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# Single Server Poisson Queueing Model with Additive Exponential Service Time Distribution

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**Abstract**

A stochastic process associated with queuing system is specified by the knowledge of (i) Arrival process (ii) Queue discipline (iii) Service process. Among these three, the service process is more important since it can be controlled by the operators of the system. A long with many other assumptions, it is customary to consider that the inter service time are Exponential. A generalization of it is Erlangian service time in which it is assumed that there are k-phase of service and each have identically distributed as Negative Exponential Distribution. But in many practical situations the service times are not identical. Hence in this paper we consider a queueing system with Poisson arrival having component of additive exponential service times. Using the probability generating function the system size distribution is derived. The system behaviour analyzed by deriving the system characteristics like, average number of customer in the system, the variability of system size, etc,. The waiting time distribution of the system is also derived. The sensitivity of the model with respect to the parameter is analyzed. It is observed that the system performance is influenced by the service time distribution parameters. This model includes M/M/1, M/E/1 models as particular cases for specific or timely value of parameters.

*Key words:* Queueing system; Erlangian service time; Additive exponential service times; Negative Exponential Distribution; M/M/1; M/E/1; Sensitivity analysis.

## 1. Introduction

In many of the queuing models it is customary to consider that the inter service times follows exponential distribution. In many practical situations the exponential assumption concerning service times being distributed may be rather limiting on its utility. In particular, in computer communications the service time of request is sum of two random variables namely, (1) entering (key in) time and (2) processing time. Each of these service times are exponentially distributed with different parameters say and as result of it the inter-service time between two customers follows an additive exponential distribution. Very little work has been observed regarding queueing models with additive exponential distribution. Hence, in this paper, we develop and analysis a single server queueing model with Poisson arrivals having additive exponential service times distribution.

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In this section, we briefly review some of the contributions in queueing models with non exponential service time in order to highlight the present work in its right perspective. Kendall (1951) used the concept of regeneration point by suitable choice of regeneration points and extracts. This method is known as embedded Markov chains. This method pioneered the M/G/1 queueing models. Keillson and Koharian (1960) developed the supplementary variable technique for analyzing the M/G/1 queueing model. This technique is very popular in analyzing the non-Markovian queueing models. Heymans (1968) considered the economic behavior of an M/G/1 queueing system that operates under the cost structure, a server start-up cost, a server shut-down cost, a cost per unit time when the server is busy and a holding cost per unit time spent in the system for each customer. The author proved that for a single server queue, there is a stationary optimal operating policy. Levy and Yechiah (1975) considered the utilization of idle time of the server in a M/G/1 for some additional work in a secondary queue. Two types of vacation policies *viz.,* M/G/1/Vs and M/G/1/Vm with exhaustive service are also studied.

Bohm (1992) considered an M/G/1 queueing model with N-policy operating. The server start up only if a queue of a prescribed length was built up. For this model, the time dependent distribution of the queue length is given by renewal arguments without resorting to integral transform techniques. Movaghar (1998) studied a queueing system where customers have strict deadlines until the beginning of their service. An analytical method is given for the analysis of a class of such queues, namely, M(n)/M/m/{rm FCFS} + {rm G} models. The principal measure of performance is the probability measure induced by the offered waiting time.

Hisashi and Brian (2001) studied the loss models in the traffic engineering of traditional telephone exchanges. These models were generalized to the loss networks, which provide models for resource-sharing in multi-service telecommunication networks. The authors introduced a generalized class of models, queueing-loss networks, which captures both queueing and loss aspects of a system. Choudhury *et. al.* (2004) considered an $M_x$/M/1 queueing model under a threshold policy with vacation process, where the server takes a sequence of vacations, till the server returns to find at least some prespecified number of customers (threshold) observed after each grand vacation.

EI-Paoumy (2008) derived the analytical solution of the queue: $M_x$/M/2/N for batch arrival system with balking, reneging and two heterogeneous servers. A modified queue discipline is used with a more general condition. The steady-state probabilities and measures of effectiveness are derived. El-Paoumy and Ismail (2009) studied $M_x/E_k$/I/N with balking and reneging queueing model in which, (i) Units arrive in batches of random size with the inter arrival times of batches following negative exponential distribution. (ii) The queue discipline is FCFS, it being assumed that the batches are pre-ordered for service purpose. (iii) The service time distribution is Erlangian with K stages. Recurrence relations connecting the various probabilities are derived. Measures of effectiveness as L and $L_q$ are deducted and some special cases are presented.

