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Mathematical Model for Spread of COVID-19 Virus using Fractional Order Approach

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Abstract

The COVID-19 pandemic has exhibited unprecedented and adverse effects on global health and living patterns. To understand and predict the spread of any disease, the Susceptible-Infectious-Recovered (SIR) model has been extensively used. However, the SIR model is failing to accurately predict the dynamics of complex systems. Therefore, this research proposes the development of a modified fractional order derivative for modelling the COVID-19 epidemic. The total population is divided into four classes with exclusive consideration of quarantined individuals. Attributes such as positivity, boundedness of solution, and stability of a model at disease-free equilibrium are thoroughly studied. The obtained results are utilized to predict the progression of COVID-19 through modelling.

Key words: Mathematical Model; Epidemic model; COVID-19 virus; Fractional calculus; Stability.

AMS Subject Classifications: 26A33, 34A30, 92-10, 00A71, 34D20.

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1. Introduction

Predicting the projection of COVID-19 till remains a challenge that demands the integration of epidemiological models, statistical analyses, and real-time data streams. The COVID-19 pandemic has wrought profound and multifaceted effects on the world, touching nearly every aspect of human life. Economically, it has triggered widespread disruptions, leading to job losses, business closures, and supply chain bottlenecks, exacerbating inequalities and pushing millions into poverty. Socially, it has imposed isolation, disrupted education, and strained healthcare systems to their limits, with far-reaching implications for mental health and well-being. Therefore modelling of spread of COVID-19 is much necessary to predict the spread disease. There are various modelling approaches to forecast the spread of disease such as SIR, statistical, machine learning.

Mathematical modeling of epidemics was first introduced by W.O. Kermack and A.G

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McKendrick in year 1927 Kermack and McKendrick (1927). Since then many Mathematicians, Life sciences scientist and Medical professionals have used such type of model for studying the dynamics of various infectious diseases. By developing the model helps in predicting the infections. In Silva and Torres (2014) authors have developed a mathematical model on TB-HIV syndemic and treatment, in this model they divided total population into ten partitions and discussed about the positivity, boundedness of the solution and also discussed the stability. In Diethelm (2013); Solanke and Pachpatte (2019, 2021) the mathematical models on TB, dengue and Swine flu diseases have been studied.

COVID-19 virus is the one of family members of coronavirus, which are RNA viruses can be mild to lethal. It is harmful because of the risk factor, as some strains can kill up to 30 percent of affected people. As of June 22, 2021, there have been 180101870 confirmed cases and 3902501 deaths worldwide. Fever, dry cough, dyspnea, diarrhoea, sore throat and other symptoms are prevalent. It can infect cats, dogs, camels, and horses, among other animals web (a,b). The first incidence of COVID-19 was discovered in China in 2019. The first case of COVID-19 in India was detected by a person travelling from China to Kerala in the last week of January 2020.

Recently many researchers all over the world have started developing the Models for studying the behaviour of COVID-19. In Brandenburg (2020); Vega (2020) authors have developed a SIR model on COVID-19 for piecewise quadratic growth and discussed about the lockdowns due to increased infections of COVID-19 disease. Also in Shaikh *et al.* (2020) authors have discussed a mathematical model on COVID-19 formed by using fractional order derivative and also discussed the stability using the Laplace transform method. The SEIR model on COVID-19 disease have developed and studied local stability and global stability by some researchers Ssematimba *et al.* (2021); Tiwari *et al.* (2020); Wang *et al.* (2020). Some authors have developed the model by including class of quarantine or isolated population Krishna (2020); Mnganga and Zachariah (2020); Peter *et al.* (2021); Tanga *et al.* (2020), in Lina *et al.* (2020) different classes are provided for cumulative cases and deaths occurred due to COVID-19. The environmental changes such as population, air changes due to COVID-19 disease and effect of lockdown occurred due to COVID-19 Kerimray *et al.* (2020); Xu *et al.* (2020). In Makade *et al.* (2020), they have discussed about the most influential parameter for the spread of COVID-19 disease. Spread of COVID-19 active infection cases in three countries India, Italy and United States Of America(USA) have studied Pachpatte *et al.* (2021). Some logistic models, dynamic models and on spatial density are also developed on COVID-19 Abusam *et al.* (2020); Adekunle *et al.* (2020); Alzahrani *et al.* (2021); Al-Khani *et al.* (2020); Vaz and Torres (2021); Zaitri *et al.* (2021).

