\beta=\frac{a_2-1}{\beta}-b_2$$
$$L_\alpha=\frac{n}{\alpha}-\sum\ln(1+x_i)+(\beta-1)\sum\frac{(1+x_i)^{-\alpha}}{1-(1+x_i)^{-\alpha}}\ln(1+x_i)$$
$$L_{\alpha\alpha}=\frac{-n}{\alpha^2}-(\beta-1)\sum\frac{(1+x_i)^{-\alpha}(\ln(1+x_i))^2}{[1-(1+x_i)^{-\alpha}]^2}$$
$$L_{\alpha\alpha\alpha}=\frac{2n}{\alpha^3}-(\beta-1)\sum\frac{\left((1+x_i)^{3\alpha}-(1+x_i)^{-\alpha}\right)}{[1-(1+x_i)^{-\alpha}]^4}(\ln(1+x_i))^3$$
$$L_{\alpha\beta}=\sum\frac{(1+x_i)^{-\alpha}\ln(1+x_i)}{[1-(1+x_i)^{-\alpha}]}=L_{\beta\alpha}$$
$$L_{\alpha\alpha\beta}=-\sum\frac{(1+x_i)^{-\alpha}(\ln(1+x_i))^2}{[1-(1+x_i)^{-\alpha}]^2}=L_{\beta\alpha\alpha}=L_{\alpha\beta\alpha}$$
$$L_\beta=\frac{n}{\beta}+\sum\ln(1-(1+x_i)^{-\alpha})$$
$$L_{\beta\beta}=\frac{-n}{\beta^2}$$
$$L_{\beta\beta\beta}=\frac{2n}{\beta^3}$$
$$L_{\beta\beta\alpha}=L_{\beta\alpha\beta}=L_{\alpha\beta\beta}=0$$

If α and β are orthogonal then $\sigma_{ij}=0$ for $i\neq j$ and $\sigma_{ij}=-\frac{1}{L_{ij}}$ for $i=j$.

After evaluation of all U-terms, L-terms, and p-terms at the point $(\hat{\alpha},\hat{\beta})$ and using the above expression, the approximate Bayes estimator of α under SELF is,

$$\hat{\alpha}^L_{SELF}=\hat{\alpha}+\hat{u}_\alpha\hat{p}_\alpha\hat{\sigma}_{\alpha\alpha}+0.5(\hat{u}_\alpha\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\beta\beta}+\hat{u}_\alpha\hat{\sigma}^2_{\alpha\alpha}\hat{L}_{\alpha\beta\beta}) \tag{37}$$

and similarly the Bayes estimate fir β under SELF is, $u_\beta=1, u_{\alpha\alpha}=u_{\alpha\beta}=u_{\beta\alpha}=u_{\beta\beta}=0$ and remaining L-terms and p-terms will be same as above. Thus we have

$$\hat{\beta}^L_{SELF}=\hat{\beta}+\hat{u}_\beta\hat{p}_\beta\hat{\sigma}_{\beta\beta}+0.5(\hat{u}_\beta\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\alpha\beta}+\hat{u}_\beta\hat{\sigma}^2_{\beta\beta}\hat{L}_{\beta\beta\beta}) \tag{38}$$

• Bayes estimator under LINEX loss function

The approximate Bayes estimator of α under LINEX is evaluated by taking $u(\alpha,\beta) = e^{-h\alpha}, h > 0,\ u_\alpha = -he^{-h\alpha}, u_{\alpha\alpha} = h^2 e^{-h\alpha}, u_{\alpha\beta} = u_{\beta\alpha} = u_{\beta\beta} = 0$ and remaining L-terms and p-terms will be same as above. Thus we have

$$\hat{\alpha}^L_{LINEX} = \hat{\alpha} + \hat{u}_\alpha \hat{p}_\alpha \hat{\sigma}_{\alpha\alpha} + 0.5(\hat{u}_\alpha \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\beta\beta} + \hat{u}_\alpha \hat{\sigma}^2_{\alpha\alpha}\hat{L}_{\alpha\beta\beta}) \tag{39}$$

and similarly the Bayes estimate fir β under LINEX is evaluated by taking $u(\alpha,\beta) = e^{-h\beta}, h > 0,\ u_\beta = -he^{-h\beta}, u_{\beta\beta} = h^2 e^{-h\beta}, u_{\alpha\beta} = u_{\beta\alpha} = u_{\alpha\alpha} = 0$ and remaining L-terms and p-terms will be same as above. Thus we have

$$\hat{\beta}^L_{LINEX} = \hat{\beta} + \hat{u}_\beta \hat{p}_\beta \hat{\sigma}_{\beta\beta} + 0.5(\hat{u}_\beta \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\alpha\beta} + \hat{u}_\beta \hat{\sigma}^2_{\beta\beta}\hat{L}_{\beta\beta\beta}) \tag{40}$$