Fralix and Zwart (2010) studied a conjecture "the distribution of the number of jobs in the system of a symmetric M/G/1 queue at a fixed time is independent of the service discipline if the system starts empty". Their arguments are based on a time-reversal argument for regenerative processes. Down *et. al.* (2011) discussed the dynamic server control in a two-class service system with abandonments. Two models are considered. In the first case, rewards are received upon service completion, and there are no abandonment costs (other than the lost opportunity to gain rewards). In the second, holding costs per customer per unit time are accrued, and each abandonment involves a fixed cost. Both cases are considered under the discounted or average reward/cost criterion. Chydzinski and Adamczyk (2019) studied Queues with the dropping function and general service time Firstly, a stability condition, more general than the well-known $\rho < 1$, is proven. Secondly, the formulas for the queue size distribution, loss ratio and mean duration of the busy period, are derived. Thirdly, numerical examples are given, including optimizations of the shape of the dropping function with regard to the combined cost of the queue size and loss ratio.

Dudin *et.al.* (2021) studied the single-server multi-class queue with unreliable service, batch correlated arrivals, customers impatience, and dynamical change of priorities. Using the embedded Markov chain technique the probability generating function of the system size distribution under steady state condition is derived. The system performance of like the probability of system emptiness, the average no of customers in the system and in the queue, the variance of the number of customers in the system, Laplace transformation of waiting time distribution of the customers in the system, the average waiting time of the customers in the system and queue ,the variance of the waiting time distribution etc., are derived. The sensitivity of the model with respect to parameters is studied through numerical illustration. This model includes the M/M/1 model when $1/\theta_1 \to 0$ this also includes M/$E_2$/1 model if $\theta_2 \to \theta_1$.

**Additive exponential service time distribution**

The additive exponential distribution was introduced as a sum of two different exponential variates. The general procedure for obtaining the probability distribution function for two independent different exponential random variables is through Jacobian transformation or inverse theorem of characteristic functions. This distribution also includes exponential if one of the parameters tends to zero. Consider two univariate continues random variables $T_1$ and $T_2$ which follow Exponential distributions with parameters $\theta_1$ and $\theta_2$ respectively. Then the addition of these two random variables $T = T_1 + T_2$ is having an Additive exponential distribution with probability density function

$$f(t) = \frac{\left(e^{-\left(\frac{t}{\theta_1}\right)} - e^{-\left(\frac{t}{\theta_2}\right)}\right)}{\theta_1 - \theta_2} \qquad \theta_1 > \theta_2 > 0;\;\; t > 0$$

**Properties of additive exponential distribution**

i) If $\theta_1 \to \theta_2$ then the above probability density function gives Gamma distribution with parameters $\theta_2$ as $\theta_1 \to \theta_2$

ii) The cumulative distribution of the additive exponential distribution is,

$$= \frac{\left(e^{-\left(\frac{t}{\theta_1}\right)} - e^{-\left(\frac{t}{\theta_2}\right)}\right)}{\theta_1 - \theta_2} \qquad \theta_1 > \theta_2 > 0;\;\; t > 0$$

iii) The mean of the distribution is,

$$Mean = \theta_1 + \theta_2$$

iv) The variance of the distribution is given by,

$$\mu_2 = (\theta_1^2 + \theta_2^2)$$

v) Moment generating function of the distribution is given by,

$$M_t(x) = \frac{1}{(1 - t\theta_1)(1 - t\theta_2)}$$

vi) Characteristic function of the distribution is given by,

$$\phi_t(x) = \frac{1}{(1 - it\theta_1)(1 - it\theta_2)}$$

vii) The $r^{th}$ raw moment of the distribution is,

$$\mu_r' = \int_0^\infty t^r.b(t).dt = \frac{r!}{\theta_1 - \theta_2}\left(\theta_1^{r+1} - \theta_2^{r+1}\right)$$

viii) The $r^{th}$ cumulant of the distribution is,

$$k_r = \frac{(r-1)!}{\theta_1 - \theta_2}\left(\theta_1^{r+1} - \theta_2^{r+1}\right)$$

ix) The first four central moments of this distribution are,

$$\mu_1 = 0,\ \ \mu_2 = (\theta_1^2 + \theta_2^2),\ \ \mu_3 = 2(\theta_1^3 + \theta_2^3),\ \ \mu_4 = 9\theta_1^4 + 6\theta_1^2\theta_2^2 + 9\theta_2^4$$

x) The skewness of the distribution is,

$$= 4\frac{(\theta_1^3 + \theta_2^3)^2}{(\theta_1^2 + \theta_2^2)^3}$$

This distribution is positively skewed distribution.

