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On Uniform Truncated Poisson Distribution and its Applications

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Abstract

The Uniform Truncated Poisson distribution defined on the interval $[0, 1]$ is studied in detail and has shown that this distribution is derivable in three different ways. Analytical properties of this distribution are derived and estimation problems are addressed. Real data sets are modeled using this distribution. Generalization of the distribution on any finite interval is also considered and properties are studied.

Key words: Truncated uniform distribution; Truncated Poisson distribution; Estimation.

AMS Subject Classifications: 62E15, 60E99

1. Introduction

Theoretical probability distribution gives us a law according to which different values of the random variables are distributed with specified probabilities which can be expressed mathematically. Recent studies on probability distributions are mainly concerned with support either in the real line or positive real line. Distributions on finite intervals are less considered by the researchers. But we know that many of our real data sets are lying in finite intervals. Moreover many of the organisms in biology, experimental results in physics, chemistry, *etc.* show a uniform pattern in $[0, 1]$. Some recent distributions defined on $[0, 1]$ are available in the research papers, Altawil (2019) and Hassan *et al.* (2020). Rescaling a data into $[0, 1]$ is useful in machine learning and image processing. The most elegant and common method widely used in these fields are min-max scaling procedure. This is an alternative method to z-score normalization. By the min-max transformation discussed in this paper any random variable with support on a real line can be transformed into $[0, 1]$ and further analysis can be done. Also in neural networks $[0, 1]$ data is required for normalizing pixel intensities. As mentioned in Weigend and Gershenfeld (1993) and Yu *et al.* (2006) normalization has an important role in the data management. By this transformation all the features are kept same, but it results in smaller standard deviations of the observations, which minimizes the outlier effect. So an attempt is made to study a distribution with support on $[0, 1]$ which was mentioned in Hao and Godbole (2014). More recently a new distribution with support on $[0, 1]$ called unifed distribution has been introduced in Quijano Xacur (2019) which can be used as the response distribution for a generalized linear model. When the index parameter is unity this family gives the distribution we study in this

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paper. We further explore this distribution by introducing it in another way and bringing together the relevant properties and results concerning it. We used a compounding method for the derivation of this distribution. Derivations of new discrete and continuous distributions compounding two distributions have been discussed by several authors, see for instance uniform-geometric distribution in Akdogan *et al.* (2016), binomial-Poisson distribution in Hu *et al.* (2007), and Weibull-power series distribution in Morais and Barreto-Souza (2011). Similar distributions can also been seen in Adamidis and Loukas (1998), Kus (2007), Tahmasbi and Rezaei (2008), and Chahkandi and Ganjali (2009). We have some well-known distributions like beta distribution and power function distribution with support on $[0, 1]$. These distributions are found to have useful applications in several real life situations in reliability, time series, *etc.* So we have made a comparison of the distribution studied in this paper with these well known distributions. We could also use this new distribution in the modeling of time series data. So an advanced level model diagnosis in non-linear and volatile time series data using this distribution will be quite interesting in the near future.

This paper is organized as follows. In Section 2, uniform truncated Poisson distribution is introduced and its properties are studied. Transformations are considered and corresponding distributions are derived in Section 3. The estimation of the parameter is done in Section 4 and numerical illustrations are given therein. Asymptotic properties of the estimators are also delineated in the same section. A generalization of this new distribution with support on any finite interval is done in Section 5. Application to real data sets is given in Section 6 followed by a concluding Section.

2. Uniform Truncated Poisson Distribution

Distributions defined on [0,1] are not very common in literature and the most widely used distributions belonging to this category are power function distribution and beta distribution. Several applications of the distributions defined on [0,1] have been portrayed in the introduction part. The applications of such distributions in neural networks, pixel intensities, artificial intelligence, physics, engineering, time series *etc.* are the motivation for this present study. Also for the variates in [0,1] like percentages or fractions, we have only few studies on regression/time series models. In this context some of the notable works are Kieschnick and McCullough (2003), Jara *et al.* (2013), Ristic and Popovic (2000), Rocha and Cribari-Neto (2009) and Bayer *et al.* (2018). So the distribution studied in this paper may be applied in the advanced fields of the areas mentioned above even though we have illustrated some of the applications in the last section of this paper. Now we consider the distribution on [0,1] mentioned in Hao and Godbole (2014) and propose its construction in three different ways. These methods are described below.

Method 1: We consider a transformation of the truncated uniform distribution to form a new random variable defined on [0,1]. Let U be a random variable following truncated uniform distribution with probability density function (pdf),

$$g(u) = \frac{1}{e^{\theta} - 1}, \ \ 0 \le u \le e^{\theta} - 1.$$

Consider the transformation,

$$X = \frac{\log(1 + U)}{\theta}. \tag{1}$$

Then the probability density function of X is

$$f(x) = \frac{1}{e^{\theta}-1}\theta e^{\theta x}, \quad 0 \leq x \leq 1, \theta \neq 0. \tag{2}$$

It may be noted that when $\theta = 0$ the distribution is uniform in [0,1].