Simple mathematical model to investigate the transmission and regulation of the novel coronavirus disease (COVID-19) from human to human has been done in Ahmed *et al.* (2021). The researchers used mathematical epidemiology principles to model, how people are exposed to and infected with the disease, as well as their possible future recovery. Both the ordinary differential equation (ODE) and the fractional differential equation were used in the mathematical study. It is critical for health practitioners and the rest of the world to understand and predict infected individuals in order to plan for citizens' health concerns and to control the spread rate with limited supply. The simulation's data is based on the spread of disease in Nigeria.

Riyapan *et al.* (2021) have proposed and examined nonlinear mathematical model in terms of understanding the dynamics of the COVID-19 epidemic in Thailand. The formulated model's equilibrium point was determined. The basic reproduction number pertaining to the model was also calculated using the next generation matrix approach. Haq *et al.* (2023) have created a vaccination model by including the vaccine class and other factors that are crucial for immunizing those who are susceptible.

Motivated by the above work, a mathematical model for COVID-19 disease containing fractional order derivative and the properties about their solution are studied.

2. Mathematical model

2.1. Preliminaries

Now, in this section, some basic terminology that will come in useful during our discussions.

Definition 1: Podlubny (1999); Zhou (2014). The fractional calculus in classical form is given by the Riemann–Liouville integral which can be defined as

$$ {}_aD_t^{-\tau}(u(t)) = {}_a I_t^{\tau}(u(t)) = \frac{1}{\Gamma(\tau)} \int_a^t (t-\varsigma)^{\tau-1} u(\varsigma) d\varsigma, \tag{1} $$

where $t > a$.

Definition 2: Shaikh *et al.* (2020); Podlubny (1999); Zhou (2014). The Caputo fractional derivative operator of order τ where $\tau \geq 0$ and $n \in N \cup \{0\}$ can be given as

$$ D_t^{\tau}(u(t)) = \frac{1}{\Gamma(n-\tau)} \int_0^t (t-\varsigma)^{n-\tau-1} \frac{d^n}{dt^n} u(\varsigma) d\varsigma, \tag{2} $$

where $n-1 \leq \tau < n$.

2.2. Mathematical model

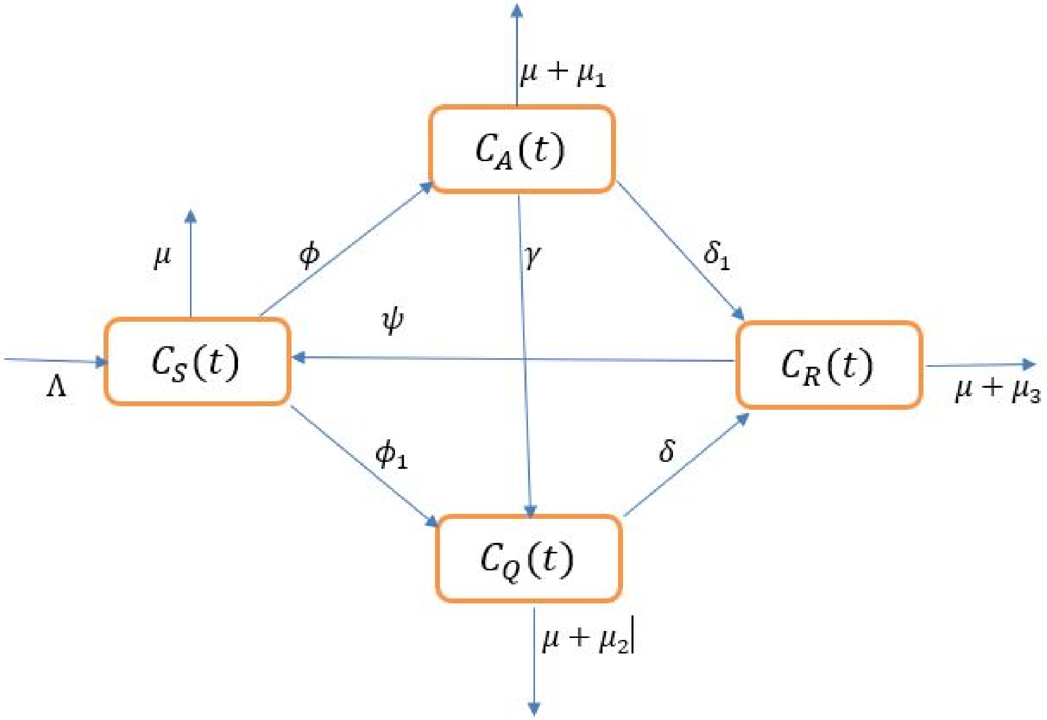
In this section present a model with class of quarantined individuals. Suppose that the entire population is organised into four classes, each of which is mutually exclusive, meaning that no individual may be assigned to more than one. Divided the population in four classes as three partitions are not sufficient to study the asymptomatic individuals and for this fifth partition is not required. Use $N(t)$ for total population is function of time t. Define four classes as