xi) The kurtosis of the distribution is,

$$= 9 - 12\frac{(\theta_1\theta_2)^2}{(\theta_1^2 + \theta_2^2)}$$

## 2. Single server poisson queueing model with additive exponential service time distribution

In this section, a single server infinite capacity Poisson queueing system having FIFO discipline in which the arrivals follows a Poisson process with parameter $\lambda$ is considered. It is also assumed that the inter-service times follows an additive exponential service time distribution with parameters $\theta_1$ and $\theta_2$. The probability density function of inter-service times is,

$$f(t) = \frac{\left(e^{-\left(\frac{t}{\theta_1}\right)} - e^{-\left(\frac{t}{\theta_2}\right)}\right)}{\theta_1 - \theta_2} \qquad \theta_1 > \theta_2 > 0;\ \ t > 0 \tag{1}$$

Following the heauristic arguments of Gross and Harris (1974) the queueing model is analyzed. The embedded stochastic process X($t_i$),where, X denotes the number in the system and $t_1,\ t_2,\ t_3 \ldots$,are the successive times of completion of service. Since, $t_i$ is the completion time of the $i^{th}$ customer, then X($t_i$) is the number of customers the $i^{th}$ customer leaves behind as he departs. Since, the state space is discrete, $X_i$ represents the number of customers remaining in the system as the $i^{th}$ customer departs. Then for all n > 0 one can have.

$$X_{n+1} = \begin{cases} X_n - 1 + A_{n+1} & ; \quad X_n \geq 1 \\ A_{n+1} & ; \quad X_n = 0 \end{cases} \tag{2}$$

where $X_n$ is the number in the stem at the $n^{th}$ departure point and $A_{n+1}$ is the number of customers who arrived during the service time, $S^{n+1}$ of the $(n+1)^{th}$ customer.

The random variable $S^{n+1}$ by assumption is independent of previous service times and the length of the queue, since arrivals are Poissonian, the ransom variable $A_{n+1}$ depends only on S and not on the queue or on the time of service initiation. Then,

$$P\{A = a\} = \int_0^{\infty} P\{A = a|S = t\}dB(t) \tag{3}$$

$$\text{and} \qquad P\{A = a|S = t\} = \frac{e^{-\lambda t}(\lambda t)^a}{a!} \tag{4}$$

so that,

$$\begin{aligned} P\{X_{n+1} = j|X_n = i\} &= P\{A = j - i + 1\} \\ &= \begin{cases} \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^{(j-i+1)}}{(j-i+1)!}\, dB(t) & ; \quad (j \geq i-1, i \geq 1) \\ 0 & ; \quad (j \geq i-1, i \geq 1) \end{cases} \end{aligned} \tag{5}$$

If a departing customer leaves an empty system, the system state remains zero until an arrival comes. Thus the transition probabilities for the case i=0 are identical to those for i=1. Let $p_{ij}$ denote the probability that the system size immediately after a departure point is j given that the system size after previous departure was i. $k_n$ is the probability that there are n arrivals during a service time t.

Then,

$p_{ij}$ = Pr {system size immediately after a departure point j | system size after previous departure was i}

= P$\{X_{n+1}$=j | $X_n$=i }

$$P_{ij} = \int_0^\infty \left( \frac{e^{-t\left(\lambda + \frac{1}{\theta_1}\right)} - e^{-t\left(\lambda + \frac{1}{\theta_2}\right)}}{\theta_1 - \theta_2} \right) \frac{(\lambda t)^{(j-i+1)}}{(j-i+1)!}\, dt; \quad j \geq i-1, i \geq 1 \tag{6}$$

Therefore, $k_n$ = P{n arrivals during the service time S=t}

$$k_n = \frac{\lambda^n}{\theta_1 - \theta_2} \left( \frac{\theta_1}{(\theta_1 \lambda + 1)} \right)^{n+1} - \frac{\lambda^n}{\theta_1 - \theta_2} \left( \frac{\theta_2}{(\theta_2 \lambda + 1)} \right)^{n+1} \tag{7}$$

Therefore,

$$p = [p_{ij}] = \begin{bmatrix} k_0 & k_1 & k_2 & \dots \\ k_0 & k_1 & k_2 & \dots \\ 0 & k_0 & k_1 & \dots \\ 0 & 0 & k_0 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \tag{8}$$

Assuming that the system is in steady state, and $p_{ij}$= $\pi_j$, then,

$$p = \pi_0 k_i + \sum_{j=1}^{i+1} \pi_i k_{i-j+1} \qquad (i = 0, 1, 2, ....) \tag{9}$$

where, $\pi_j$ is the probability of j customers in the system at departure point after steady state is reached.

