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Optimal Row-Column Designs with Three Rows

J. P. Morgan[1] **and Siriluck Jermjitpornchai**[2]
[1] *Department of Statistics*
Virginia Tech, Blacksburg, Virginia, United States
[2] *Department of Mathematics*
Mahasarakham University, Mahasarakham, Thailand

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Abstract

E and MV optimality results are established for several series of 3-rowed row-column designs. All of these designs are generalized binary in rows, and their column component designs have completely symmetric information matrices. Included are optimal designs which are nonbinary in columns, and which are superior to any competitor that is binary in columns. The optimality of designs with BIBD column components is extended beyond that of regular Youden designs.

Key words: Block design; Completely symmetric design; E optimality; MV optimality.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

Block designs are useful for experiments where it is important to eliminate an identified source of heterogeneity in experimental units. In many experimental situations, the position that a unit occupies within a block can also affect observed responses. In such cases, row-column designs can often be used to additionally eliminate heterogeneity in this second, orthogonal direction. Applications of row-column designs range from agriculture to psychology to industry and beyond, and an abundance of examples can be found in design textbooks, for example Bailey (2008) or Dean *et al.* (2017).

Consider then bk experimental units arranged in a $k \times b$ array to which v treatments are to be assigned. Determining optimal row-column designs, aside from situations where treatments can either be equally replicated in each row or equally replicated in each column (which includes such well-known designs as Latin squares and regular Youden designs), has proven to be a challenging task. The limited results up to about 1989 are summarized in the monograph Shah and Sinha (1989), and from there to 2015 in Morgan (2015). There has been little progress since.

We proceed with the standard, additive linear model with uncorrelated, equi-variable errors, and in which the expected response arising from the unit in row l, column j is the sum of the effect of row l, the effect of column j, and the effect of the treatment employed in

Corresponding Author: J. P. Morgan
Email: jpmorgan@vt.edu

that cell. With this model, the $v \times v$ treatment information matrix C_d for a $k \times b$ row-column design d can be written (see, for instance, Shah and Sinha, 1989, p. 66) as

$$C_d = diag(r_{d1}, \ldots, r_{dv}) - \frac{1}{k}N_d N_d' - \frac{1}{b}M_d M_d' + \frac{1}{bk}r_d r_d' \qquad (1)$$

where

- N_d denotes the $v \times b$ treatment-column incidence matrix whose entries n_{dij} are nonnegative integers indicating the number of times treatment i occurs in column j,
- M_d denotes the $v \times k$ treatment-row incidence matrix whose entries m_{dil} are nonnegative integers indicating the number of times treatment i occurs in row l,
- r_d denotes the $v \times 1$ vector with entries r_{di}, where r_{di} is the replication of treatment i in design d (*i.e.* the number of experimental units to which treatment i is assigned).

The ith row sum of both N_d and M_d is r_{di}. The matrices $N_d N_d'$ and $M_d M_d'$ are respectively the column-concurrence and row-concurrence matrices for d, with entries denoted by $\lambda_{dii'}$ and $\mu_{dii'}$ respectively. The individual entries of C_d can be displayed thusly:

$$c_{dii'} = r_{di}\delta_{ii'} - \frac{1}{k}\lambda_{dii'} - \frac{1}{b}\mu_{dii'} + \frac{1}{bk}r_{di}r_{di'} \qquad (2)$$

where $\delta_{ii'}$ is the Kronecker delta.

C_d is known to be symmetric and nonnegative definite with zero row and column sums. Hence the rank of C_d satisfies $r(C_d) \leq v - 1$. Here only designs with $r(C_d) = v - 1$ are considered. These are exactly the designs for which every contrast $\sum_i l_i \tau_i$ ($\sum_i l_i = 0$) is estimable, commonly termed the connected designs. For given v, b, and k, $\mathcal{D}(v, b, k)$ will denote the class of all connected $k \times b$ row-column designs for v treatments.

For a given class $\mathcal{D}(v, b, k)$, define the *replication target* r by $r = \lfloor \frac{bk}{v} \rfloor$, and so $bk = vr + p$ where $p \in \{0, 1, \ldots, v - 1\}$ is the *plot excess*, that is, the number of experimental units available beyond that needed for equal replication. Also define $\lambda = \lfloor \frac{r(k-1)}{(v-1)} \rfloor$ as the *concurrence target* for the column component design. Then $\lambda(v - 1) = r(k - 1) + q$ where $q \in \{0, 1, \ldots, v - 1\}$ is the *(column) concurrence excess* for a treatment replicated r times. In this paper we study settings $\mathcal{D}(v, b, k)$ for which $q = 0$, which is henceforth assumed, allowing frequent use of the equality $r(k - 1) = \lambda(v - 1)$.

Each row-column design $d \in \mathcal{D}(v, b, k)$ can be associated with two block designs: the column component design d_N and the row component design d_M, having respective information matrices $C_{d_N} = diag(r_{d1}, \ldots, r_{dv}) - \frac{1}{k}N_d N_d'$ and $C_{d_M} = diag(r_{d1}, \ldots, r_{dv}) - \frac{1}{b}M_d M_d'$. The assignment of treatment i in design d is said to be generalized binary in columns (rows) if $n_{dij} \in \{\lfloor \frac{k}{v} \rfloor, \lfloor \frac{k}{v} \rfloor + 1\}$ for all $j = 1, \ldots b$ ($m_{dil} \in \{\lfloor \frac{b}{v} \rfloor, \lfloor \frac{b}{v} \rfloor + 1\}$ for all $l = 1, \ldots k$). The component block design d_N (d_M) is called generalized binary if the assignment of every treatment to columns (rows) is generalized binary. A generalized binary assignment in which the two counts are 0 and 1 is said to be binary.

Following Kiefer (1975), the treatment information matrix for a design is said to be completely symmetric if its elements are constant on the diagonal and constant off the diagonal. The condition $q = 0$ allows the possibility that C_{d_N} is completely symmetric, which will play an important role in the optimality proofs in this article. When C_{d_N} is completely symmetric, d_N will be termed a completely symmetric design, or CSD for short.

From (1), C_d can be written as

$$\begin{aligned} C_d &= C_{d_N} - \frac{1}{b}M_d(I_k - \frac{1}{k}J_k)M_d' \qquad (3)\\ &\leq C_{d_N} \qquad (4) \end{aligned}$$

where "$\leq$" in (4) is with respect to the Loewner ordering, by virtue of the fact that $M_d(I_k - \frac{1}{k}J_k)M_d'$ is nonnegative definite. These relationships will be useful in what follows. It can be immediately noted that for any optimality criterion respecting the Loewner ordering, a design d whose column component design is optimal, and for which $M_dM_d' - \frac{1}{k}rr' = M_d(I_k - \frac{1}{k}J_k)M_d' = 0$, is optimal over $\mathcal{D}(v, b, k)$. Now $M_d(I_k - \frac{1}{k}J_k)M_d' = 0$ if, and only if, m_{dil} is constant in l for each i, so $m_{dil} = m_{di}$ (say) for $i = 1 \ldots, v$; row-column designs fulfilling this condition are said to be of *Youden type* (Das and Dey, 1990). If further m_{di} is constant in i, the design is said to be *row-regular*. This is the basis for proving optimality of regular Youden designs, which are row-regular row-column designs with optimal, completely symmetric, column component designs (see pp. 66-7 of Shah and Sinha, 1989). The settings explored in this article preclude row-regularity for designs that are equireplicate or nearly so.