$C_s(t)$- Class of individuals Susceptible at time t,
$C_Q(t)$-Class of individuals Quarantined time t,
$C_A(t)$- Class of individuals asymptomatic (infected individuals not having symptoms) and not quarantined at time t,
$C_R(t)$- Class of individuals recovered at time t with or without medical treatment.

Total population $N(t)$ is given by
$N(t) = C_s(t) + C_A(t) + C_Q(t) + C_R(t)$.

We denote Λ is the birth rate, μ is the natural death rate, μ_1 is the death rate of asymptomatic people due to COVID-19, similarly μ_2 is the death rate of quarantine people due to COVID-19. Some deaths are occurred in recovered class due to post COVID-19 diseases at the rate of μ_3. In this consider that very less number of asymptotic individuals recover without treatment means most of them have to pass through the quarantine class. Here β_2 denotes effective contact rate with infected people, which comprises 2 parameters as ϕ is the rate at which the people get infected but not having symptoms (not quarantined) and ϕ_1 is the rate at which people are infected and having symptoms and they are quarantined. The asymptomatic peoples recovers at the rate of δ_1 without any treatment and they are quarantined at the rate γ when they becomes symptomatic and tested positive. Quarantined people recovers at the rate of δ and recovered people will become susceptible at the rate ψ after some time due loss of immunity.

Above system can be represented in model format as follows:



$$\frac{dC_S(t)}{dt} = \Lambda + \psi C_R(t) - (\phi + \phi_1 + \mu)C_S(t), \tag{3}$$

$$\frac{dC_A(t)}{dt} = \phi C_S(t) - (\gamma + \delta_1 + \mu + \mu_1)C_A(t), \tag{4}$$

$$\frac{dC_Q(t)}{dt} = \phi_1 C_S(t) + \gamma C_A(t) - (\delta + \mu + \mu_2)C_Q(t), \tag{5}$$

$$\frac{dC_R(t)}{dt} = \delta C_Q(t) + \delta_1 C_A(t) - (\psi + \mu + \mu_3)C_R(t), \tag{6}$$

with initial conditions

$$C_S(0) \geq 0, C_A(0) \geq 0, C_Q(0) \geq 0, C_R(0) \geq 0. \tag{7}$$

Construct a mathematical model using fractional derivative.

The above system can written using Caputo fractional derivative operator as follows

$$D_t^{\tau}(C_S(t)) = \Lambda + \psi C_R(t) - (\phi + \phi_1 + \mu)C_S(t), \tag{8}$$

$$D_t^{\tau}(C_A(t)) = \phi C_S(t) - (\gamma + \delta_1 + \mu + \mu_1)C_A(t), \tag{9}$$

$$D_t^{\tau}(C_Q(t)) = \phi_1 C_S(t) + \gamma C_A(t) - (\delta + \mu + \mu_2)C_Q(t), \tag{10}$$

$$D_t^{\tau}(C_R(t)) = \delta C_Q(t) + \delta_1 C_A(t) - (\psi + \mu + \mu_3)C_R(t), \tag{11}$$

with the initial conditions

$$C_S(t) \geq 0, C_A(t) \geq 0, C_Q(t) \geq 0, C_R(t) \geq 0. \tag{12}$$

2.3. Properties of model

Now in this section we study the positivity and boundedness properties of the solution of the system. Let $\{(C_S, C_A, C_Q, C_R) \in \mathbb{R}_+^4\}$ be any solution of system 8 - 11 with initial conditions 12. Now, let us assume the region $\omega = \{(C_S, C_A, C_Q, C_R) \in \mathbb{R}_+^4 : 0 \leq N(t) \leq \frac{\Lambda}{\mu}\}$.