Method 2: The distribution specified in (2) can be derived as a solution of the first order differential equation as given below. We have used this method keeping in mind that the radioactive decay is associated with a differential equation and an exponential random variable is an example for it. So we are trying to construct a distribution with an initial value at time zero as a function of θ but the limit of the initial value function at time point zero is 1. This initial value function $\frac{\theta}{e^{\theta}-1}$ is monotone decreasing in θ. Let

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \theta y = \frac{\theta}{e^{\theta}-1} \tag{3}$$

be the first order differential equation and choose $y = F(x)$.
That is

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} - \theta F(x) = \frac{\theta}{e^{\theta}-1}.$$

Solving we get

$$F(x) = \frac{e^{\theta x}-1}{e^{\theta}-1}$$

$$\bar{F}(x) = 1 - F(x) = \frac{e^{\theta}-e^{\theta x}}{e^{\theta}-1}$$

and hence $f(x)$ is of the form (2). So this distribution is the solution of the first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \theta y = \frac{\theta}{e^{\theta}-1}.$$

Method 3: Random minimum or maximum of N independent and identically distributed (i.i.d) random variables are studied in Louzada *et al.* (2011), Kus (2007), Cancho *et al.* (2011) and several other papers. It may be noted that Hao and Godbole (2014) has introduced the uniform-Poisson model, deriving it as given below. They have applied the method mentioned above and considered only a few properties in that paper. So using the procedure used there, assuming the random variable N to be truncated Poisson with probability mass function

$$P(N=n) = \frac{e^{-\theta}\theta^{N}}{N!(1-e^{-\theta})}, \quad N = 1, 2, ...$$

and $X_1, X_2, ..., X_N$ i.i.d $U[0,1]$ with distribution function F(.), the distribution of $X = \min_{1\leq i\leq N}(X_i)$ is,

$$g(X=x) = f(x)\sum_{n=1}^{\infty} n\,(F(x))^{N-1}\,P(N=n)$$

which is exactly the same as (2). Hence we call this random variable X with pdf (2) as Uniform Truncated Poisson distribution (θ) denoted as UTPD(θ).

It is quite interesting to note that UTPD(θ) is derived in three different ways. Now we look at the properties of this new distribution.

2.1. Properties

1. The survival function is $\frac{e^{\theta}-e^{\theta x}}{e^{\theta}-1}$.

2. The hazard function, $h\left(x\right)=\frac{f(x)}{\bar{F}(x)}=\frac{\theta e^{\theta x}}{e^{\theta}-e^{\theta x}}=\frac{\theta}{e^{\theta(1-x)}-1}$.

   It can be seen that for all θ values, the distribution has an increasing failure rate (IFR).

3. The characteristic function, $\phi_X\left(t\right)=\frac{\theta}{\left(e^{\theta}-1\right)}\frac{1}{(\theta+it)}\left(e^{\theta+it}-1\right)$.

4. The r^{th} moment of UTPD is given by

$$\begin{aligned}E(X^r) \quad = \quad & \frac{e^{\theta}}{e^{\theta}-1}\left[1-\frac{r}{\theta}+\frac{r(r-1)}{\theta^2}-\frac{r(r-1)(r-2)}{\theta^3}\right.\\ & \left.+\frac{r(r-1)(r-2)(r-3)}{\theta^4}-...+(-1^r)\frac{r(r-1)(r-2)(r-3)...1}{\theta^r}\right]\\ & +\frac{1}{e^{\theta}-1}(-1)^{r+1}\frac{r(r-1)(r-2)(r-3)...1}{\theta^r}, \quad \text{for } r\text{=1, 2, ...}\end{aligned}$$

5. Mean= $\frac{e^{\theta}(\theta-1)+1}{\theta\left(e^{\theta}-1\right)}$.

6. Variance= $\frac{e^{\theta}}{e^{\theta}-1}\left(1-\frac{2}{\theta}+\frac{2}{\theta^2}\right)-\frac{2}{\theta^2\left(e^{\theta}-1\right)}-\left(\frac{e^{\theta}(\theta-1)+1}{\theta\left(e^{\theta}-1\right)}\right)^2$.
   For $\theta=1$, Mean=$\frac{1}{(e-1)}$, $Variance=\frac{e^2-3e+1}{(e-1)^2}$.

7. The p^{th} quantile is given by $x_p=\frac{1}{\theta}\log\left\{1+p\left(e^{\theta}-1\right)\right\}$, $0\leq p\leq 1$.

8. Entropy, a measure of the uncertainty associated with the random variable is given by $H\left(X\right)=\frac{-\theta}{\theta-1}\left\{\frac{\ln\theta e^{\theta}}{\theta}-\ln\theta\left(\frac{1}{\theta}\right)+\frac{e^{\theta}}{\theta}-\frac{1}{\theta^2}\left(e^{\theta}-1\right)-\ln\left(e^{\theta}-1\right)\right\}$.

9. Odds ratio : Odds ratios are often used in the medical literature.

   (a) The odds ratio of surviving beyond time, $\phi^{+}=\frac{\bar{F}(X)}{F(X)}=\frac{e^{\theta}-e^{\theta x}}{e^{\theta x}-1}$.

   (b) The odds ratio of failure by time, $\phi^{-}=\frac{F(X)}{\bar{F}(X)}=\frac{e^{\theta x}-1}{e^{\theta}-e^{\theta x}}$.

   The density function, distribution function, and the hazard function for different values of θ are plotted in Figures 1 to 3 respectively. From the density plots, it is clear that the positive value of the parameter θ confirms the left-skewed behavior and a negative value indicates the right-skewed behavior. So it is a distribution on $[0,1]$, which can be used for modeling left or right skewed data sets. When the value of θ is positive and increases the density function becomes more peaked but less left-skewed. But the behavior is just the opposite when θ is negative. Even though from Figure 3 it is clear that the distribution has IFR for different values of θ, the behavior of the hazard function doesn't vary much. The nature of this distribution is actually very similar to the power function distribution. It means a comparison with power function distribution will be quite interesting. For illustrating this, the density plots of UTPD and power function distribution are drawn together in Figure 4. As $\theta > 0$ and increases UTPD coincides with power function distribution. A comparison with beta distribution is also interesting since beta distribution is a flexible distribution with wide applications. But we know that the failure rate function of the beta (p, q) distribution is increasing only if $p \geq 1$, and the comparison will be meaningful only under this particular case. So we have not given much importance to this part in this study.