• Bayes estimator under entropy loss function

The approximate Bayes estimator of α under ELF is evaluated by taking $u(\alpha,\beta) = e^{\alpha^{-1}},\ u_\alpha = -\frac{e^{\alpha^{-1}}}{\alpha^2}, u_{\alpha\alpha} = \frac{e^{\alpha^{-1}}}{\alpha^3}\left[\frac{1}{\alpha} + 2\right], u_{\alpha\beta} = u_{\beta\alpha} = u_{\beta\beta} = 0$ and remaining L-terms and p-terms will be same as above. Thus we have

$$\hat{\alpha}^L_{ELF} = \hat{\alpha} + \hat{u}_\alpha \hat{p}_\alpha \hat{\sigma}_{\alpha\alpha} + 0.5(\hat{u}_\alpha \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\beta\beta} + \hat{u}_\alpha \hat{\sigma}^2_{\alpha\alpha}\hat{L}_{\alpha\beta\beta}) \tag{41}$$

and similarly the Bayes estimate for β under LINEX is evaluated by taking $u(\alpha,\beta) = e^{\beta^{-1}}$, $u_\beta = -\frac{e^{\beta^{-1}}}{\beta^2}, u_{\beta\beta} = \frac{e^{\beta^{-1}}}{\beta^3}\left[\frac{1}{\beta} + 2\right], u_{\alpha\beta} = u_{\beta\alpha} = u_{\alpha\alpha} = 0$ and remaining L-terms and p-terms will be same as above. Thus we have

$$\hat{\beta}^L_{ELF} = \hat{\beta} + \hat{u}_\beta \hat{p}_\beta \hat{\sigma}_{\beta\beta} + 0.5(\hat{u}_\beta \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\alpha\beta} + \hat{u}_\beta \hat{\sigma}^2_{\beta\beta}\hat{L}_{\beta\beta\beta}) \tag{42}$$

4.3.2. Bayesian estimation by using uniform prior under different loss functions

• Bayes estimator under squared error loss function

After substitution, the equation (15) reduces like Lindleys integral, therefore, for the Bayes estimates of the parameter α under squared error loss function are,

$$u(\alpha,\beta) = \alpha$$
$$L(\alpha,\beta) = n\ln\alpha + n\ln\beta - (\alpha+1)\sum\ln(1+x_i) + (\beta-1)\sum\ln(1-(1+x_i)^{-\alpha})$$
$$G(\alpha,\beta) = \ln k_1 + \ln k_2$$

It may be verified that,

$$u_\alpha = 1, u_{\alpha\alpha} = u_{\alpha\beta} = u_{\beta\alpha} = u_{\beta\beta} = 0$$
$$p_\alpha = p_\beta = 0$$
$$L_\alpha = \frac{n}{\alpha} - \sum \ln(1+x_i) + (\beta-1)\sum \frac{(1+x_i)^{-\alpha}}{1-(1+x_i)^{-\alpha}} \ln(1+x_i)$$
$$L_{\alpha\alpha} = \frac{-n}{\alpha^2} - (\beta-1)\sum \frac{(1+x_i)^{-\alpha}(\ln(1+x_i))^2}{[1-(1+x_i)^{-\alpha}]^2}$$
$$L_{\alpha\alpha\alpha} = \frac{2n}{\alpha^3} - (\beta-1)\sum \frac{\left((1+x_i)^{3\alpha} - (1+x_i)^{-\alpha}\right)}{[1-(1+x_i)^{-\alpha}]^4}(\ln(1+x_i))^3$$
$$L_{\alpha\beta} = \sum \frac{(1+x_i)^{-\alpha}\ln(1+x_i)}{[1-(1+x_i)^{-\alpha}]} = L_{\beta\alpha}$$
$$L_{\alpha\alpha\beta} = -\sum \frac{(1+x_i)^{-\alpha}(\ln(1+x_i))^2}{[1-(1+x_i)^{-\alpha}]^2} = L_{\beta\alpha\alpha} = L_{\alpha\beta\alpha}$$
$$L_\beta = \frac{n}{\beta} + \sum \ln(1-(1+x_i)^{-\alpha})$$
$$L_{\beta\beta} = \frac{-n}{\beta^2}$$
$$L_{\beta\beta\beta} = \frac{2n}{\beta^3}$$
$$L_{\beta\beta\alpha} = L_{\beta\alpha\beta} = L_{\alpha\beta\beta} = 0$$

If α and β are orthogonal then $\sigma_{ij} = 0$ for $i \neq j$ and $\sigma_{ij} = -\frac{1}{L_{ij}}$ for $i = j$.