The determination of optimal row-column designs is pursued here in terms of two distinct optimality criteria. Let $z_{d0} = 0 < z_{d1} \leq z_{d2} \leq \ldots \leq z_{dv-1}$ denote the eigenvalues of the information matrix C_d associated with $d \in \mathcal{D}(v, b, k)$. The E optimality criterion aims to maximize the smallest positive eigenvalue z_{d1} of C_d. In terms of contrast variances, an E-optimal design minimizes, over $d \in \mathcal{D}$, the maximum of $Var_d(\widehat{\sum_{i=1}^{v} l_i\tau_i})$ over all possible choices of normalized ($\sum_i l_i^2 = 1$) contrast coefficients. The MV criterion requires minimizing, over $d \in \mathcal{D}$, the maximum variance over all elementary treatment contrasts $\widehat{\tau_i - \tau_{i'}}$. That is, an MV-optimal design minimizes over $d \in \mathcal{D}$ the quantity

$$\Upsilon_d = \max_{i \neq i'} \frac{Var_d(\widehat{\tau_i - \tau_{i'}})}{\sigma^2}.$$

The E and MV criteria are both minimax criteria, and both criteria respect the Loewner ordering.

This article examines row-column designs with $k = 3$ from the perspectives of the E and MV optimality criteria. Section 2 develops a series of bounds that are useful in constructing the optimality arguments. The main results are in Section 3. Concluding discussion comprises Section 4.

2. Preliminaries

Useful lemmas, to be employed in the optimality proofs in Section 3, are established here. At the heart of these are two results, stated first, due to M. Jacroux. For information matrix $C_d = (c_{dii'})$ define the quantity $\theta_{dii'}$ for each $i \neq i'$ by $\theta_{dii'} = c_{dii} + c_{di'i'} - 2c_{dii'}$.

From (2), $\theta_{dii'}$ may be written as

$$\theta_{dii'} = r_{di} + r_{di'} - \frac{1}{bk}[b\sum_{j=1}^{b}(n_{dij} - n_{di'j})^2 + k\sum_{l=1}^{k}(m_{dil} - m_{di'l})^2] + \frac{(r_{di} - r_{di'})^2}{bk} \tag{5}$$

Lemma 1: (Jacroux, 1982) Let $d \in \mathcal{D}(v,b,k)$ have information matrix $C_d = (c_{dii'})$.

(i) If M is a subset of $\{1,2,\ldots,v\}$ of size m, $1 \le m \le v-1$, then
$z_{d1} \le (v/m(v-m))(\sum_{i\in M} c_{dii} + \sum_{i\in M}\sum_{i'(\neq i)\in M} c_{dii'})$;

(ii) $z_{d1} \le \theta_{dii'}/2$ for $i, i' = 1,\ldots,v$ $(i \neq i')$.

Moreover, in light of (4), d in the right-hand side of each of these two inequalities can be replaced by d_N, providing two additional (and possibly less sharp) upper bounds for z_{d1}.

Lemma 2: (Jacroux, 1983) Let $d \in \mathcal{D}(v,b,k)$ have information matrix $C_d = (c_{dii'})$. For any $i \neq i'$,

$$(1/\sigma^2)Var_d(\widehat{\tau_i - \tau_{i'}}) \ge \frac{4}{\theta_{dii'}} \ge \frac{4}{\theta_{d_N ii'}}.$$

Proofs for Lemmas 1 and 2 may be found in the cited papers. Proofs for the lemmas that follow appear in the appendix.

Lemma 3: A row-column design $d \in \mathcal{D}(v,b,3)$ for which the column design is a CSD with $c_{d_N^* ii} = \frac{r(k-1)}{k}$ for all $i = 1,\ldots,v$ satisfies

$$z_{d1} = \frac{\lambda v}{3} - \frac{1}{2b}\left(a_{1d} + \sqrt{2a_{2d} - a_{1d}^2}\right)$$

where $a_{1d} = tr(B_d)$ and $a_{2d} = tr(B_d^2)$ for the matrix $B_d = M_d M_d' - \frac{1}{k} r_d r_d'$.

Lemma 4: A row-column design $d \in \mathcal{D}(v,b,3)$ for which $r_{di} \le r-1$ for some i satisfies

$$\theta_{dii'} \le \frac{2\lambda v}{3} + \frac{2[v(q-1)+p]}{3(v-1)}.$$

for some $i' \neq i$.

Lemma 5: A row-column design $d \in \mathcal{D}(v,b,k)$ for which some treatment with replication r is nonbinary in columns satisfies

$$\theta_{dii'} \le \frac{2vr(k-1)}{k(v-1)} - \frac{2}{k} + \frac{p(k-1)-4}{k(v-1)}$$

for some $i \neq i'$.

Lemma 6: For the row-column setting $\mathcal{D}(v,b,k)$ with $p \le v-2$, let $d \in \mathcal{D}(v,b,k)$ have $r_{di} \ge r$ for all i. If two treatments $i \neq i'$ with $r_{di} = r = r_{di'}$ have $\lambda_{dii'} \le \lambda - 1$, then

$$\theta_{dii'} \le \frac{2(\lambda v + q - 1)}{k}.$$

Lemma 7: Let $d \in \mathcal{D}(v, b, 3)$ satisfy

(i) $r_{d1} = \ldots = r_{d,v-p} = r$,

(ii) $r_{d,v-p+1} = \ldots = r_{dv} = r + 1$, and

(iii) the treatments with replication $r + 1$ are generalized binary in rows.

If some treatment with replication r is not generalized binary in rows, where $r \equiv 1 \pmod 3$ and $p < (v + 3)/2$, or where $r \equiv 2 \pmod 3$, then

$$\theta_{dii'} \quad \leq \quad 2r - \frac{1}{3b}[b\sum_{j=1}^{b}(n_{dij} - n_{di'j})^2 + 18]$$

for some $i \neq i'$ with $i, i' \leq v - p$.

Lemma 8: Let $d \in \mathcal{D}(v, b, 3)$ satisfy

(i) $r_{d1} = \ldots = r_{d,v-p} = r$, and

(ii) $r_{d,v-p+1} = \ldots = r_{dv} = r + 1$.

If some treatment with replication $r+1$ is not generalized binary in rows, where $r \equiv 1$ (mod 3) and $p < (v + 3)/2$, or where $r \equiv 2 \pmod 3$ and $p < (v + 6)/4$, then

$$\theta_{dii'} \quad \leq \quad 2r + 1 - \frac{1}{3b}[b\sum_{j=1}^{b}(n_{dij} - n_{di'j})^2 + 14]$$

for some $i \leq v - p$ and some $i' \geq v - p + 1$.

3. Main Results

As stated in Section 1, design classes $\mathcal{D}(v, b, k)$ for which the concurrence excess is $q = 0$ are the focus in the theorems to follow. Optimality results for the E and MV criteria will now be derived for the plot excess p taking the values $p = 1$ and $p = 0$. The same techniques can be successfully applied for some larger p, but for reasons of space are not pursued here.

3.1. Designs with $p = 1$

Theorem 1: Let $\mathcal{D}(v, b, 3)$ be the class of $3 \times b$ row-column designs for which the plot excess is $p = 1$, the concurrence excess is $q = 0$, and $b > v + 2$. If the target replication r satisfies $r \equiv 1 \pmod 3$ and $d^* \in \mathcal{D}(v, b, 3)$ satisfies

(i) the column design of d^* is a CSD with $c_{d^*_N ii} = \frac{r(k-1)}{k}$ for all $i = 1, \ldots, v$, and

(ii) the row design of d^* is generalized binary, then d^* is E-optimal in $\mathcal{D}(v, b, 3)$, and any design failing either (i) or (ii) is E-inferior to d^*. In particular, any design which is binary in columns is E-inferior to d^*.

Proof: Condition (i) implies that $r_{d^*1} = \ldots = r_{d^*v-1} = r$ and thus $r_{d^*v} = r + 1$. It then further implies that treatment v is nonbinary in columns of d^*.