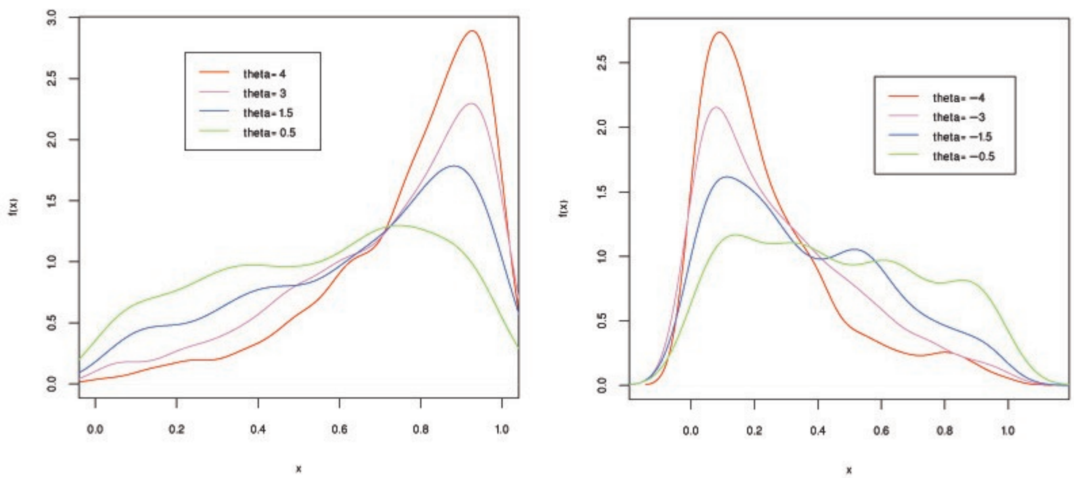

Figure 1: Density plots of UTPD for various values of θ

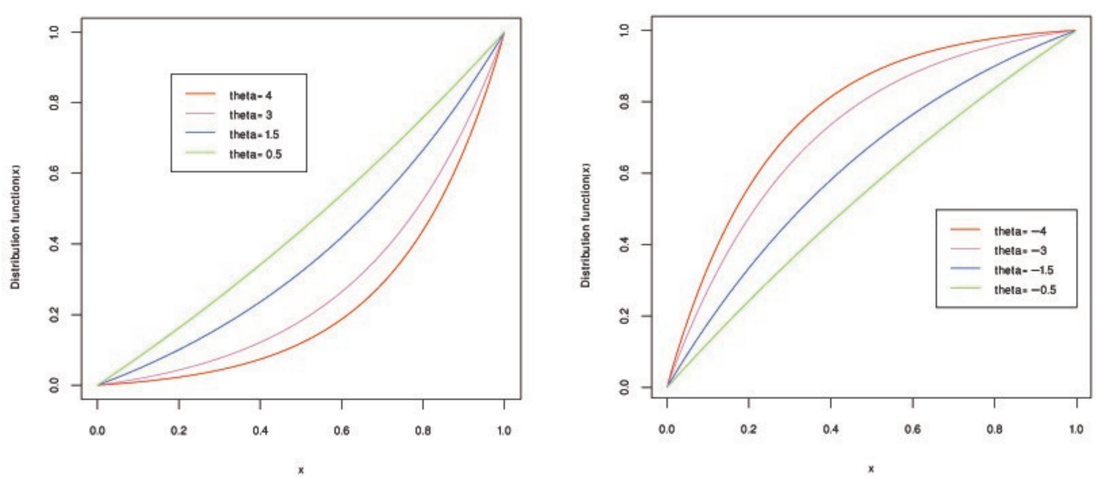

Figure 2: Distribution function of UTPD for various values of θ

10. Skewness and Kurtosis

Using the quantile function given in property 7, the first, second and third quantiles are $x_{0.25}$, $x_{0.50}$, $x_{0.75}$ respectively.

Bowley's measure of skewness,

$$S = \frac{x_{0.75} + x_{0.25} - 2x_{0.50}}{x_{0.75} - x_{0.25}}$$
$$= \frac{log[\frac{\frac{1}{16}((3e^\theta+1)(e^\theta+3))}{(\frac{1}{2}(e^\theta+1))^2}]}{log[\frac{3e^\theta+1}{e^\theta+3}]}.$$

The kurtosis is measured by the method introduced by Moors (1988). He derived this measure using octiles, where the octiles E_i are defined as,

$$P(X < E_i) \leq \frac{i}{8}$$

and

$$P(X > E_i) \leq 1 - \frac{i}{8}.$$

Using octiles the measure of kurtosis,

$$K = \frac{(E_7 - E_5) + (E_3 - E_1)}{E_6 - E_2}.$$

These two measures are given in Table 1 and the observations we made from the density plots regarding skewness and kurtosis are very well established numerically in this table. It is clear that the distribution is symmetric for θ, and the values of the kurtosis are the same for both the negative and positive values of the parameter.