After evaluation of all U-terms, L-terms, and p-terms at the point $(\hat{\alpha}, \hat{\beta})$ and using the above expression, the approximate Bayes estimator of α under SELF is,

$$\hat{\alpha}^L_{SELF} = \hat{\alpha} + \hat{u}_\alpha \hat{p}_\alpha \hat{\sigma}_{\alpha\alpha} + 0.5(\hat{u}_\alpha \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\beta\beta} + \hat{u}_\alpha \hat{\sigma}^2_{\alpha\alpha}\hat{L}_{\alpha\beta\beta}) \tag{43}$$

and similarly the Bayes estimate for β under SELF is, $u_\beta = 1, u_{\alpha\alpha} = u_{\alpha\beta} = u_{\beta\alpha} = u_{\beta\beta} = 0$ and remaining L-terms and p-terms will be same as above. Thus we have

$$\hat{\beta}^L_{SELF} = \hat{\beta} + \hat{u}_\beta \hat{p}_\beta \hat{\sigma}_{\beta\beta} + 0.5(\hat{u}_\beta \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\alpha\beta} + \hat{u}_\beta \hat{\sigma}^2_{\beta\beta}\hat{L}_{\beta\beta\beta}) \tag{44}$$

- **Bayes estimator under LINEX loss function**

The approximate Bayes estimator of α under LINEX is evaluated by taking $u(\alpha,\beta) = e^{-h\alpha}, h > 0,\ u_\alpha = -he^{-h\alpha}, u_{\alpha\alpha} = h^2 e^{-h\alpha}, u_{\alpha\beta} = u_{\beta\alpha} = u_{\beta\beta} = 0$ and remaining L-terms and p-terms will be same as above. Thus we have

$$\hat{\alpha}^L_{LINEX} = \hat{\alpha} + \hat{u}_\alpha \hat{p}_\alpha \hat{\sigma}_{\alpha\alpha} + 0.5(\hat{u}_\alpha \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\beta\beta} + \hat{u}_\alpha \hat{\sigma}^2_{\alpha\alpha}\hat{L}_{\alpha\beta\beta}) \tag{45}$$

and similarly the Bayes estimate fir β under LINEX is evaluated by taking $u(\alpha,\beta) = e^{-h\beta}, h > 0,\ u_\beta = -he^{-h\beta}, u_{\beta\beta} = h^2 e^{-h\beta}, u_{\alpha\beta} = u_{\beta\alpha} = u_{\alpha\alpha} = 0$ and remaining L-terms and p-terms will be same as above. Thus we have

$$\hat{\beta}^L_{LINEX} = \hat{\beta} + \hat{u}_\beta \hat{p}_\beta \hat{\sigma}_{\beta\beta} + 0.5(\hat{u}_\beta \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\alpha\beta} + \hat{u}_\beta \hat{\sigma}^2_{\beta\beta}\hat{L}_{\beta\beta\beta}) \tag{46}$$

• Bayes estimator under entropy loss function

The approximate Bayes estimator of α under ELF is evaluated by taking $u(\alpha,\beta)=e^{\alpha^{-1}}$, $u_\alpha=-\frac{e^{\alpha^{-1}}}{\alpha^2}, u_{\alpha\alpha}=\frac{e^{\alpha^{-1}}}{\alpha^3}\Big[\frac{1}{\alpha}+2\Big], u_{\alpha\beta}=u_{\beta\alpha}=u_{\beta\beta}=0$ and remaining L-terms and p-terms will be same as above. Thus we have

$$\hat{\alpha}^L_{ELF}=\hat{\alpha}+\hat{u}_\alpha\hat{p}_\alpha\hat{\sigma}_{\alpha\alpha}+0.5(\hat{u}_\alpha\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\beta\beta}+\hat{u}_\alpha\hat{\sigma}^2_{\alpha\alpha}\hat{L}_{\alpha\beta\beta}) \quad (47)$$