For each treatment i, define the vector of row counts m_{di} for design d as $m_{di} = (m_{di1}, m_{di2}, m_{di3})$, the i^{th} row of M_d. With appropriate labeling of treatments and ordering of rows, condition (ii) implies that $m_{d^*v} = (\frac{r+2}{3}, \frac{r+2}{3}, \frac{r-1}{3})$ and so $m_{d^*i} = (\frac{r+2}{3}, \frac{r-1}{3}, \frac{r-1}{3})$ for $i = 1, \ldots, \frac{v-2}{3}$; $m_{d^*i} = (\frac{r-1}{3}, \frac{r+2}{3}, \frac{r-1}{3})$ for $i = \frac{v+1}{3}, \ldots, \frac{2(v-2)}{3}$; and $m_{d^*i} = (\frac{r-1}{3}, \frac{r-1}{3}, \frac{r+2}{3})$ for $i = \frac{2v-1}{3}, \ldots, v-1$. With M_{d^*} so determined, write $v_1 = v_2 = (v-2)/3$, $v_3 = (v+1)/3$ and $v_4 = 1$ for the frequencies of its four distinct rows. It follows that the symmetric matrix $B_{d^*} = M_{d^*}(I_3 - \frac{1}{3}J_3)M'_{d^*}$ is the partitioned block matrix with diagonal blocks $\frac{2}{3}J_{v_g v_g}$ for $g = 1, \ldots, 4$, and blocks $-\frac{1}{3}J_{v_g v_h}$ for $g < h \leq 3$, $\frac{1}{3}J_{v_g v_4}$ for $g = 1, 2$, and $-\frac{2}{3}J_{v_3 v_4}$ above the diagonal. It is now simple to calculate $a_{1d^*} = tr(B_{d^*}) = \frac{2v}{3}$ and $a_{2d^*} = tr(B^2_{d^*}) = \frac{2(v^2+4)}{9}$, so that by Lemma 3,

$$z_{d^*1} = \frac{\lambda v}{3} - \frac{(v+2)}{3b}.$$

Let d be any other design with $r_{d1} \leq \ldots \leq r_{dv}$. Suppose d has $r_{di} \leq r - 1$ for some $i = 1, \ldots, v$. Then d is E-inferior to d^*, as by Lemma 1, and Lemma 4 with $p = 1$,

$$z_{d1} \leq \frac{\lambda v}{3} + \frac{v(q-1)+p}{3(v-1)} = \frac{\lambda v}{3} - \frac{1}{3} < \frac{\lambda v}{3} - \frac{(v+2)}{3b} = z_{d^*1}, \tag{6}$$

the last inequality due to the condition $b > v + 2$.

So assume d has $r_{di} \geq r$ for all i; with appropriate labeling of treatments, $r_{di} = r_{d^*i}$ for all i. If d has some treatment with replication r nonbinary in columns, Lemmas 1 and 5 give

$$z_{d1} \leq \frac{vr(k-1)}{k(v-1)} - \frac{1}{k} + \frac{p(k-1)-4}{2k(v-1)} = \frac{\lambda v}{3} - \frac{1}{3} - \frac{1}{3(v-1)} < \frac{\lambda v}{3} - \frac{(v+2)}{3b} = z_{d^*1} \tag{7}$$

and again d is E-inferior to d^*.

So now suppose d has treatments $1, \ldots, v-1$ each binary in columns. If $\lambda_{dii'} \leq \lambda - 1$ for some $i, i' \leq v - 1$ $(i \neq i')$, then by Lemmas 1 and 6,

$$z_{d1} \leq \frac{\lambda v}{3} - \frac{1}{3} < \frac{\lambda v}{3} - \frac{(v+2)}{3b} = z_{d^*1} \tag{8}$$

and again d is E-inferior to d^*.

Thus for d to be E-admissible, it must have $r_{d1} = \ldots = r_{dv-1} = r$, and all treatments replicated r times are binary in columns with $\lambda_{dii'} = \lambda$ for all $i \neq i'$ where $i, i' = 1, \ldots, v-1$. If in addition treatment v is binary in columns then $\sum_{i=1}^{v-1} \lambda_{dvi} = 2(r+1)$, implying

$$\sum_{i=1}^{v-2} \sum_{i'=i+i}^{v-1} \lambda_{dii'} \;=\; \sum_{i=1}^{v-1} \sum_{i'=i+i}^{v} \lambda_{dii'} - \sum_{i=1}^{v-1} \lambda_{dvi} \;=\; 3b - 2(r+1) \;=\; (v-2)r - 1,$$

the last equality because $bk = 3b = vr + 1$. Thus the average concurrence among treatments replicated r times is

$$\frac{(v-2)r-1}{(v-1)(v-2)/2} = \frac{(v-2)r-1}{(v-2)r/\lambda} = \lambda - \frac{1}{(v-2)r}$$

so that $\lambda_{dii'} < \lambda$ for some $i \neq i'$, $i, i' < v$. As shown above (see (8)), this implies d is E-inferior to d^*, so d cannot have treatment v binary in columns.

Now $c_{dvv} \leq c_{d_N vv} = r + 1 - (\sum_{j=1}^{b} n_{dvj}^2)$ and nonbinarity of treatment v in columns implies $\sum_{j=1}^{b} n_{dvj}^2 \in \{r+3, r+5, r+7\ldots\}$. If $\sum_{j=1}^{b} n_{dvj}^2 \geq r+5$ then $c_{dvv} \leq (r+1)-(r+5)/3 = 2(r-1)/3$ and by Lemma 1(i),

$$z_{d1} \leq \left(\frac{v}{v-1}\right) c_{dvv} = \frac{2v(r-1)}{3(v-1)} = \frac{\lambda v(r-1)}{3r}$$

$$\Rightarrow 3r(z_{d^*1} - z_{d1}) \geq \lambda vr - \frac{(v+2)r}{b} - \lambda v(r-1) = \lambda v - \frac{(v+2)r}{b} = \lambda + 2r - \frac{(v+2)r}{b} > 0,$$

again invoking $b > v + 2$. So d must have $\sum_{j=1}^{b} n_{dvj}^2 = r + 3 \Rightarrow c_{d_N vv} = 2r/3 = c_{d_N ii}$ for $i = 1, \ldots, v-1$. The same argument as in (8) implies $\lambda_{dvi} \geq \lambda$ for $i < v$ and hence $\lambda_{dii'} \geq \lambda$ for all $i \neq i' \Rightarrow \lambda_{dii'} = \lambda$ for all $i \neq i'$. This establishes that E-admissibility of d requires $C_{d_N} = C_{d_N^*}$, that is, d and d^* have identical information matrices for their column component designs. If C_d differs from C_{d^*}, it can do so only in the term $M_d M_d'$.

If the row design of d is generalized binary in rows then d fulfills the conditions of the theorem, that is, d is a version of d^*. So suppose d is not generalized binary in rows. Then $a_{1d} = \sum_i (\mu_{dii} - \frac{r_{di}^2}{k}) \geq (2 + \sum_i \mu_{d^*ii}) - (\sum_i \frac{r_{d^*i}^2}{k}) = a_{1d^*} + 2 = \frac{2(v+3)}{3}$. By Lemma 3,

$$z_{d1} = \frac{\lambda v}{3} - \frac{1}{2b}(a_{1d} + \sqrt{2a_{2d} - a_{1d}^2}) \leq \frac{\lambda v}{3} - \frac{1}{2b} a_{1d} \leq \frac{\lambda v}{3} - \frac{(v+3)}{3b} < \frac{\lambda v}{3} - \frac{(v+2)}{3b} = z_{d^*1},$$

completing the proof. □

The strict inequality $b > v+2$, employed in equations (6) to (8), serves only to guarantee that any design failing either condition (i) or (ii) will be E-inferior to d^*. With that note, E optimality of d^* also holds for $b = v + 2$. Following Theorem 2, it will be shown that when $b = v + 2$ it is possible to find other E-optimal designs not satisfying (i) and (ii). This issue does not arise in the MV optimality proof of Theorem 2, which otherwise covers the same settings $\mathcal{D}(v, b, k)$ as in Theorem 1.

Theorem 2: Let $\mathcal{D}(v, b, 3)$ be the class of $3 \times b$ row-column designs for which the plot excess is $p = 1$, the concurrence excess is $q = 0$, and $b \geq v + 2$. If the target replication r satisfies $r \equiv 1 \pmod 3$ and $d^* \in \mathcal{D}(v, b, 3)$ satisfies

(i) the column design of d^* is a CSD with $c_{d_N^* ii} = \frac{r(k-1)}{k}$ for all $i = 1, \ldots, v$, and

(ii) the row design of d^* is generalized binary, then d^* is MV-optimal in $\mathcal{D}(v,b,3)$, and any design failing either (i) or (ii) is MV-inferior to d^*. In particular, any design which is binary in columns is MV-inferior to d^*.