Table 1: Skewness and Kurtosis

Parameter:θ	Skewness	Kurtosis
0.5	-0.0613	1.0114
2	-0.1953	1.1385
5	-0.2579	1.2925
8	-0.2616	1.3055
-0.5	0.0613	1.0114
-2	0.1953	1.1385
-5	0.2579	1.2925
-8	0.2616	1.3055

Remark 1: This distribution is useful in machine learning specifically for the normalization used for the data representation, further processing and accuracy. The usual transformation used for this purpose is

$$\frac{x_i - min(x_i)}{max(x_i) - min(x_i)}. \tag{4}$$

Later in our real data analysis part described in the last section of this paper, we explain the use of this distribution in such transformations.

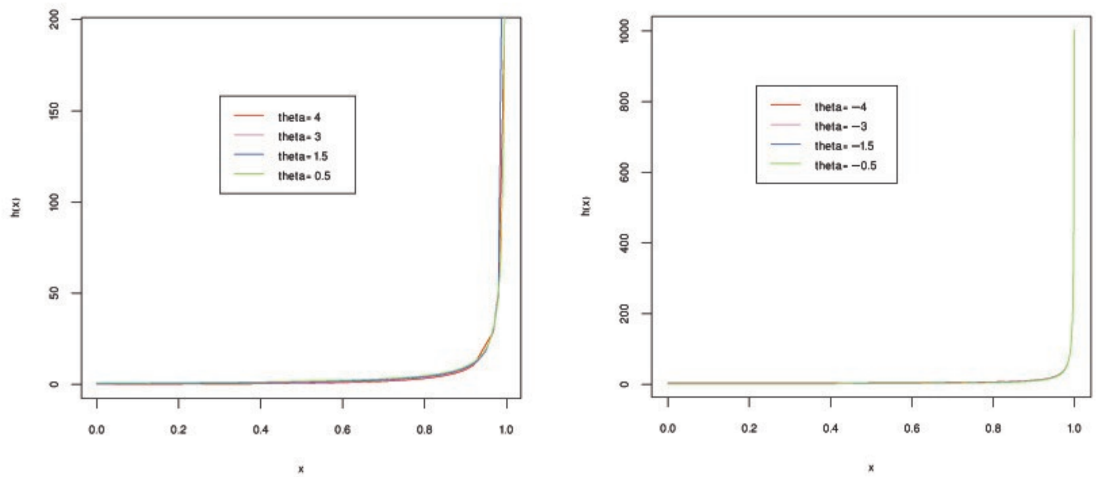

Figure 3: Hazard function of UTPD for various values of θ

2.2. Distribution of order statistics

Assume that $X_1, X_2, X_3, ..., X_n$ are independent random variables following UTPD with parameter θ. The pdf of $\min\left(X_1, X_2, X_3, ..., X_n\right)$ is given by

$$f_{X_{(1)}}\left(x\right) = \frac{n\theta e^{\theta x}}{\left(e^{\theta}-1\right)^{n}}\left(e^{\theta}-e^{\theta x}\right)^{n-1}$$

and the pdf of $\max\left(X_1, X_2, X_3..., X_n\right)$ is given by

$$f_{X_{(n)}}\left(x\right) = \frac{n\theta e^{\theta x}}{\left(e^{\theta}-1\right)^{n}}\left(e^{\theta x}-1\right)^{n-1}.$$

In the next section, we describe some transformed distributions, which seems very similar to some familiar distributions but with different domains.

3. Transformed Distributions

Here we consider some random variables generated through the transformations of (2) and derive their distributions.

Result 1: Considering the transformation

$$U = -\log X,$$

where X follows UTPD with density function given in (2), the pdf of U is

$$g(u) = \frac{\theta}{e^{\theta}-1}e^{-u}e^{\theta e^{-u}}, \qquad 0 \leq u < \infty, \tag{5}$$

which is the Weibull-Poisson distribution by Morais and Barreto-Souza (2011).