Now we prove positivity of system 8 - 11 with initial conditions 12 in our next theorem.

Theorem 1: Let $\{(C_S, C_A, C_Q, C_R) \in \mathbb{R}_+^4\}$ be any solution of system 8 - 11 with initial conditions 12. Consider

$$\omega = \left\{(C_S, C_A, C_Q, C_R) \in \mathbb{R}_+^4 : 0 \leq N(t) \leq \frac{\Lambda}{\mu}\right\}, \tag{13}$$

then $C_S(0) \geq 0, C_A(0) \geq 0, C_Q(0) \geq 0, C_R(0) \geq 0$.

Proof: We will prove our result by contradiction, suppose on contrary that for some point $\tilde{t} > 0$, the $C_A(t) = 0$, *i.e.* $C_A(\tilde{t}) = 0$ and $C_S(t) \geq 0, C_Q(t) \geq 0, C_R(t) \geq 0$ (given).

Then from equation 9 we have

$$D_t^{\tau}(C_A(t)) > 0, \tag{14}$$

which is not true.

Thus, $C_A(t) \geq 0$ for all $t > 0$.

Similarly, one can prove that, $C_S(t) \geq 0$, $C_Q(t) \geq 0$, $C_R(t) \geq 0$ for all time $t > 0$. $\square$

Now proof of a boundedness of system 8 - 11 with initial conditions 12 is in next theorem.

Theorem 2: If $N(t)$ is the total population given by

$$N(t) = C_S(t) + C_A(t) + C_Q(t) + C_R(t),$$

then

$$D_t^\tau(N(t)) \leq \Lambda - \mu N(t), \tag{15}$$

where, Λ- birth rate, μ- death rate.

Proof: Since,

$$N(t) = C_S(t) + C_A(t) + C_Q(t) + C_R(t),$$

from the equations of system 8 - 11, we have

$$\begin{aligned} D_t^\tau(N(t)) =& D_t^\tau(C_S(t)) + D_t^\tau(C_A(t)) + D_t^\tau(C_Q(t)) + D_t^\tau(C_R(t)), \\ =& \Lambda + \psi C_R(t) - (\phi + \phi_1 + \mu)C_S(t) + \phi C_S(t) - (\gamma + \delta_1 + \mu + \mu_1)C_A(t) \\ & + \phi_1 C_S(t) + \gamma C_A(t) - (\delta + \mu + \mu_2)C_Q(t) + \delta C_Q(t) + \delta_1 C_A(t) \\ & - (\psi + \mu + \mu_3)C_R(t), \\ =& \Lambda + \psi C_R(t) - \phi C_S(t) - \phi_1 C_S(t) - \mu C_S(t)) + \phi C_S(t) - \gamma C_A(t) \\ & - \delta_1 C_A(t) - \mu C_A(t) - \mu_1 C_A(t) + \phi_1 C_S(t) + \gamma C_A(t) - \delta C_Q(t) \\ & - \mu C_Q(t) - \mu_2 C_Q(t) + \delta C_Q(t) + \delta_1 C_A(t) - \psi C_R(t) - \mu C_R(t) \\ & - \mu_3 C_R(t), \\ =& \Lambda - \mu C_S(t)) - \mu C_A(t) - \mu_1 C_A(t) - \mu C_Q(t) - \mu_2 C_Q(t) - \mu C_R(t) \\ & - \mu_3 C_R(t). \end{aligned}$$

Since $N(t) = C_S(t) + C_A(t) + C_Q(t) + C_R(t)$,

$$D_t^\tau(N(t)) = \Lambda - \mu N(t) - \mu_1 C_A(t) - \mu_2 C_Q(t) - \mu_3 C_R(t) \ \leq \ \Lambda - \mu N(t).$$

Therefore, conclude that $N(t)$ is bounded for all $t > 0$ and every solution of system 8 - 11 with initial conditions 12 is bounded. ☐

3. Stability analysis

In this section stability of the system 8 - 11 with initial conditions 12 is studied. Now give some basic definitions of stability analysis given in Remsing (2006).