Let

$$K(z) = \sum_{n=0}^{\infty} k_i z^i \tag{10}$$

$$\pi(z) = \sum_{n=0}^{\infty} \pi_i z^i \qquad (|z| \leq 1) \tag{11}$$

are generating functions of $\pi_n$ and $k_n$ respectively.

Hence,

$$K(z) = \sum_{i=0}^{\infty} \frac{\lambda^i}{\theta_1 - \theta_2} \left( \frac{\theta_1}{(\theta_1 \lambda + 1)} \right)^{i+1} z^i - \sum_{i=0}^{\infty} \frac{\lambda^i}{\theta_1 - \theta_2} \left( \frac{\theta_2}{(\theta_2 \lambda + 1)} \right)^{i+1} z^i \tag{12}$$

After simplification, we get

$$K(z) = \frac{\theta_1}{(\theta_1 - \theta_2)(1 + \theta_1 \lambda (1-z))} - \frac{\theta_2}{(\theta_1 - \theta_2)(1 + \theta_2 \lambda (1-z))} \tag{13}$$

Therefore, the probability generating function of system size distribution for the M/G/1 model under consideration as,

$$\pi(z) = \frac{(1 - K'(z))(1 - z)K(z)}{K(z) - z} \tag{14}$$

Differentiating the equation (13) with respect to z and taking z=1 we get

$$\frac{dK(z)}{dz} = \left[\left(\frac{\theta_1}{\theta_1 - \theta_2}\right)\left(\frac{\theta_1\lambda}{(1 + \theta_1\lambda(1 - z))^2}\right) - \left(\frac{\theta_2}{\theta_1 - \theta_2)}\right)\left(\frac{\theta_2\lambda}{(1 + \theta_2\lambda(1 - z))^2}\right)\right]_{/z=1}$$

This implies,

$$\rho = K'(z)_{/z=1} = \lambda(\theta_1 + \theta_2) \tag{15}$$

Substituting equations (13) and (15) in equation (14) we get,

$$\pi(z) = \frac{\dfrac{(1-\rho)(1-z)}{(\theta_1 - \theta_2)}\left[\dfrac{\theta_1}{1 + \theta_1\lambda(1-z)} - \dfrac{\theta_2}{1 + \theta_2\lambda(1-z)}\right]}{\left[\left(\dfrac{1}{(\theta_1 - \theta_2)}\right)\left[\dfrac{\theta_1}{1 + \theta_1\lambda(1-z)} - \dfrac{\theta_2}{1 + \theta_2\lambda(1-z)}\right] - z\right]} \tag{16}$$

## 3. System characteristics

In this section we derive and analyze the performance of the queueing model. The probability that there are n customers in the system at any arbitrary time is, coefficient of $z^n$,

$$p_n = A\left[\sum_{j=0}^{n/2} B(n-j)C(j) - p\sum_{j=0}^{(n/2)-1} B(n-j-1)C(j)\right], \text{where n is even}$$

$$p_n = A\left[\sum_{j=0}^{(n+1/2)} B(n-j)C(j) - p\sum_{j=0}^{(n-1/2)} B(n-j-1)C(j)\right], \text{where n is odd}$$

From the equation (16) the probability generating function of the number of customers in system is

$$\pi(z) = \frac{\dfrac{(1-\rho)(1-z)}{(\theta_1 - \theta_2)}\left[\dfrac{\theta_1}{1 + \theta_1\lambda(1-z)} - \dfrac{\theta_2}{1 + \theta_2\lambda(1-z)}\right]}{\left[\left(\dfrac{1}{(\theta_1 - \theta_2)}\right)\left[\dfrac{\theta_1}{1 + \theta_1\lambda(1-z)} - \dfrac{\theta_2}{1 + \theta_2\lambda(1-z)}\right] - z\right]} \tag{17}$$