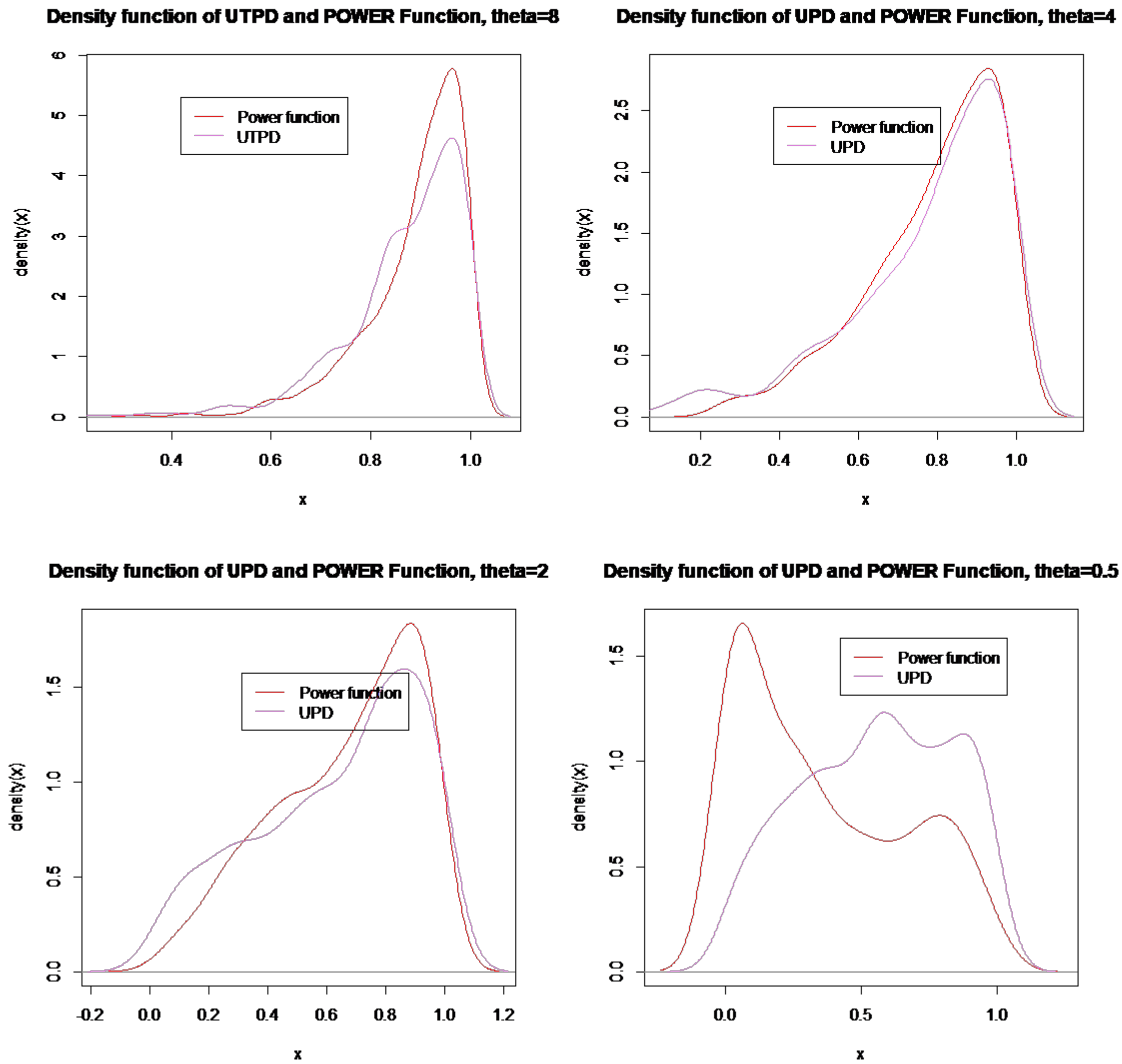

Figure 4: Comparison of UTPD and power function distribution

Result 2: When we take a power transformation

$$V = X^{\frac{1}{\beta}} \tag{6}$$

the density function of V becomes

$$g(v) = \frac{\theta\beta}{e^{\theta}-1} v^{\beta-1} e^{\theta v^{\beta}}, \quad 0 \leq v \leq 1, \tag{7}$$

which has the form of the Weibull distribution, but domain is quite different.

Result 3: The probability density function of $W = \frac{1}{X}$, where X follows UTPD is

$$h(w) = \frac{\theta}{e^{\theta}-1} e^{\frac{\theta}{w}} \frac{1}{w^2}, \quad 1 \leq w < \infty. \tag{8}$$

Estimation of the parameter of the UTPD is done in the next section.

4. Estimation of the Parameter

For the estimation of the parameter, we employ the maximum likelihood (ML) method and the method of moments (MM), and comparisons are made with numerical illustrations.

4.1. Maximum likelihood estimation (MLE)

Suppose a sample of size n is taken from UTPD with pdf (2). By taking logarithm of the likelihood function and finding the derivative with respect to θ, we have a nonlinear equation

$$\frac{\partial \log L}{\partial \theta} = 0 \Rightarrow \frac{n}{\theta} - \frac{ne^{\theta}}{e^{\theta}-1} + \sum_{i=1}^{n} x_i = 0,$$

which can be solved numerically to estimate the parameter.

4.2. Method of moments

Another method used for the estimation of the parameter is the method of moments. Equating the first raw moment to the corresponding sample moment, the following equation is obtained, and solving the same for θ results in the estimate.

$$\frac{\sum_{i=1}^{n} x_i}{n} = \frac{e^{\theta}(\theta-1)+1}{\theta(e^{\theta}-1)}.$$

4.3. Large sample properties

The asymptotic properties of the ML estimators, assuming the usual regularity conditions are provided in this section.

Property 1: The ML estimator $\hat{\theta}$ is asymptotically normally distributed with mean θ and variance $\frac{1}{nI(\theta)}$ where $I(\theta)$ is the well known information matrix.

Proof: We have the log likelihood function

$$\log L = n\log\theta - n\log\left(e^{\theta}-1\right) + \theta\sum_{i=1}^{n} x_i.$$

Then

$$\frac{\partial^2 logL}{\partial\theta^2} = \frac{-n}{\theta^2} + \frac{ne^{\theta}}{(e^{\theta}-1)^2}.$$

If we denote the gradient of log L, the score statistic as $S(\theta)$, and $-\frac{\partial^2 logL}{\partial\theta^2}$ as $K(\theta)$, then the above equation can be written as,

$$K(\theta) = -S'(\theta) = \frac{n}{\theta^2} - \frac{ne^{\theta}}{(e^{\theta}-1)^2}.$$

Also we know that,

$$S(\theta) = \tfrac{\partial logL}{\partial\theta} = \textstyle\sum_{i=1}^{n} \tfrac{\partial logf(X_i,\theta)}{\partial\theta} \quad and$$

$$K(\theta) = \sum_{i=1}^{n} K(X_i, \theta).$$

Then, $E(K(\theta)) = nI(\theta)$ where $I(\theta) = E[\frac{\partial log f(X_i,\theta)}{\partial\theta}]^2$, the information matrix.

Using Taylor's formula,

$0 = S(\hat{\theta}) = S(\theta) - K(\theta)(\hat{\theta} - \theta) + R$, where R tends to zero.