and similarly the Bayes estimate for β under LINEX is evaluated by taking $u(\alpha,\beta)=e^{\beta^{-1}}$, $u_\beta=-\frac{e^{\beta^{-1}}}{\beta^2}, u_{\beta\beta}=\frac{e^{\beta^{-1}}}{\beta^3}\Big[\frac{1}{\beta}+2\Big], u_{\alpha\beta}=u_{\beta\alpha}=u_{\alpha\alpha}=0$ and remaining L-terms and p-terms will be same as above. Thus we have

$$\hat{\beta}^L_{ELF}=\hat{\beta}+\hat{u}_\beta\hat{p}_\beta\hat{\sigma}_{\beta\beta}+0.5(\hat{u}_\beta\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\alpha\alpha\beta}+\hat{u}_\beta\hat{\sigma}^2_{\beta\beta}\hat{L}_{\beta\beta\beta}) \quad (48)$$

5. Simulation study

Next, a simulation study was conducted to investigate the performance of Bayes estimators of the unknown parameter for case I, i,e, when α is known and β is unknown under two priors discussed in this paper. The study was executed for different sample sizes specifically for n= 20,50,70,100,150,200. The observations were generated from Inverted Kumaraswamy distribution using the quantile function. For the expression of quantile function refer Al-Fattah *et al.* (2017). The Bayes estimates were obtained using LINEX, SELF and Entropy loss function. For Gamma prior the values of hyperparameters considered are (a=0.01, b=0.01). In our study 6000 samples were generated. The Bayes estimates were compared in terms of relative mean square errors.

6. Conclusion

In this paper, we estimated the unknown parameter of IKum distribution considering three different cases: (i)when α is known and β is unknown, (ii)when α is unknown and β is known, and (iii)when α and β both are unknown, using two prior distributions under three different loss functions, though simulations were carried out for caseI only. Relative MSE were also derived using the following formula:

$$MSE=\frac{\sum_{i=1}^{N}(Estimator-Truevalue)^2}{N}$$

$$RelativeMSE=\frac{MSE}{Truevalue}$$

where $N=6000$.

Concluding Remarks

∗ From Table 1, the Relative MSE of estimator under Entropy loss function is minimum for $\alpha=0.4$, $\beta=0.8$ and the Relative MSE of estimator under Squared error loss funtion is minimum for $\alpha=0.5$, $\beta=1$.

∗ From Table 2 also, estimator under Entropy loss function stands most efficient for $\alpha=0.4$, $\beta=0.8$, whereas for $\alpha=0.5$, $\beta=1$, estimator under Squared error loss function holds minimum relative error.

$*$ From the graphs it is observed that Relative MSE decreases as sample size increases.

As a future research work, this paper can be extended in many ways. The simulation results for the other two cases(when α is unknown and β is known, and when α and β both are unknown) can also be computed and the obtained results can be represented in the form of graphs. The obtained estimators can also be applied to real life data for illustrative purposes. Additionally, the estimation of entropy in this setup may also be considered.

Acknowledgements

I am indeed grateful to the Editors for their guidance and counsel. I am very grateful to the reviewer for valuable comments and suggestions of generously listing many useful references.

Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

References

Afaq, S. P., Ahmad, and Ahmed, A. (2015). Preference of priors for the exponentiated exponential distribution under different loss functions. *International Journal of Modern Mathematical Sciences*, **13**, 307–321.

Ahmad, S. P. and Fatima K. (2017). Preference of priors for the generalized inverse Rayleigh distribution under different loss functions. *Journal of Statistics Applications and Probability Letters*, **4**, 73–90.

Aldahlan, M. A., Bakoban, R. A., and Alzahrani, L. S. (2022). On estimating the parameters of the Beta inverted exponential distribution under type-II censored samples. *Mathematics*, **10**, 506–542.

Al-Fattah, A. A., E-Helbawy, A. A., and AL-Dayian G. R. (2017). Inverted Kumaraswamy distribution: Properties and estimation. *Pakistan Journal of Statistics*, **33**, 37–61.

Dey, S. (2012). Bayesian estimation of the parameter and reliability function of an inverse Rayleigh distribution. *Malaysian Journal of Mathematical Sciences*, **6**, 113–124.

El-Din, M. M. and Abu-Moussa, M. (2018). On estimation and prediction for the inverted Kumaraswamy distribution based on general progressive censored samples. *Pakistan Journal of Statistics and Operation Research*, **14**, 717–736.

Fan, J. and Gui, W. (2022). Statistical inference of inverted exponentiated Rayleigh distribution under joint progressively type-II censoring. *Entropy*, **24**, 171.