Proof: C_{d^*}, as determined in the proof of Theorem 1, has generalized group divisible structure (see *e.g.* Srivastav and Morgan, 1998) with four groups, call them V_g for $g = 1,2,3,4$ with $|V_g| = v_g$. Accordingly, $Var_{d^*}(\widehat{\tau_i - \tau_{i'}})$ depends only on group membership for i and i'. Writing $T_{d^*} = C_{d^*} + (\frac{\lambda}{3} - \frac{1}{3b})J_{vv}$, a generalized inverse of C_{d^*} is $T_{d^*}^{-1}$, from which the pairwise variances arising from d^*, displayed in Table 1, are easily found. The reader may check that these five variances satisfy $var_1 < var_4 < var_2 < var_3 < var_5$. For the purposes of this proof $var_5 = \Upsilon_{d^*}$ is rewritten as

$$\Upsilon_{d^*} = \frac{6}{\lambda v}\left[1 + \frac{4}{\lambda v b - (v+2)}\right] = \frac{6}{\lambda v}\left[\frac{\lambda v b - (v-2)}{\lambda v b - (v+2)}\right] = \frac{2}{\frac{\lambda v}{3} - \frac{4}{3}\frac{\lambda v}{\lambda v b - (v-2)}}.$$

Let d be any other design with $r_{d1} \leq \ldots \leq r_{dv}$. First suppose d has $r_{di} \leq r - 1$ for some $i = 1,\ldots,v$. Assume $r_{d1} \leq r - 1$. From Lemmas 2 and 4,

$$\Upsilon_d \geq \frac{2}{\frac{\lambda v}{3} - \frac{1}{3}} \geq \frac{2}{\frac{\lambda v}{3} - \frac{(v+2)}{3b}} > \frac{2}{\frac{\lambda v}{3} - \frac{4}{3}\frac{\lambda v}{\lambda v b - \lambda v}} > \frac{2}{\frac{\lambda v}{3} - \frac{4}{3}\frac{\lambda v}{\lambda v b - (v-2)}} = \Upsilon_{d^*} \qquad (9)$$

and therefore d^* is MV-better than d.

Next, suppose d has $r_{di} \geq r$ for $i = 1,\ldots,v$. Since $p = 1$, there are $v - 1$ treatments replicated r times and one treatment replicated $r + 1$ times. Assume $r_{d1} = \ldots = r_{dv-1} = r$ and $r_{dv} = r + 1$.

Suppose d has some treatment with replication r nonbinary in columns. Then by the same calculation as in (7), there is some $\theta_{dii'} \leq \frac{2\lambda v}{3} - \frac{2(v+2)}{3b}$. Applying Lemma 2 as shown in (9) gives $\Upsilon_d > \Upsilon_{d^*}$. The same θ bound, and thus the same result for Υ_d, is found (see (8)) if two treatments $i < i' < v$ are binary in columns with $\lambda_{dii'} \leq \lambda - 1$. Thus the column component design for d must be binary in treatments $1,\ldots,v-1$, and it must have $\lambda_{dii'} \geq \lambda$ for any two of these $v - 1$ treatments.

If treatment v is binary in columns of d, then just as shown in the proof of Theorem 2, $\lambda_{dii'} \leq \lambda - 1$ for some $i < i' < v$ and so, as established in the preceding paragraph, $\Upsilon_d > \Upsilon_{d^*}$. If treatment v is nonbinary in columns and $\sum_j n_{dvj}^2 \geq r + 5$, then as established in the proof

Table 1: Pairwise variances for d^* of Theorem 2

$i \neq i'$	$\frac{1}{\sigma^2} Var_{d^*}(\widehat{\tau_i - \tau_{i'}})$
$i, i' \in V_g,\ g = 1,2,3$	$var_1 = \frac{6}{\lambda v}$
$i \in V_1,\ i' \in V_2$	$var_2 = \frac{6}{\lambda v}[1 + \frac{3}{\lambda v b - (v-2)}]$
$i \in V_1$ or $V_2,\ i' \in V_3$	$var_3 = \frac{6}{\lambda v}[1 + \frac{3(\lambda v b - (v-1))}{(\lambda v b - (v+2))(\lambda v b - (v-2))}]$
$i \in V_1$ or $V_2,\ i' \in V_4$	$var_4 = \frac{6}{\lambda v}[1 + \frac{2(\lambda v b - 3(v-1))}{(\lambda v b - (v+2))(\lambda v b - (v-2))}]$
$i \in V_3,\ i' \in V_4$	$var_5 = \frac{6}{\lambda v}[1 + \frac{4}{\lambda v b - (v+2)}]$

of Theorem 1, $c_{dvv} \leq 2(r-1)/3$. This yields

$$\frac{\sum_{i=1}^{v-1} \lambda_{dvi}}{v-1} = \frac{3c_{dvv}}{v-1} \leq \frac{2(r-1)}{v-1} = \frac{\lambda(v-1)-2}{v-1}$$

implying that for some $i < v$, say $i = 1$, $\lambda_{dv1} \leq \lambda - 1$. Then θ_{dv1} is

$$\theta_{dv1} = \frac{2r}{3} + \frac{2(r-1)}{3} + \frac{2\lambda_{dv1}}{3} \leq \frac{4r-1+2(\lambda-1)}{3} = \frac{2\lambda v - 2}{3}$$

which upon applying Lemma 2 as shown in (9) again gives $\Upsilon_d > \Upsilon_{d^*}$. Thus $\sum_j n_{dvj}^2 = r+3$, $c_{d_N vv} = 2r/3$, and just as for E-admissibility in the proof of Theorem 2, it follows that that MV-admissibility of d requires $C_{d_N} = C_{d_N^*}$.

It remains to consider the row component design for d. If the row design of d is generalized binary in rows then d fulfills the conditions of the theorem, that is, d is a version of d^*. So suppose d is not generalized binary in rows.

First consider if treatment i, for some $i \leq v-1$, is not generalized binary in rows, but treatment v is. Note that $\sum_{j=1}^{b}(n_{dij} - n_{di'j})^2 = 2(r-\lambda)$ for any $i < i' < v$. Lemma 7 now says there is $\theta_{dii'}$ for which

$$\theta_{dii'} \;\leq\; 2r - \frac{1}{3b}[2b(r-\lambda)+18] \;=\; \frac{2(\lambda vb - 9)}{3b}.$$

Employing this inequality in Lemma 2,

$$\Upsilon_d \;\geq\; \frac{4}{\theta_{dii'}} \;\geq\; \frac{6}{\lambda v}\left(\frac{\lambda vb}{\lambda vb - 9}\right) \;=\; \frac{6}{\lambda v}\left(1 + \frac{9}{\lambda vb - 9}\right) \;>\; \frac{6}{\lambda v}\left(1 + \frac{4}{\lambda vb - (v+2)}\right) \;=\; \Upsilon_{d^*}.$$

Suppose treatment v is not generalized binary in rows. It has been established above that $\sum_{j=1}^{b} n_{dvj}^2 = r+3$, so that treatment v occurs twice in one column, and once in each of $r-1$ columns. Let treatment 1 be the treatment that appears in the block with $n_{dvj} = 2$. Then $\sum_{j=1}^{b}(n_{dvj} - n_{d1j})^2 = 2(r-\lambda)+3$. Lemma 8 now says

$$\theta_{d1v} \;\leq\; 2r+1-\frac{1}{3b}[b\sum_{j=1}^{b}(n_{dij}-n_{di'j})^2+14] \;=\; \lambda(v-1)+1-\frac{[2(r-\lambda)+3]}{3}-\frac{14}{3b} \;=\; \frac{2(\lambda vb - 7)}{3b}.$$