And finally after adjusting the terms, $\sqrt{n}(\hat{\theta} - \theta) = \frac{S(\theta)/\sqrt{n}}{K(\theta)/n}$.

By Slutsky's theorem $\hat{\theta}$ converges in distribution to $N(\theta, \frac{1}{nI(\theta)})$. □

Now the consistency property of $\hat{\theta}$ is stated below, the proof of which readily follows as in Kale (2007).

Property 2: The likelihood equation admits a consistent solution and the consistent estimator is essentially unique.

4.4. Numerical examples

Simulated samples of sizes 20, 60 and 100 from the population following UTPD for selecting the better method of estimation. For the comparison purpose of the two methods discussed above, each sample is generated 1000 times. The estimate of θ, standard error (SE), mean square error (MSE), 95% confidence intervals (CI) for the parameters and the coverage probabilities (CP) are shown in Table 2. All the simulation works and other computations are done using R-programming and the R codes are presented in the Annexure. The SE and MSE are decreasing with an increase in sample size. The coverage probabilities are increasing when the sample size is increasing. But for smaller sample sizes, the coverage probabilities of the parameters estimated using the ML method are lesser than that generated by MM. From the table, it is clear that both the ML method and MM are equally good for estimation purposes based on the MSE. Both the methods give us approximately equal values as parameter estimates.

In the next section, an attempt is made to generalize the UTPD into a general finite interval (a,b).

5. Generalized UTPD

In this section, we construct a generalization of UTPD. As we have seen in the definition of UTPD, the domain is $[0, 1]$. This can be generalized to a distribution defined on a finite interval (a, b). Let X be a continuous random variable defined on (a, b). The probability density function of X is given by

$$f(x) = \frac{\theta}{e^{\theta b} - e^{\theta a}} e^{\theta x}, \quad a < x < b, \theta \neq 0.$$

When $\theta = 0$, it becomes the uniform distribution defined on (a,b).

Properties

1. The k^{th} raw moment is given by

$$E(X^k) = C\{\frac{b^k e^{\theta b} - a^k e^{\theta a}}{\theta} - \frac{k}{\theta^2}(b^{k-1}e^{\theta b} - a^{k-1}e^{\theta a}) + \frac{(k-1)k}{\theta^3}(b^{k-2}e^{\theta b} - a^{k-2}e^{\theta a}) + ... + \frac{(-1)^k(1.2.3...k)}{\theta^{k+1}}(e^{\theta b} - e^{\theta a})\},$$

where $C = \frac{\theta}{(e^{\theta b} - e^{\theta a})}$.

2. Mean= $\frac{be^{\theta b} - ae^{\theta a}}{\left(e^{\theta b} - e^{\theta a}\right)} - \frac{1}{\theta}$.

3. Variance= $\frac{\theta}{\left(e^{\theta b} - e^{\theta a}\right)} \left\{\frac{b^2 e^{\theta b} - a^2 e^{\theta a}}{\theta} - \frac{2}{\theta^2}\left(be^{\theta b} - ae^{\theta a}\right) + \frac{2}{\theta^3}\left(e^{\theta b} - e^{\theta a}\right)\right\} - \left(\frac{be^{\theta b} - ae^{\theta a}}{\left(e^{\theta b} - e^{\theta a}\right)} - \frac{1}{\theta}\right)^2$.

4. The hazard rate function, $h\left(x\right) = \frac{\theta e^{\theta x}}{e^{\theta b} - \left\{e^{\theta x}\right\}}$.

5. The mean residual life function (MRL), $\mu(t) = \frac{1}{e^{\theta b} - e^{\theta a} - e^{\theta t} - e^{\theta a}} \left\{e^{\theta b}\left(b - t - 1\right) + e^{tb}\right\}$.

6. The quantile function is $x = \frac{1}{\theta} \ln \left[e^{\theta a}\left(1 - u\right) + ue^{\theta b}\right]$.

More interesting features are the topics for further studies.

6. Applications

In this section, the application of the distribution is illustrated by fitting the UTPD to four data sets. As mentioned in Section 2, we are comparing the UTPD and power function distributions for all these data sets.

Data Set 1: This data set is obtained from the Los Angeles Department of Water and Power (LADWP) solar incentive program, which offers incentives to offset the cost of installing a solar rooftop system in the homes/business of the people of Los Angeles. This metric measures the Net Energy Metering (NEM) installed capacity (Kilowatts), which is available in https://catalog.data.gov/dataset. The data consists of the observations from 2016 to 2018, which describes the application of UTPD in time series as well as physics. As mentioned in Remark 1, the data can be transformed using (4) to bring the data into the range [0,1]. Now we try to fit the power function and UTPD to this transformed data. By the Kolmogorov-Smirnov (K-S) distance measure and p-value given in Table 3, it is clear that both these distributions are good fit for this data set. The p value is greater than 0.05, confirming that the UTPD and power function distribution are good approximations. But we may conclude that UTPD is a better fit for this data since K-S distance is lesser but p-value is greater when compared with power function distribution.

Data Set 2: Data set 2 is the total tax and non-tax revenue of Egypt from 2002 to 2018 available in https://stats.oecd.org and these are time series observations from financial sector. Transform the data using (4) and here also we obtain the power function and UTPD as suitable fit for this data based on K-S distance and p-value given in Table 3. In the light of these two values we could identify that UTPD is a better fit for this data than power function distribution.