Flaih, A., Elsalloukh, H., Mendi E., and Milanova, M. (2012). The exponentiated inverted Weibull distribution.*Applied Mathematics and Information Sciences*, **6**, 167–171.

Hasan, M. R. and Baizid, A. R. (2016). Bayesian estimation under different loss functions using Gamma prior for the case of exponential distribution. *Jouranl of Scientific Research*, **9**, 67–78.

Iqbal, Z., Tahir, M. M., Riaz, N., Ali, S. A., and Ahmad, M. (2017). Generalized inverted Kumaraswamy distribution: Properties and application. *Open Journal of Statistics*, **7**, 645-662.

Kazmi, A. M. S., Aslam, M., and Ali. S. (2012). Preference of prior for the class of life-time distributions under different loss functions. *Pakistan Journal of Statistics*, **28**, 467–487.

Kumaraswamy, P. (1980). A generalized probability density function for double bounded random processes. *Journal of Hydrology*, **46**, 79–88.

Prakash, G. (2012). Inverted exponential distribution under a Bayesian viewpoint. *Journal of Modern Applied Statistical Methods*, **11**, 190–202.

Reshi, J. A., Ahmed, A., and Ahmad, S. P. (2014). Bayesian analysis of scale parameter of the generalized inverse Rayleigh model using different loss functions. *International Journal of Modern Mathematical Sciences*, **10**, 151–162.

Rastogi, M. K. and Oguntunde, P. E. (2019). Classical and Bayes estimation of reliability characteristics of the Kumaraswamy-inverse exponential distribution. *International Journal of System Assurance Engineering and Management*, **10**, 190–200.

Sana, Faizan, M., and Khan, A. A. (2023). Bayesian estimation using Lindley's approximation of inverted Kumaraswamy distribution based on lower record values. *TWMS Journal of Applied and Engineering Mathematics*, **13**, 65–73.

Tiao, G. G. and Cuttman, I. (1965). The inverted Dirichlet distribution with applications, *Journal of the American Statistical Association*, **60**, 793–805.

Tummala, V. M. and Sathe, P. T. (1978). Minimum expected loss estimators of reliability and parameters of certain life time distributions. *IEEE Transactions on Reliability*, **27**, 283–285.

ANNEXURE

Table 1: Bayes estimate and Relative mean square error under Gamma prior when hyperparameters (a,b)=(0.01,0.01)

Case I: α known, β unknown							
n		$\alpha = 0.4$,$\beta = 0.8$			$\alpha = 0.5$,$\beta = 1$		
		LINEX	SELF	ELF	LINEX	SELF	ELF
20	Estimate	0.83308	0.84414	0.80328	1.03981	1.0529	0.99961
	RelMSE	0.04870	0.05210	0.04490	0.05910	0.06280	0.05390
50	Estimate	0.81030	0.81922	0.80116	1.01620	1.02001	1.00010
	RelMSE	0.01730	0.01750	0.01660	0.02080	0.02140	0.02060
70	Estimate	0.80934	0.81081	0.80090	1.0109	1.0125	0.99918
	RelMSE	0.01230	0.01260	0.01210	0.01510	0.01470	0.01460
100	Estimate	0.80651	0.80865	0.80010	1.0205	0.99102	1.00040
	RelMSE	0.00824	0.00855	0.00802	0.01100	0.01010	0.01020
150	Estimate	0.80458	0.80526	0.79951	1.0141	0.99454	1.00070
	RelMSE	0.00538	0.00550	0.00528	0.00715	0.00665	0.00696
200	Estimate	0.80325	0.80363	0.80090	1.0081	0.99723	0.99930
	RelMSE	0.00414	0.00417	0.00400	0.00530	0.00498	0.00506

Table 2: Bayes estimate and Relative mean square error under Uniform prior

Case I: α known, β unknown							
n		$\alpha = 0.4$,$\beta = 0.8$			$\alpha = 0.5$,$\beta = 1$		
		LINEX	SELF	ELF	LINEX	SELF	ELF
20	Estimate	0.87870	0.88261	0.84432	1.1579	1.00160	1.0559
	RelMSE	0.06090	0.06200	0.05420	0.10100	0.05380	0.06670
50	Estimate	0.83057	0.82968	0.81540	1.0619	0.99932	1.0178
	RelMSE	0.01880	0.01900	0.01780	0.02740	0.02050	0.02200
70	Estimate	0.82186	0.82366	0.81270	1.0430	0.99687	1.0147
	RelMSE	0.01300	0.01340	0.01250	0.01780	0.01500	0.01570
100	Estimate	0.81395	0.81584	0.80961	1.0315	1.00010	1.0089
	RelMSE	0.00858	0.00860	0.00811	0.00907	0.01190	0.01020
150	Estimate	0.80922	0.81043	0.80556	1.0205	1.00060	1.0073
	RelMSE	0.00570	0.00555	0.00544	0.00738	0.00668	0.00690
200	Estimate	0.80754	0.80732	0.80328	1.0154	0.99883	1.0057
	RelMSE	0.00418	0.00435	0.00416	0.00541	0.00484	0.00519