Expending equation (1) and collecting the constant terms we get the probability that the system is empty as

$$P_0 = 1 - \lambda(\theta_1 + \theta_2) \tag{18}$$

The average number of customers in the system can be obtained as,

$$L_s = \left[\frac{d}{dz}[\pi(z)]\right]_{z=1}$$

Differentiating equation (2) and using L-Hospital rule, we get,

$$L_s = \left[\frac{[\rho - \theta\lambda^2]}{[1-\rho]}\right]$$

$$\theta_1\theta_2 = \theta, \lambda(\theta_1 + \theta_2) = \rho \tag{19}$$

The average number of customers in the queue is

$$L_q = L_s - \rho$$

$$L_q = \frac{\rho^2 - \theta\lambda^2}{1-\rho} \tag{20}$$

The variance of the number of customers in the system is given by,

$$\begin{aligned} V_s &= E(N^2 - N) + E(N) - (E(N))^2 \\ &= [\pi''(z) + \pi'(z) - [\pi'(z)]^2] \end{aligned} \tag{21}$$

Differentiating equation (1) with respect to z and using L-Hospital rule, we get the variance of the number of customers in the system as

$$V(N) = \frac{\rho - \theta\lambda^2(3 + \rho - \theta\lambda^2)}{(1-\rho)^2} \tag{22}$$

## 4. Waiting time distribution

In this section we derive the waiting time distribution of the single server Poisson arrival queueing model with additive exponential inter-service time distribution. Consider the queue discipline of the system as FIFO, following the heauristic arguments of Gross and Harris (1974) for the M/G/1 model, we derive the Laplace transformation of the waiting time distribution.

Let $B^*(s)$ be the Laplace Transformation of the inter-service time distribution and $W^*(s)$ be the Laplace transformation of the waiting time distribution. Then we have,

$$B^*(s) = \frac{1}{(s\theta_1 + 1)(s\theta_2 + 1)}$$

we have

$$K(z) = \frac{\theta_1}{(\theta_1 - \theta_2)(1 + \theta_1\lambda(1-z))} - \frac{\theta_2}{(\theta_1 - \theta_2)(1 + \theta_2\lambda(1-z))}$$

Therefore,

$$K(z) = B^*(\lambda - \lambda z) \tag{23}$$

The Laplace transformation of waiting time distribution is

$$W^*[\lambda(1-z)] = \frac{[1 - K'(1)](1-z)B^*(\lambda(1-z))}{B^*(\lambda(1-z)) - z} \tag{24}$$

where, $K'(1)$ is as given in equation (15)

Writing $\lambda(1-z)=s$, we get $z=1-\frac{s}{\lambda}$

Therefore,

$$W^*(s)=\frac{[1-K'(1)]sB^*(s)}{s-\lambda(1-B^*(s))} \tag{25}$$

From the convolution property of transformation,

$$W^*(s)=W_q^*(s).B^*(s) \tag{26}$$

where, T is the waiting time of the customer in the system and $T_q$is the time waiting time of a customer in the queue and S is the service time of the customer and $T=T_q+S$. Therefore,

$$W_q^*=\frac{[1-K'(1)]s}{s-\lambda(1-B^*(s))} \tag{27}$$

The mean waiting time of a customer in the queue is,

$$W_q=\left(\frac{d\left(W_q^*(s)\right)}{ds}\right)_{s=0}$$

$$\frac{d}{ds}\left[\frac{W_q^*(s)}{[1-K'(1)]}\right]=\frac{s-\lambda+\lambda B^*(s)-s[1+\lambda B^*(s)]}{s-\lambda[1-B^*(s)]]^2} \tag{28}$$

substituting the values of $B^*(s)$ in equation (28) and using L-Hospital rule, we get the random waiting time of a customer in the queue as

$$W_q=\frac{1}{\lambda}\left[\frac{[\theta\lambda^2-\rho]}{[1-\rho]}\right] \tag{29}$$

The waiting time of the customers in system is,

$$W_s=W_q+\rho, \quad \text{where,}\rho=\lambda(\theta_1+\theta_2)$$

Therefore,

$$W_s=\frac{\theta\lambda^2-\rho+\rho\lambda(1-\rho)}{(1-\rho)} \tag{30}$$

The variance of the waiting time of customer in the queue is,

$$V(W_q)=V_q=\left(\frac{d^2\left(W^*(s)\right)}{ds^2}\right)_{s=0}-[W_q]^2 \tag{31}$$

Therefore,

$$V_q=\frac{\rho^3(2-\rho)-2\rho\theta\lambda^2(4-\rho)-3\theta^2\lambda^4}{\lambda^2(1-\rho)^2} \tag{32}$$