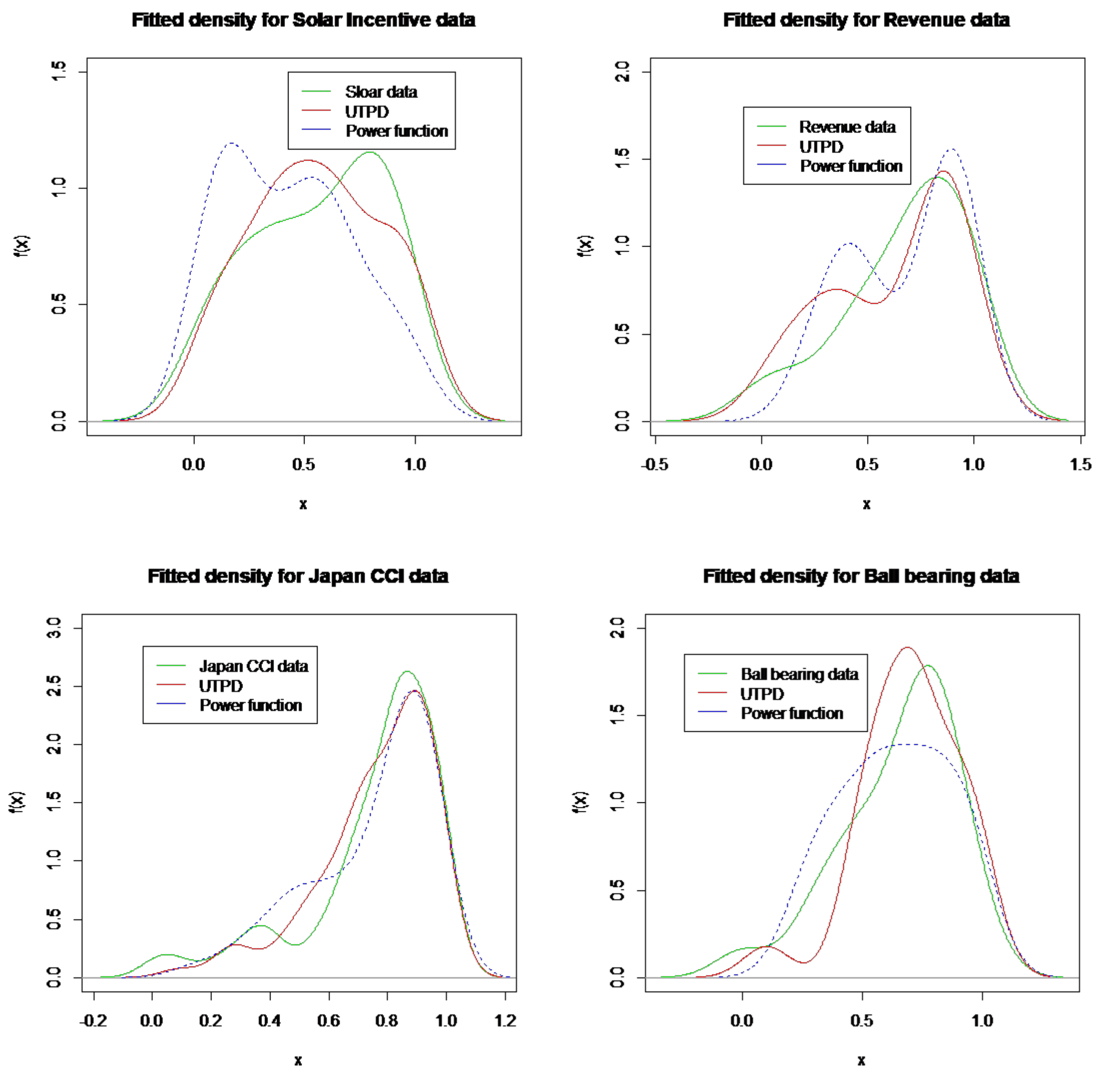

Figure 5: Density plots of fitted data sets

Data Set 3: Now we consider another time series data for describing the applications of UTPD. This is a set of observations of Japan consumer confidence index from January 2014 to March 2021. (Ref: https://stats.oecd.org). Again after suitable transformation we get the power function and UTPD as suitable fit for this transformed data from the values in Table 3. But since the p-value is higher and K-S distance is lesser for UTPD than power function, it is clear that UTPD is a better fit.

Data Set 4: The fourth data set we consider is the ball bearing data taken from Lawless (2003) to employ it in the engineering field. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test and they are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, and 173.40. As mentioned in Remark 1, the data can be transformed using (4) to $[0, 1]$. The K-S distance and p-value given in Table 3, reveal that UTPD is a better fit for this data than power function distribution.

The densities of the original data sets together with the fitted densities plotted in Figure 5 reveal that UTPD is a good fit for all the data sets considered.