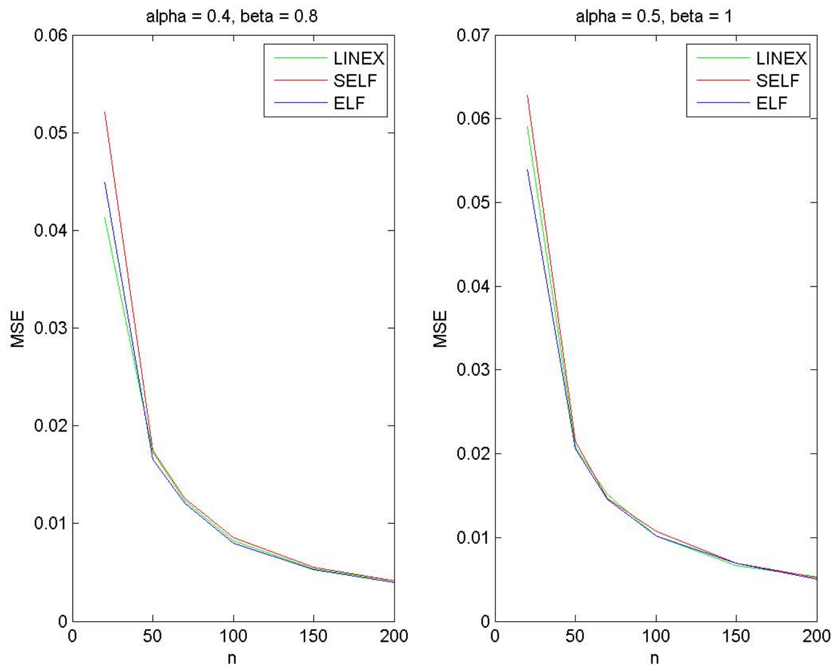

Figure 1: Relative Mean Square Error of β under Gamma prior

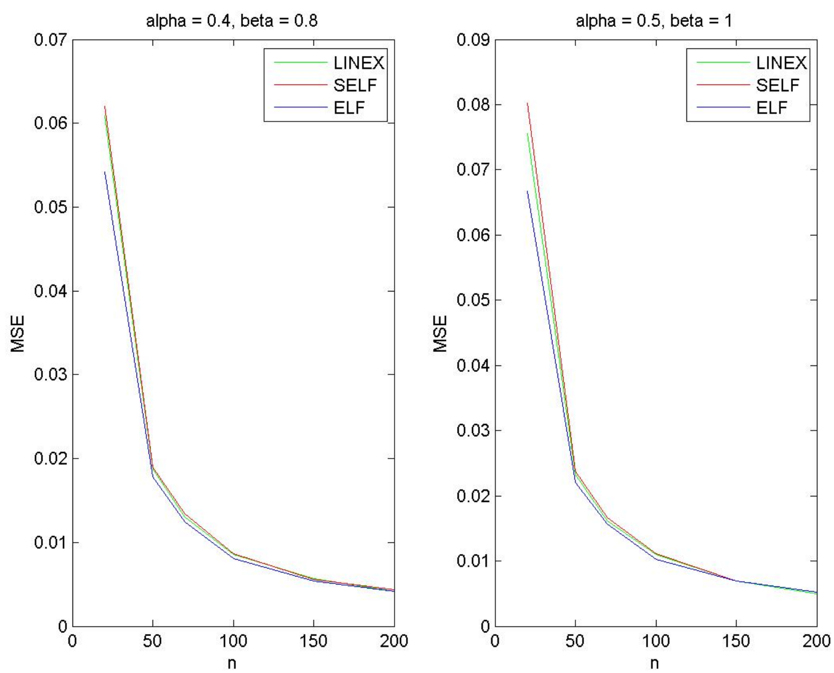

Figure 2: Relative Mean Square Error of β under Uniform prior