**Table 1: Values of $P_0$ and $(1-P_0)$ for different values of $\lambda$, $1/\theta_1$ and $1/\theta_2$**

| $\lambda$ | $1/\theta_1$ | $1/\theta_2$ | $P_0$ | $1-P_0$ |
|---|---|---|---|---|
| 2 | 11 | 11 | 0.636 | 0.364 |
| 2 | 11 | 13 | 0.664 | 0.336 |
| 2 | 11 | 15 | 0.685 | 0.315 |
| 2 | 11 | 17 | 0.701 | 0.299 |
| 2 | 11 | 19 | 0.713 | 0.287 |
| 4 | 11 | 15 | 0.273 | 0.727 |
| 4 | 13 | 15 | 0.329 | 0.671 |
| 4 | 15 | 15 | 0.370 | 0.630 |
| 4 | 17 | 15 | 0.401 | 0.599 |
| 4 | 19 | 15 | 0.426 | 0.574 |
| 1 | 15 | 15 | 0.867 | 0.133 |
| 2 | 15 | 15 | 0.733 | 0.267 |
| 3 | 15 | 15 | 0.600 | 0.400 |
| 4 | 15 | 15 | 0.467 | 0.533 |
| 5 | 15 | 15 | 0.333 | 0.667 |

## 5. Sensitivity analysis

In this section, the performance of the queueing mode is discussed through a numerical illustrations. Different values of the parameter are considered for the given value of $\lambda$=1,2,3,4,5, $1/\theta_1$=11,13,15,17,19 and $1/\theta_2$=11,13,15,17,19. The probability that the system is empty and the probability the service is busy are computed and presented in Table 1.The relation between the parameters and probability of the idleness are shown in the figure 1.

From Table 1, it is observed that the probability of emptiness is highly influenced by

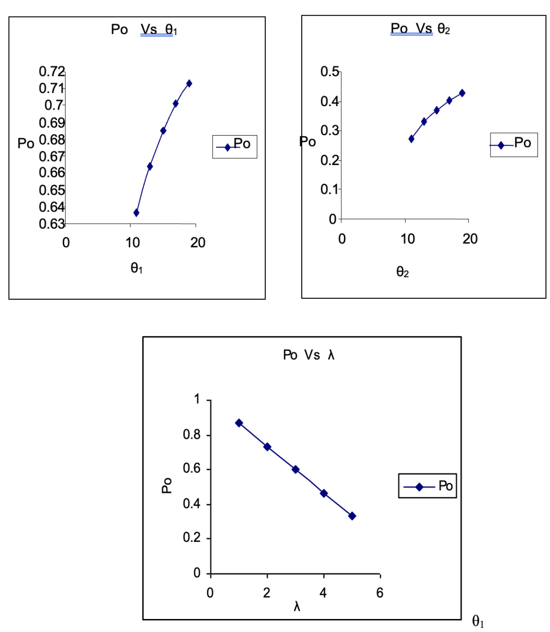


**Figure 1: Relation between probability of emptiness and input parameters**

the model parameters. As the mean arrival rate $\lambda$ varies from 1 to 5, the probability that the emptiness in the system is decreasing from 0.867 to 0.333 when other parameters are fixed at $1/\theta_1$=15 and $1/\theta_2$=15. The service time parameter $1/\theta_1$ increases from 11 to 19, the probability that the emptiness in the system increasing from 0.273 to 0.426 when other parameter are fixed at $\lambda = 4$ and $1/\theta_2 = 15$. The service time parameter $1/\theta_2$ increases from 11 to 19, the probability that the system is empty is in the system increasing from 0.636 to 0.713 when other parameter are fixed at $\lambda = 2$ and $1/\theta_1 = 11$.

For different values of the parameter the average number of customers in the system, average number of customers in the queue and the variance of the number of customers in the system are computed and presented in Table 2.The relation between the parameters and the performance measures in the figure 2. From Table 2, it is observed that the performance measures of the queueing model are significantly influenced by the parameters of the model. As the mean arrival rate $\lambda$ varies from 1 to 5, the average number of customers in the system is increasing. The same phenomenon is observed with respective the average number of customers in the queue for the given values of the other parameters.

When the parameter $1/\theta_1$ increases from 11 to 19, the average number of customers in the system is decreasing from 2.182 to 1.169 for fixed values of $\lambda$=4, $1/\theta_2$ =15. Similarly the value of average number of customers in the queue is decreasing from 1.937 to 0.773. It is observed that as $\lambda$ increases the variance of the number of customers in system is increasing from given values of the other parameters when $1/\theta_1$ is increasing the variance of the number of the customers in system is decreasing for fixed values of the other parameters. When $1/\theta_2$ is increasing the variance of the number of the customers in system is decreasing for fixed values of the other parameters.