Table 2: Parameter Estimates

Sample size (n)	Parameter (θ)	Method	Estimate ($\hat{\theta}$)	SE	MSE	CI	CP
20	4	MLE	4.12	1.110	1.307	(3.15,5.09)	0.627
		MM	4.10	0.252	1.288	(1.88,6.37)	0.957
60		MLE	4.07	0.625	0.385	(3.12,5.02)	0.880
		MM	4.03	0.080	0.394	(2.80,5.26)	0.948
100		MLE	4.01	0.482	0.234	(3.07,4.96)	0.955
		MM	4.02	0.048	0.246	(3.06,4.98)	0.954
20	3	MLE	3.06	0.968	1.020	(2.21,3.91)	0.620
		MM	3.09	0.233	1.098	(1.05,5.14)	0.938
60		MLE	3.06	0.553	0.303	(2.22,3.90)	0.876
		MM	3.04	0.072	0.312	(1.95,4.13)	0.950
100		MLE	3.01	0.424	0.183	(2.18,3.84)	0.950
		MM	3.01	0.043	0.190	(2.15,3.86)	0.947
20	1.5	MLE	1.56	0.835	0.740	(0.83,2.29)	0.622
		MM	1.54	0.196	0.770	(-0.17,3.26)	0.950
60		MLE	1.50	0.475	0.231	(0.78,2.22)	0.861
		MM	1.55	0.063	0.147	(0.79,2.23)	0.942
100		MLE	1.50	0.367	0.147	(0.78,2.22)	0.942
		MM	1.49	0.036	0.130	(0.78,2.20)	0.947
20	0.5	MLE	0.52	0.792	0.653	(-0.16,1.22)	0.610
		MM	0.51	0.180	0.653	(-1.07,2.09)	0.950
60		MLE	0.50	0.452	0.209	(-0.18,1.19)	0.868
		MM	0.50	0.058	0.208	(-0.38,1.40)	0.957
100		MLE	0.49	0.349	0.125	(-0.20,1.16)	0.949
		MM	0.48	0.034	0.119	(-0.18,1.16)	0.950
20	-2	MLE	-2.06	0.870	0.780	(-2.83,-1.31)	0.631
		MM	-2.10	0.203	0.840	(-3.89,-0.31)	0.954
60		MLE	-2.04	0.495	0.263	(-2.79,-1.28)	0.865
		MM	-2.04	0.062	0.234	(-2.98,-1.09)	0.948
100		MLE	-2.01	0.382	0.148	(-2.78,-1.26)	0.941
		MM	-2.03	0.036	0.132	(-2.74,-1.33)	0.952

Table 3: Fitting of real data sets

	Distribution	Parameter	K-S distance	p-value
Dataset 1	UTPD	0.75	0.0967	0.9991
	Power function	1.28	0.1290	0.9634
Dataset 2	UTPD	2.29	0.1176	0.9999
	Power function	2.08	0.1764	0.7631
Dataset 3	UTPD	3.64	0.0919	0.8585
	Power function	3.04	0.1149	0.6164
Dataset 4	UTPD	1.91	0.1421	0.7657
	Power function	1.86	0.2173	0.6487

7. Conclusion

In this paper, we have studied in detail the uniform truncated Poisson distribution as the solution of a first order differential equation and derived the same from the truncated uniform distribution. Comparisons with some well known distributions are done. The expressions for moments, distributions of the order statistics, *etc.* are further derived. Some transformed distributions are also studied. Some of the estimation procedures of the parameter are discussed. The newly constructed distribution is applied on real data. Characterizations and further applications of UTPD in time series, regression and reliability are the topics for further studies.

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ANNEXURE

A1: R Code for simulation studies using MLE method

```
m=1000
n=20
para=4
x<-list(mode = "vector", length = m)
z<-list(mode = "vector", length = m)
z<-list(mode = "vector", length = m)
out<-list(mode = "vector", length = m)
est<-list(mode = "vector", length = m)
dut<-function(x,a=4)((a/(exp(a)-1))*exp(a*x))
put<-function(x,a=4)(1-(1/(exp(a)-1)*(exp(a)-exp(a*x))))
qut<-function(u,a=4)((1/a)*log(1+u*(exp(a)-1)))
rut<-function(n,a=4)qut(runif(n),a)
for(i in 1:m)
{
x[[i]]<-runif(n)
z[[i]]<-qut(x[[i]])
fn <- function(theta)
sum(log((exp(theta)-1)/theta)-theta*z[[i]])
out[[i]]<-nlm(fn, theta<-0.1, hessian=TRUE)
out
}
w<-vector(mode = "numeric", length = m)
setheta=vector(mode="numeric", length = m) #standard error of theta
for(j in 1:m)
{
w[j]=out[[j]]$estimate
setheta[j]=sqrt(diag(solve(out[[j]]$hessian)))
}
#Confidence intervals
lcla<-vector(mode = "numeric", length = m)
ucla<-vector(mode = "numeric", length = m)
for( i in 1:m)
{
lcla[i] <- (w[i] - 1.96*setheta[i])
ucla[i] <- (w[i] + 1.96*setheta[i])
}
#Coverage Probability
covera<-vector(mode = "numeric", length = m)
for(i in 1:m)
{
covera[i] <- (lcla[i]<=4)&(ucla[i]>=4)
}
```

A2: R Code for simulation studies using the method of moments

```
for(i in 1:m)
{
x[[i]]<-runif(n)
z[[i]]<-qut(x[[i]])
z[[i]]<-na.omit(z[[i]])
func=function(theta){
(sum(z[[i]])/n)- ((exp(theta)*(theta-1)+1)/(theta*(exp(theta)-1)))
}
 Result[[i]]=BFfzero_2(func,-10,100)
}
# w1<-vector(mode = "numeric", length = m)
sew1<-vector(mode = "numeric", length = m)
for(j in 1:m)
{
w1[j]=Result[[j]]$root
}
#Confidence intervals
lclaw1<-vector(mode = "numeric", length = m)
uclaw1<-vector(mode = "numeric", length = m)
for( i in 1:m)
{
lclaw1[i] <- (w1[i] - 1.96*sqrt(var(w1)))
uclaw1[i] <- (w1[i] + 1.96*sqrt(var(w1)))
}
#Coverage Probability
coveragew1<-vector(mode = "numeric", length = m)
for(i in 1:m)
{
coveragew1[i]<-(lclaw1[i]<=4)&(uclaw1[i]>=4)
}
```