By Lemma 2,

$$\Upsilon_d \;\geq\; \frac{4}{\theta_{div}} \;\geq\; \frac{6}{\lambda v}\left(\frac{\lambda vb}{\lambda vb - 7}\right) \;=\; \frac{6}{\lambda v}\left(1 + \frac{7}{\lambda vb - 7}\right) > \frac{6}{\lambda v}\left(1 + \frac{4}{\lambda vb - (v+2)}\right) \;=\; \Upsilon_{d^*}.$$

□

Example 1: Consider these designs $d_1, d_2 \in \mathcal{D}(5,7,3)$:

$$d_1: \begin{array}{ccccccc} 5 & 1 & 3 & 2 & 4 & 4 & 3 \\ 5 & 2 & 1 & 5 & 2 & 3 & 4 \\ 4 & 5 & 5 & 3 & 1 & 1 & 2 \end{array} \qquad d_2: \begin{array}{ccccccc} 4 & 1 & 1 & 2 & 5 & 4 & 3 \\ 1 & 2 & 5 & 3 & 2 & 3 & 4 \\ 5 & 5 & 3 & 1 & 4 & 5 & 2 \end{array}$$

Design d_1 satisfies the conditions of Theorems 1 and 2, and so is E-optimal and MV-optimal in $\mathcal{D}(5,7,3)$. As discussed following the proof of Theorem 1, since $b = v+2$ the E optimality may not be unique. Design d_2 fails both conditions (i) and (ii) of Theorem 1, but has the same E value as d_1. Thus d_2 is also E-optimal, even while being MV-inferior to d_1.

Example 2: Consider this design d^* for $v = 8, b = 19$, and $k = 3$:

1	2	3	4	5	6	7	1	3	4	5	8	8	7	2	6	1	2	7
2	3	4	5	6	7	1	8	2	8	4	5	6	1	7	3	3	4	8
4	5	6	7	1	2	3	2	8	3	8	6	1	4	5	7	5	6	8

All conditions of Theorem 1 and Theorem 2 are satisfied, so d^* is E- and MV-optimal in $\mathcal{D}(8,19,3)$. Any design that is binary in columns, or which is row-regular, is inferior to d^* on both of these criteria.

The column component designs for the design in Example 2, and for d_1 in Example 1, are two instances of an infinite series of completely symmetric block designs with $k = 3$ constructed by Morgan and Uddin (1995). Those designs have, for each $t \geq 1$, parameters

$$v = 3t+2,\ b = 3t^2+3t+1,\ k = 3,\ r = 3t+1,\ \lambda = 2,\ p = 1,\ q = 0. \tag{10}$$

Also, each of these block designs has $c_{dii} = r(k-1)/k = 2r/3$ for all i, so if the blocks are arranged as columns of a $3 \times b$ row-column design, condition (i) of the theorems will be satisfied. To satisfy condition (ii), treatments must then be ordered in each column so that each treatment appears in each row either t or $t+1$ times. Examples 1 and 2 demonstrate that this can be done for $t = 1, 2$. That it can be done for all t can be proven using systems of distinct representatives, illustrating application of a result due to Das and Dey (1989).

Lemma 9: (Das and Dey, 1989) If a block design with v treatments in b blocks of size k has treatment replication numbers $r_i = km_i$ for integer values m_i and $i = 1,\dots v$, then the blocks can be arranged as columns of a $k \times b$ row-column design so that, for $i = 1,\dots v$, treatment i occurs in each row m_i times.

The row-column arrangement guaranteed by Lemma 9 is a Youden type design.

The Morgan and Uddin (1995) construction of CSDs with parameters (10) is divided into four cases. One of those (their Case 2(a)) will be employed here to show how condition (ii) can be achieved; the other three cases are handled similarly. Designs in this case comprise the subseries of (10) with $t = 4w+2$ having $v = 12w+8$ and $r = 12w+7$. The treatment symbols in that construction are ∞_1, ∞_2, and the integers (mod $12w+6$). To apply Lemma 9, partition the blocks of this subseries into four subdesigns as shown in Table 2. Counting replications in each of the subdesigns S_1, S_2, and S_3 shows that, by Lemma 9, each can can be arranged as a Youden type design on the treatments involved. Thus taken together, these three subdesigns form a $3 \times (b-(v+1)/3)$ row-column design on all v treatments that is row-regular, all treatments being replicated $r-1$ times. Adding the $(v+1)/3$ columns of S_4 as displayed in Table 2, for which each treatment except ∞_1 appears exactly once, gives the required design satisfying condition (ii) of the theorems.

Example 2 contains the Case 2(a) design with $w = 0$. In that example, the first seven columns (making a Youden design) are subdesign S_1, the next nine (making a Youden type design) are subdesign S_3, and the last three are subdesign S_4. Since $w = 0$, subdesign S_2 is empty.

For other CSDs having block size $k = 3$ with plot excess $p \geq 1$, to which the methods of this article can be applied, see Morgan and Srivastav (2002).

3.2. Designs with $p = 0$

Theorem 3: Let $\mathcal{D}(v, b, 3)$ be the class of $3 \times b$ row-column designs for which the plot excess p and the concurrence excess q are both zero. If $d^* \in \mathcal{D}(v, b, 3)$ satisfies

(i) the column design of d^* is a BIBD, and

(ii) the row design of d^* is generalized binary,

then d^* is E-optimal and MV-optimal in $\mathcal{D}(v, b, 3)$.

Proof: If $r = bk/v$ is a multiple of 3 then d^* is a regular GYD, for which the result is already known. So only $r = \lambda(v-1)/2$ which is not a multiple of 3 need be considered. Condition (i) then implies that v is a multiple of 3, see (15) and (16) below. Write $r_1 = \lfloor r/3 \rfloor$ and $r_2 = r_1 + 1$.

Let $m_{di} = (m_{di1}, m_{di2}, m_{di3})$. For any equireplicated d which satisfies (ii), $m_{dil} \in \{r_1, r_2\}$ for every i and l, that is, m_{di} is some permutation of (r_1, r_1, r_2) or (r_1, r_2, r_2) as $r \equiv 1 \pmod 3$ or $r \equiv 2 \pmod 3$. In either case, there are exactly $\frac{v}{3}$ treatments corresponding to each of the three permutations, so that generalized binarity in rows induces a grouping of the treatments. By "treatments in group g" is meant those i for which m_{dig} is the distinct member of m_{di}, $g = 1, 2, 3$. For generalized binary d, $\mu_{dii'} = \mu_1 = \frac{r^2-1}{3}$ if i, i' are in different groups, and $\mu_{dii'} = \mu_1 + 1$ otherwise. Since the column design for d^* is a BIBD and d^* is generalized binary in rows, C_{d^*} has group divisible structure. In addition, B_{d^*} may be written in partitioned block form with matrices $\frac{2}{3} J_{\frac{v}{3}, \frac{v}{3}}$ along the main diagonal and matrices $-\frac{1}{3} J_{\frac{v}{3}, \frac{v}{3}}$ along the off-diagonal. It follows that $tr(B_{d^*}) = \frac{2v}{3}$ and $tr(B_{d^*}^2) = \frac{2v^2}{9}$. By Lemma 3, and the fact that

Table 2: Partition of blocks for CSDs with $t = 4w+2$ in (10). Blocks are displayed as rows for compactness. All integers are reduced mod $(12w+6)$.