For the different values of parameters the values of the average waiting time of customer in system, the average waiting time of customer in queue, the variance of the waiting of the customer in the queue are computed and given the Table 3. The relation between the

**Table 2: Values of $L_s$,$L_q$ and $V_s$ for different values of $\lambda$, $1/\theta_1$ and $1/\theta_2$**

| $\lambda$ | $1/\theta_1$ | $1/\theta_2$ | $L_s$ | $L_q$ | $V_s$ |
|---|---|---|---|---|---|
| 2 | 11 | 11 | 0.519 | 0.207 | 0.626 |
| 2 | 11 | 13 | 0.463 | 0.169 | 0.551 |
| 2 | 11 | 15 | 0.425 | 0.145 | 0.502 |
| 2 | 11 | 17 | 0.397 | 0.128 | 0.467 |
| 2 | 11 | 19 | 0.376 | 0.115 | 0.442 |
| 4 | 11 | 15 | 2.182 | 1.937 | 3.387 |
| 4 | 13 | 15 | 1.702 | 1.369 | 2.528 |
| 4 | 15 | 15 | 1.443 | 1.073 | 2.105 |
| 4 | 17 | 15 | 1.280 | 0.893 | 1.855 |
| 4 | 19 | 15 | 1.169 | 0.773 | 1.690 |
| 1 | 15 | 15 | 0.149 | 0.020 | 0.159 |
| 2 | 15 | 15 | 0.339 | 0.097 | 0.338 |
| 3 | 15 | 15 | 0.600 | 0.266 | 0.738 |
| 4 | 15 | 15 | 0.990 | 0.609 | 1.318 |
| 5 | 15 | 15 | 1.667 | 1.332 | 2.444 |

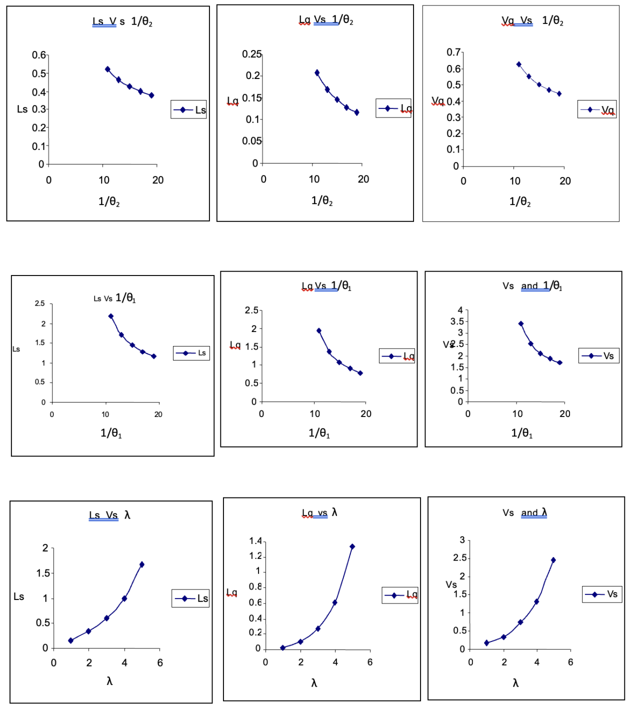


**Figure 2: Relation between probability of emptiness and input parameters**

parameters and the performance measures in the figure 3.

From the table 3 it is observed that the model parameters have a significant influence on the waiting time of the customer in the system and the queue. As the mean arrival rate $\lambda$ is increasing then the average waiting time of the customer in the queue and the average