Definition 3: Remsing (2006) An equilibrium state $x = 0$ is said to be stable, if for any positive scalar ϵ there exists a positive scalar δ such that $||x(t_0)|| < \delta$ implies $||x(t)|| < \epsilon$ for all $t \geq t_0$.

Definition 4: Remsing (2006) An equilibrium state $x = 0$ is said to be asymptotically stable, if it is stable and if in addition $x(t) \to 0$ as $t \to \infty$.

The system 8 - 11 with initial conditions 12 is said to have disease free equilibrium (No disease) if

$$\Sigma_o = (C_{S0}, C_{A0}, C_{Q0}, C_{R0}) = \left(\frac{\Lambda}{\mu + \phi + \phi_1}, 0, 0, 0\right). \tag{16}$$

The Endemic Equilibrium is given by

$$\Sigma_* = (C_{S*}, C_{A*}, C_{Q*}, C_{R*}), \tag{17}$$

with $C_{A*} > 0$, $C_{Q*} > 0$, $C_{R*} > 0$ for $R_0 > 1$, where R_0 is the basic reproduction number for the system 8 - 11 with initial conditions 12.

The basic reproduction number is the average number of new infections due to a single individual when in contact with susceptible population.Silva and Torres (2014). Now in our next theorem gives the result on the stability of the system 8 - 11 with initial conditions 12.

Theorem 3: The disease free equilibrium Σ_0 is locally asymptotically stable if $R_0 < 1$.

Proof: The disease-free equilibrium Σ_0 is locally asymptotically stable for $R_0 < 1$, if all the eigenvalues of the Jacobian Matrix of the system of equations 8 - 11 here denoted by $M_T(\Sigma_0)$ computed at the disease free equilibrium Σ_0, given by 16 have negative real parts Remsing (2006); Benerjee (2014).

The Jacobian Matrix of the system of equations 8 - 11 at disease-free equilibrium is given by

$$J_0 = \begin{pmatrix} -\phi - \phi_1 - \mu & 0 & 0 & 0 \\ \phi & -\gamma - \delta_1 - \mu - \mu_1 & 0 & 0 \\ \phi_1 & \gamma & -\delta - \mu - \mu_2 & 0 \\ 0 & \delta_1 & \delta & -\psi - \mu - \mu_3 \end{pmatrix}.$$

The eigenvalues of this matrix are the roots of the equation $|J_0 - \lambda I| = 0$, consider

$|J_0 - \lambda I|$

$$= \begin{pmatrix} -\phi - \phi_1 - \mu - \lambda & 0 & 0 & 0 \\ \phi & -\gamma - \delta_1 - \mu - \mu_1 - \lambda & 0 & 0 \\ \phi_1 & \gamma & -\delta - \mu - \mu_2 - \lambda & 0 \\ 0 & \delta_1 & \delta & -\psi - \mu - \mu_3 - \lambda \end{pmatrix},$$

$$= (-\phi - \phi_1 - \mu - \lambda) \begin{pmatrix} -\gamma - \delta_1 - \mu - \mu_1 - \lambda & 0 & 0 \\ \gamma & -\delta - \mu - \mu_2 - \lambda & 0 \\ \delta_1 & \delta & -\psi - \mu - \mu_3 - \lambda \end{pmatrix},$$

$$= (-\phi - \phi_1 - \mu - \lambda)\Bigg\{(\gamma - \delta_1 - \mu - \mu_1, -\lambda)\Bigg[(-\delta - \mu - \mu_2 - \lambda)(-\psi - \mu - \mu_3 - \lambda)\Bigg]\Bigg\},$$

$$= -(\lambda + \phi + \phi_1 + \mu)\Bigg\{-(\lambda + \gamma + \delta_1 + \mu + \mu_1)\Bigg[(\lambda + \delta + \mu + \mu_2)(\lambda + \psi + \mu + \mu_3)\Bigg]\Bigg\},$$

$$= (\lambda + \phi + \phi_1 + \mu)\Bigg\{(\lambda + \gamma + \delta_1 + \mu + \mu_1)\Bigg[(\lambda + \delta + \mu + \mu_2)(\lambda + \psi + \mu + \mu_3)\Bigg]\Bigg\}.$$