Subdesign	Blocks
S_1	blocks of a BIBD($k = 3$, $\lambda = 1$) on the $v-1$ treatments ∞_2 and $1, 2, \ldots, 12w+6$
S_2	$(j,\ 5w{+}3{-}i{+}j,\ 5w{+}2{+}i{+}j)$ and $(j,\ 3w{+}1{-}i{+}j,\ 3w{+}1{+}i{+}j)$ for $i = 1, 2, \ldots, w$ and $j = 1, 2, \ldots, 12w+6$
S_3	$(\infty_1,\ i,\ 3w{+}1{+}i)$ for $i = 1, 2, \ldots, 12w+6$, $(\infty_2,\ j,\ 6w{+}3{+}j)$ for $i = 1, 2, \ldots, 6w+3$.
S_4	$(j,\ 4w{+}2{+}j,\ 8w{+}4{+}j)$ for $j = 1, 2, \ldots, 4w+2$, and $(\infty_1, \infty_1, \infty_2)$

$r > 3$ (see (15) and (16) below),

$$z_{d^*1} \;=\; \frac{\lambda v}{3} - \frac{2}{\lambda(v-1)} \;=\; \frac{\lambda v}{3} - \frac{1}{r} \;>\; \frac{\lambda v - 1}{3}. \tag{11}$$

Calculation of pairwise variances $\frac{1}{\sigma^2} Var_{d^*}(\widehat{\tau_i - \tau_{i'}})$ with d^* follows easily from the generalized group divisible form of B_{d^*} (and hence of C_{d^*}). They take only two values: $var_1 = \frac{6}{\lambda v}$, which is the "same group" variance, and $var_2 = \frac{6}{\lambda v}[1 + \frac{3}{\lambda vb - v}]$, which is the "different group" variance. This gives

$$\Upsilon_{d^*} \;=\; \frac{6}{\lambda v}\left(1 + \frac{3}{\lambda vb - v}\right) \;=\; \frac{2}{\frac{\lambda v}{3} - \frac{\lambda v}{\lambda vb - (v-3)}} \;<\; \frac{2}{\frac{\lambda v - 1}{3}}. \tag{12}$$

Now consider any $d \in \mathcal{D}(v,b,k)$ that does not satisfy (i). If d has $r_{di} \le r-1$ for some i, then putting $p = 0$ in Lemma 4, there is $i' \ne i$ such that

$$\theta_{dii'} \;\le\; \frac{2\lambda v}{3} - \frac{2v}{3(v-1)} \;<\; \frac{2(\lambda v - 1)}{3}. \tag{13}$$

With (13), it is immediate from Lemma 1(ii) and (11), and from Lemma 2 and (12), that d is E-inferior and MV-inferior to d^*.

Next, suppose d has $r_{di} = r$ for all i but is nonbinary in columns. By Lemma 5 with $p = 0$, there is $i \ne i'$ for which

$$\theta_{dii'} \;\le\; \frac{2\lambda v}{3} - \frac{2}{3} - \frac{4}{3(v-1)} \;<\; \frac{2(\lambda v - 1)}{3}. \tag{14}$$

The bound (14) is the same as found in (13), so again Lemmas 1(ii) and 2 show d is E-inferior and MV-inferior to d^*.

So now suppose d is equireplicate and binary in columns, but has $\lambda_{dii'} < \lambda$ for some $i \ne i'$, say $\lambda_{d12} \le \lambda - 1$. Now Lemma 6 with $p = 0$ says $\theta_{dii'} \;\le\; \frac{2(\lambda v-1)}{3}$ and yet again Lemmas 1(ii) and 2 show immediately that d is E-inferior and MV-inferior to d^*.

Now consider any d which satisfies (i) but not (ii). Since d is not generalized binary in rows,

$$a_{1d} = \sum_i (\mu_{dii} - \frac{r_{di}^2}{k}) \ge (2 + \sum_i \mu_{d^*ii}) - (\sum_i \frac{r_{d^*i}^2}{k}) = a_{1d^*} + 2 = \frac{2v}{3} + 2 = \frac{2(v+3)}{3}.$$

By Lemma 3,

$$z_{d1} = \frac{\lambda v}{3} - \frac{1}{2b}(a_{1d} + \sqrt{2a_{2d} - a_{1d}^2}) \le \frac{\lambda v}{3} - \frac{1}{2b} a_{1d} \le \frac{\lambda v}{3} - \frac{(v+3)}{3b} < \frac{\lambda v}{3} - \frac{1}{r} = z_{d^*1}.$$

Thus d^* is E-better than d.

To complete the proof, note that by virtue of (i), $\sum_{j=1}^{b} (n_{dij} - n_{di'j})^2 = 2(r - \lambda)$ for every $i \ne i'$. Lemma 7 with $p = 0$ says

$$\theta_{dii'} \;\le\; 2r - \frac{1}{3b}[2b(r-\lambda)b + 18] \;=\; \frac{2b(2r+\lambda) - 18}{3b} \;=\; \frac{2(\lambda vb - 9)}{3b}$$

for some $i \neq i'$. By Lemma 2,

$$\Upsilon_d \ \geq \ \frac{6}{\lambda v}\left(\frac{\lambda v b}{\lambda v b-9}\right) \ = \ \frac{6}{\lambda v}\left(1+\frac{9}{\lambda v b-9}\right) \ > \ \frac{6}{\lambda v}\left(1+\frac{4}{\lambda v b-(v+2)}\right) \ = \ \Upsilon_{d^*}.$$

and therefore d^* is MV-better than d. □

The BIBDs with $k = 3$ for which r is not a multiple of 3 are those with parameters

$$v = 6t+3,\ b = \frac{\lambda v(v-1)}{6},\ k=3,\ r=\frac{\lambda(v-1)}{2},\ \lambda \equiv 1 \text{ or } 2 \pmod 3, \tag{15}$$

and those with parameters

$$v = 6t,\ b = \frac{\lambda v(v-1)}{6},\ k=3,\ r=\frac{\lambda(v-1)}{2},\ \lambda \equiv 2 \text{ or } 4 \pmod 6, \tag{16}$$

and any $t \geq 1$. All of these designs are known (Hanani, 1961). Invoking Theorem 3.2 of Agrawal (1966) (also see Chai, 1998), they can all be arranged as row-column designs satifying condition (ii) of Theorem 3. Thus all produce row-column designs that are E-optimal and MV-optimal. Moreover, as can be seen in the proof, these designs are superior, with respect to both of these optimality criteria, to any design failing either condition (i) or (ii) of Theorem 3.

Example 3: The two designs shown here:

1	1	3	3	4	2	2	5	5	6
4	2	1	1	6	3	3	6	4	5
2	5	5	6	1	4	6	2	3	4

and

1	1	5	2	9	6	8	3	3	2	4	7
9	4	1	6	7	4	2	8	5	3	5	6
3	7	8	1	5	9	9	4	6	7	2	8

are E-optimal and MV-optimal in $\mathcal{D}(6, 10, 3)$ and $\mathcal{D}(9, 12, 3)$, respectively.

4. Discussion

An often employed property in determining optimal designs is maximal trace of the information matrix over the class of competing designs. The designs shown here to be E-optimal and MV-optimal do not have this property. The upper bound for $tr(C_d)$, achieved by Youden type designs with binary column components, is $2b$. Designs d^* satisfying Theorems 1 and 2 have $tr(C_{d^*}) = 2b - 2(v+b)/3b$, and those for Theorem 3 have $tr(C_{d^*}) = 2b-(2v/3b)$. All designs satisfying any of Theorems 1 to 3 are superior to any competing, maximal trace design. Indeed, they are superior to any competing design having larger trace, be that maximal or not. This is a property, see Kiefer (1975), shared with the nonregular generalized Youden designs, which also enjoy optimality properties without being of Youden type. The Youden designs are, however, binary or generalized binary in both rows and columns, while the designs satisfying Theorems 1 and 2 have nonbinarity in columns.

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APPENDIX

A. Proof of Lemma 3

Proof: From (3), the information matrix C_d for design d is

$$C_d = C_{d_N} - \frac{1}{b}B_d$$

where $B_d = M_d(I_3 - \frac{1}{3}J_3)M_d'$ is nonnegative definite. By assumption, C_{d_N} has $v-1$ eigenvalues equal to $\frac{\lambda v}{3}$. Since C_{d_N} is completely symmetric, and since B_d and C_{d_N} each have zero row sums, it follows that $C_d\mathbf{u} = \frac{\lambda v}{3}\mathbf{u} - \frac{1}{b}B_d\mathbf{u}$, where $\mathbf{u}$ is any eigenvector of B_d satisfying $\mathbf{u'1} = 0$. Thus the nonzero eigenvalues of C_d are $\frac{\lambda v}{3} - \frac{1}{b}\xi_i$ where $\xi_{d1} \geq \xi_{d2} \geq \ldots \geq \xi_{d,v-1}$ are the eigenvalues of B_d corresponding to eigenvectors $\mathbf{u}$ as identified above.