**Table 3: Values of $W_s$,$W_q$ and $V_q$ for different values of $\lambda$, $1/\theta_1$ and $1/\theta_2$**

| $\lambda$ | $1/\theta_1$ | $1/\theta_2$ | $W_s$ | $W_q$ | $V_q$ |
|---|---|---|---|---|---|
| 2 | 11 | 11 | 0.442 | 0.078 | 0.025 |
| 2 | 11 | 13 | 0.399 | 0.064 | 0.018 |
| 2 | 11 | 15 | 0.370 | 0.005 | 0.015 |
| 2 | 11 | 17 | 0.348 | 0.049 | 0.012 |
| 2 | 11 | 19 | 0.331 | 0.044 | 0.011 |
| 4 | 11 | 15 | 1.091 | 0.364 | 0.220 |
| 4 | 13 | 15 | 0.929 | 0.258 | 0.124 |
| 4 | 15 | 15 | 0.833 | 0.203 | 0.085 |
| 4 | 17 | 15 | 0.769 | 0.170 | 0.064 |
| 4 | 19 | 15 | 0.732 | 0.149 | 0.052 |
| 1 | 15 | 15 | 0.149 | 0.015 | 0.003 |
| 2 | 15 | 15 | 0.303 | 0.036 | 0.008 |
| 3 | 15 | 15 | 0.467 | 0.067 | 0.016 |
| 4 | 15 | 15 | 0.648 | 0.114 | 0.033 |
| 5 | 15 | 15 | 0.867 | 0.200 | 0.076 |

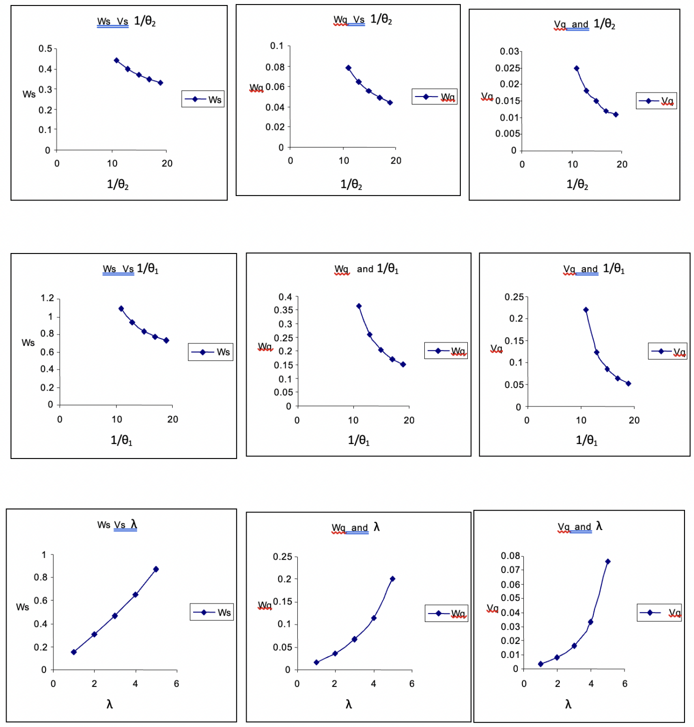


**Figure 3: Relation between $W_s$, $W_q$ and input parameters**

waiting time of customer in the system are increasing when the other parameters remain fixed. It is observed that as the parameter $1/\theta_1$ is increasing from 11 to 19, the average waiting time of the customer in the system and the average waiting time of the customer in the queue are decreasing from 1.091 to 0.732 and 0.364 to 0.149 respectively, for fixed values of other parameters. It is observed that as the parameter $1/\theta_2$ is increasing from 11 to 19, the average waiting time of the customer in the system and the average waiting time of the customer in the queue are decreasing from 0.442 to 0.331 and 0.078 to 0.044 respectively, for fixed values of other parameters. It is further observed that when the mean arrival rate $\lambda$ increases the variance of the waiting time of a customer in the system is increasing when other parameters remain fixed.

## 6. Conclusion

Developed and analyzed a single sever queueing model with Additive exponential service time distribution having Poisson arrivals. Here it is assumed that the queue discipline is FIFO. Using the embedded Markov technique the probability generating function of the queue size distribution under steady state condition is derived. The performance measures of the model like, the average number of customers in the system, the average number of customers in the queue, the probability of emptiness of the system, the probability that the server is busy, the variance of the number of the customers in system, the Laplace transformation of the waiting time distribution of a customer in the system, the average waiting time of a customer in the system, the average waiting time of customer in the queue, the variance of the waiting time of the customers are derived explicitly. The effect of the variation of the input parameter of the model on the performance measures is studied through numerical analysis. It is observed that the model parameter has significant influence on the average number of customers in the system and the average waiting time of a customer in the system and in the queue. This model also includes the M/M/I and M/$E_2$/I models as particular cases for limiting values of the parameters.

This model includes several of the earlier models as particular cases for specific or limiting values of the the parameters

If $1/\theta_1 \to 0$ then this includes M/M/1 queueing model
If $\theta_1 \to \theta_2 \to 0$ then this includes M/$E_2$/1 queueing model

The performance measures of both Exponential and Additive exponential distributions were differ.

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