Next, since B_d is nonnegative definite, and since $r(B_d) \leq r(I_3 - \frac{1}{3}J_3) = 2$, it follows that $\xi_{di} = 0$ for $i \geq 3$.

Write $a_{1d} = tr(B_d) = \xi_{d1} + \xi_{d2}$ and $a_{2d} = tr(B_d^2) = \xi_{d1}^2 + \xi_{d2}^2$. Solving these two equations for the eigenvalues in terms of the trace terms gives

$$\xi_{d1} = \frac{1}{2}[a_{1d} + \sqrt{2a_{2d} - a_{1d}^2}] \quad \text{and} \quad \xi_{d2} = \frac{1}{2}[a_{1d} - \sqrt{2a_{2d} - a_{1d}^2}].$$

so that $z_{d1} = \frac{\lambda v}{3} - \frac{\xi_{d1}}{b}$ as claimed. □

B. Proof of Lemma 4

Proof: For any $d \in \mathcal{D}(v, b, k)$ (for any $k \geq 2$),

$$\begin{aligned} \theta_{dii'} \leq \theta_{d_N ii'} &= c_{d_N ii} + c_{d_N i'i'} - 2c_{d_N ii'} \\ &= r_{di} - \frac{\lambda_{dii}}{k} + r_{di'} - \frac{\lambda_{di'i'}}{k} + \frac{2\lambda_{dii'}}{k} \qquad (17) \\ &\leq \frac{k-1}{k}(r_{di} + r_{di'} + \frac{2\lambda_{dii'}}{k-1}), \qquad (18) \end{aligned}$$

the last inequality because $\lambda_{dii} \geq r_{di}$ and $\lambda_{di'i'} \geq r_{di'}$. Now write $\bar{\theta}_{di} = \sum_{i' \neq i} \theta_{dii'}/(v-1)$. Then under the conditions of the lemma, setting $k = 3$,

$$\begin{aligned} \bar{\theta}_{di} &\leq \frac{2}{3(v-1)}\sum_{i' \neq i}(r_{di} + r_{di'} + \lambda_{dii'}) \leq \frac{2}{3(v-1)}[(v-1)r_{di} + (vr + p - r_{di}) + 2r_{di}] \\ &\leq \frac{2}{3(v-1)}[v(r-1) + (vr + p)] = \frac{2}{3(v-1)}[\lambda v(v-1) + v(q-1) + p]. \end{aligned}$$

Since for some $i' \neq i$, $\theta_{dii'} \leq \bar{\theta}_{di}$, the result is proven. □

C. Proof of Lemma 5

Proof: Suppose treatment i having $r_{di} = r$ is nonbinary in columns. So $\lambda_{dii} \geq r + 2$ and

$\sum_{i'\neq i}\lambda_{dii'} \leq r(k-1)-2$. Then from (17),

$$\begin{aligned}\sum_{i'\neq i}^{v}\theta_{dii'} &\leq \sum_{i'\neq i}^{v}[r_{di}-\frac{\lambda_{dii}}{k}+r_{di'}-\frac{\lambda_{di'i'}}{k}+\frac{2\lambda_{dii'}}{k}] \\ &\leq (v-1)r-\frac{(r+2)}{k}+(\frac{k-1}{k})[(v-1)r+p]+\frac{2[r(k-1)-2]}{k} \\ &= \frac{1}{k}\left\{2vr(k-1)-2(v-1)+[p(k-1)-4]\right\}.\end{aligned}$$

Since for some i', $\theta_{dii'} \leq \bar{\theta}_{di.} = \sum_{i'\neq i}^{v}\theta_{dii'}/(v-1)$, the result follows. □

D. Proof of Lemma 6

Proof: From (18),

$$\theta_{dii'} \leq \frac{k-1}{k}(r_{di}+r_{di'}+\frac{2\lambda_{dii'}}{k-1}) \leq \frac{2}{k}[r(k-1)+\lambda-1].$$

□

E. Proof of Lemma 7

Proof: For any two treatments $i \neq i'$ having replication r, equation (5) says

$$\theta_{dii'} = 2r-\frac{1}{3b}[b\sum_{j=1}^{b}(n_{dij}-n_{di'j})^2+3\sum_{l=1}^{3}(m_{dil}-m_{di'l})^2].$$

It will be shown that for some such $i \neq i'$, $\sum_{l=1}^{3}(m_{dil}-m_{di'l})^2 \geq 6$, establishing the result. This is done in two cases, depending on the r (mod 3) value.

Case 1: $r \equiv 1 \pmod 3$ and $p < (v+3)/2$.

Suppose treatment 1 is not generalized binary in rows. Then without loss of generality (WLOG), $m_{d11} \leq (r-4)/3$. There are two subcases to consider.

Case 1(a): $m_{di1} \geq (r+2)/3$ for some $i \leq v-p$. So take $m_{d21} \geq (r+2)/3$ and write $\overline{m}_{i(1)} = \sum_{l=2}^{3} m_{dil}/2$. We have

$$\overline{m}_{1(1)} \geq \frac{r-\frac{r-4}{3}}{2} = \frac{r+2}{3} \quad \text{and} \quad \overline{m}_{2(1)} \leq \frac{r-\frac{r+2}{3}}{2} = \frac{r-1}{3}$$

so that

$$\begin{aligned}\sum_{l=1}^{3}(m_{d1l}-m_{d2l})^2 &\geq (m_{d11}-m_{d21})^2+2(\overline{m}_{1(1)}-\overline{m}_{2(1)})^2 \\ &\geq \left(\frac{(r-4)}{3}-\frac{(r+2)}{3}\right)^2+2\left(\frac{(r+2)}{3}-\frac{(r-1)}{3}\right)^2 = 6.\end{aligned}$$

Case 1(b): $m_{di1} \leq (r-1)/3$ for $i = 2, \ldots, v-p$. Then

$$\sum_{i=1}^{v} m_{di1} = b = \frac{vr+p}{3} \leq \frac{r-4}{3} + (v-p-1)\frac{r-1}{3} + \sum_{i=v-p+1}^{v} m_{di1}$$
$$\Rightarrow \sum_{i=v-p+1}^{v} m_{di1} \geq \frac{v+pr+3}{3}$$

Also, since treatments $i > v-p$ are generalized binary in rows, $\sum_{i=v-p+1}^{v} m_{di1} \leq p(r+2)/3$. Hence we must have

$$p(r+2)/3 \geq \frac{v+pr+3}{3} \quad \Leftrightarrow \quad p \geq \frac{v+3}{2}$$

contradicting the requirement on p and thus showing that Case 1(b) cannot occur.

Case 2: $r \equiv 2 \pmod 3$

Suppose treatment 1 is not generalized binary in rows. Then WLOG $m_{d11} \geq (r+4)/3$. Again there are two subcases to consider.

Case 2(a): $m_{di1} \leq (r-2)/3$ for some $i \leq v-p$. So take $m_{d21} \leq (r-2)/3$ and with the notation employed in Case 1(a),

$$\overline{m}_{1(1)} \leq \frac{r - \frac{r+4}{3}}{2} = \frac{r-2}{3} \quad \text{and} \quad \overline{m}_{2(1)} \geq \frac{r - \frac{r-2}{3}}{2} = \frac{r+1}{3}$$

so that

$$\sum_{l=1}^{3}(m_{d1l} - m_{d2l})^2 \geq (m_{d11} - m_{d21})^2 + 2(\overline{m}_{1(1)} - \overline{m}_{2(1)})^2$$
$$\geq \left(\frac{(r+4)}{3} - \frac{(r-2)}{3}\right)^2 + 2\left(\frac{(r-2)}{3} - \frac{(r+1)}{3}\right)^2 = 6.$$

Case 2(b): $m_{di1} \geq (r+1)/3$ for $i = 2, \ldots, v-p$. Since treatments $i > v-p$ are generalized binary in rows, $m_{di1} = (r+1)/3$ for $i = v-p+1, \ldots, v$. Thus

$$3\sum_{i=1}^{v} m_{di1} = bk = vr+p \geq 3\left[\frac{r+4}{3} + (v-p-1)\frac{r+1}{3} + p\frac{(r+1)}{3}\right]$$
$$= vr+p+3,$$

a contradiction, showing that Case 2(b) cannot occur. □

F. Proof of Lemma 8

Proof: For any two treatments $i \neq i'$ having replication r and $r+1$ respectively, equation (5) says

$$\theta_{dii'} = 2r+1 - \frac{1}{3b}[b\sum_{j=1}^{b}(n_{dij} - n_{di'j})^2 + 3\sum_{l=1}^{3}(m_{dil} - m_{di'l})^2 - 1].$$

It will be shown that for some such $i \neq i'$, $\sum_{l=1}^{3}(m_{dil} - m_{di'l})^2 \geq 5$, establishing the result. This is done in two cases, depending on the $r \pmod 3$ value.

Case 1: $r \equiv 1 \pmod 3$ and $p < (v+3)/2$.

Suppose treatment v is not generalized binary in rows. Then WLOG $m_{dv1} \geq (r+5)/3$. There are two subcases to consider.

Case 1(a): $m_{di1} \leq (r-1)/3$ for some $i \leq v-p$. So take $m_{d11} \leq (r-1)/3$ and write $\overline{m}_{i(1)} = \sum_{l=2}^{3} m_{dil}/2$. We have

$$\overline{m}_{1(1)} \;\geq\; \frac{r - \frac{r-1}{3}}{2} \;=\; \frac{2r+1}{6} \quad \text{and} \quad \overline{m}_{v(1)} \;\leq\; \frac{r - \frac{r+5}{3}}{2} \;=\; \frac{r-1}{3}$$

so that

$$\begin{aligned}\sum_{l=1}^{3}(m_{d1l} - m_{dvl})^2 \;&\geq\; (m_{d11} - m_{dv1})^2 + 2(\overline{m}_{1(1)} - \overline{m}_{v(1)})^2 \\ &\geq\; \left(\frac{(r-1)}{3} - \frac{(r+5)}{3}\right)^2 + 2\left(\frac{(r-1)}{3} - \frac{(2r+1)}{6}\right)^2 \;=\; 4\frac{1}{2},\end{aligned}$$

and hence the integer $\sum_{l=1}^{3}(m_{d1l} - m_{dvl})^2$ is at least 5.

Case 1(b): $m_{di1} \geq (r+2)/3$ for $i = 1, \ldots, v-p$. Here p must satisfy $p \geq 2$, for otherwise,

$$3\sum_{i=1}^{v} m_{di1} = bk = vr + p \;\;\geq\;\; 3\left[(v-1)\frac{r+2}{3} + \frac{(r+5)}{3}\right] \;=\; vr + 2v + 3,$$

a contradiction. It is now claimed that $m_{di1} \leq \frac{r-4}{3}$ for some $i \geq v-p+1$, for if not, employing the condition $p < (v+3)/2$,

$$\begin{aligned}0 \;=\; 3[(\sum_{i=1}^{v} m_{di1}) - b] \;&=\; 3\left[(v-p)\frac{r+2}{3} + (p-1)\frac{(r-1)}{3} + \frac{(r+5)}{3} - \frac{(vr+p)}{3}\right] \\ &=\; 2v - 4p + 6 \;>\; 2v - 4\frac{v+3}{2} + 6 \;=\; 0,\end{aligned}$$

another contradiction. Hence WLOG $m_{dv-1,1} \leq (r-4)/3$, so that with the same notation as in Case 1(a),

$$\overline{m}_{1(1)} \;\leq\; \frac{r - \frac{r+2}{3}}{2} \;=\; \frac{r-1}{3} \quad \text{and} \quad \overline{m}_{v-1(1)} \;\geq\; \frac{r+1-\frac{r-4}{3}}{2} \;=\; \frac{2r+7}{6}.$$

Consequently,

$$\begin{aligned}\sum_{l=1}^{3}(m_{d1l} - m_{dv-1,l})^2 \;&\geq\; (m_{d11} - m_{dv-1,1})^2 + 2(\overline{m}_{1(1)} - \overline{m}_{v-1(1)})^2 \\ &\geq\; \left(\frac{(r+2)}{3} - \frac{(r-4)}{3}\right)^2 + 2\left(\frac{(r-1)}{3} - \frac{(2r+7)}{6}\right)^2 \;=\; 8\frac{1}{2}.\end{aligned}$$

Case 2: $r \equiv 2 \pmod 3$

Suppose treatment v is not generalized binary in rows. Then WLOG $m_{dv1} \geq (r+4)/3$. Again there are two subcases to consider.

Case 2(a): $m_{di1} \leq (r-2)/3$ for some $i \leq v-p$. So take $m_{d11} \leq (r-2)/3$ and with the notation employed in Case 1(a),

$$\overline{m}_{1(1)} \;\geq\; \frac{r-\frac{r-2}{3}}{2} \;=\; \frac{r+1}{3} \quad \text{and} \quad \overline{m}_{v(1)} \;\leq\; \frac{r+1-\frac{r+4}{3}}{2} \;=\; \frac{2r-1}{6}$$

so that

$$\begin{aligned}\sum_{l=1}^{3}(m_{d1l}-m_{dvl})^2 &\geq (m_{d11}-m_{dv1})^2+2(\overline{m}_{1(1)}-\overline{m}_{v(1)})^2 \\ &\geq \left(\frac{(r-2)}{3}-\frac{(r+4)}{3}\right)^2+2\left(\frac{(r+1)}{3}-\frac{(2r-1)}{6}\right)^2 \;=\; 4\frac{1}{2}\end{aligned}$$

implying $\sum_{l=1}^{3}(m_{d1l}-m_{dvl})^2 \geq 5$.

Case 2(b): $m_{di1} \geq (r+1)/3$ for $i=1,\ldots,v-p$. As in Case 1(b), p must satisfy $p \geq 2$, for otherwise

$$3\sum_{i=1}^{v} m_{di1} = bk = vr+p \;\;\geq\;\; 3\left[(v-1)\frac{r+1}{3}+\frac{(r+4)}{3}\right] \;=\; vr+v+3,$$

a contradiction. It is now claimed that $m_{di1} \leq \frac{r-5}{3}$ for some $i \in \{v-p+1,\ldots,v-1\}$, for if not, employing the condition $p < (v+6)/4$,

$$\begin{aligned}0 \;=\; 3[(\sum_{i=1}^{v} m_{di1})-b] &= 3\left[(v-p)\frac{r+1}{3}+(p-1)\frac{(r-2)}{3}+\frac{(r+4)}{3}-\frac{(vr+p)}{3}\right] \\ &= v-4p+6 \;>\; v-4\frac{v+6}{4}+6 \;=\; 0,\end{aligned}$$

another contradiction. Hence WLOG $m_{dv-1,1} \leq (r-5)/3$, so that with the same notation as in Case 1(a),

$$\overline{m}_{1(1)} \;\leq\; \frac{r-\frac{r+1}{3}}{2} \;=\; \frac{2r-1}{6} \quad \text{and} \quad \overline{m}_{v-1(1)} \;\geq\; \frac{r+1-\frac{r-5}{3}}{2} \;=\; \frac{r+4}{3}.$$

Consequently,

$$\begin{aligned}\sum_{l=1}^{3}(m_{d1l}-m_{dv-1,l})^2 &\geq (m_{d11}-m_{dv-1,1})^2+2(\overline{m}_{1(1)}-\overline{m}_{v-1(1)})^2 \\ &\geq \left(\frac{(r+1)}{3}-\frac{(r-5)}{3}\right)^2+2\left(\frac{(2r-1)}{6}-\frac{(r+4)}{3}\right)^2 \;=\; 8\frac{1}{2}.\end{aligned}$$

□