Let

$$\begin{aligned}
&\phi + \phi_1 + \mu = a,\\
&\gamma + \delta_1 + \mu + \mu_1 = b,\\
&\delta + \mu + \mu_2 = c,\\
&\psi + \mu + \mu_3 = d.
\end{aligned}$$

Therefore

$$\begin{aligned}
|J_0 - \lambda I| &= (\lambda + a)\Bigg\{(\lambda + b)\Bigg[(\lambda + c)(\lambda + d)\Bigg]\Bigg\},\\
&\Rightarrow (\lambda + a)(\lambda + b)(\lambda + c)(\lambda + d) = 0,\\
&\Rightarrow \lambda = -a, \lambda = -b, \lambda = -c, \lambda = -d.
\end{aligned}$$

Since all the parameters are greater than or equal to 0, $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$. Thus, all the eigenvalues of this Jacobian Matrix are negative.

The system of equations 8 - 11 with initial conditions 12 is stable. Thus a mathematical model 8 - 11 with initial conditions 12 is asymptotically stable at disease free equilibrium for $R_0 < 1$. □

4. Results and discussions

In this section verification of results with data of COVID-19 of India to this system has been done. The plot shows the curves for the Asymptomatic, Quarantine, Recovered population for COVID-19 disease in India. Here the parameters are as below, which are obtained from real data, MATLAB software is used for the numerical solution of the model.

Table 1: Parameters

Parameter	Value	Parameter	Value
Λ	0.0000563447	μ	0.0000194123
ϕ	0.000000125927	ϕ_1	0.00000029383
ψ	0.9999805877	μ_1	0.0118594279
δ_1	0.9881165471	γ	0.000046128
δ	0.8813863087	μ_2	0.118594279
μ_3	0	N	1380004385
$C_A(0)$	1	$C_Q(0)$	1
$C_R(0)$	0	.	.

For prediction of the spread of infection, the important parameters are β_2(effective contact rate with infected people), ϕ(the rate at which people got infected and not having any symptoms), ϕ_1(the rate at which people got infected and and having symptoms). For control of spread of infection the value of these parameter should be minimum.

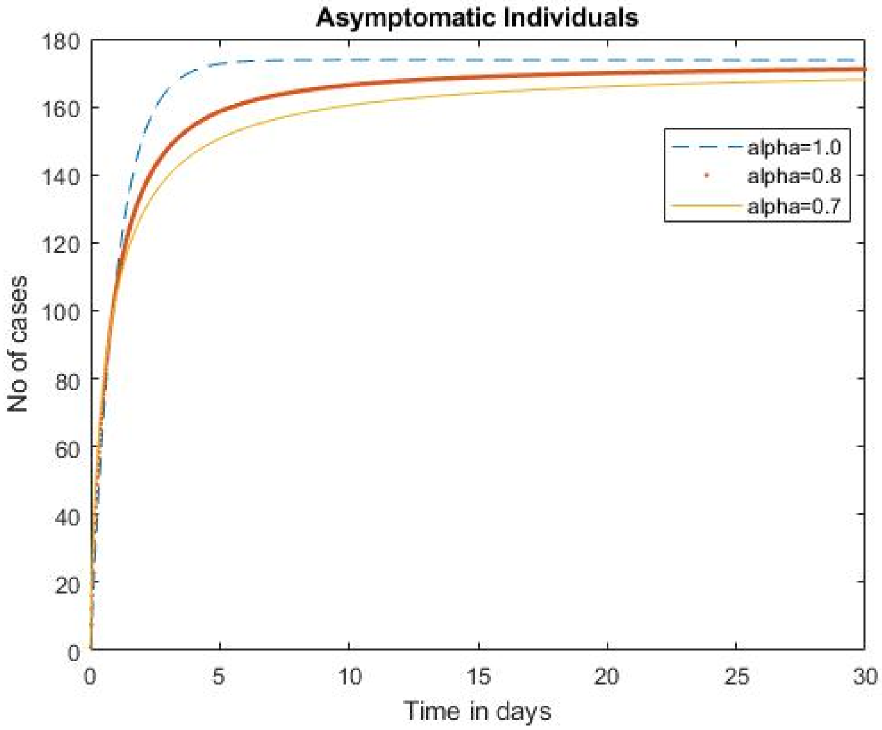


Figure 1: Graph for asymptomatic population

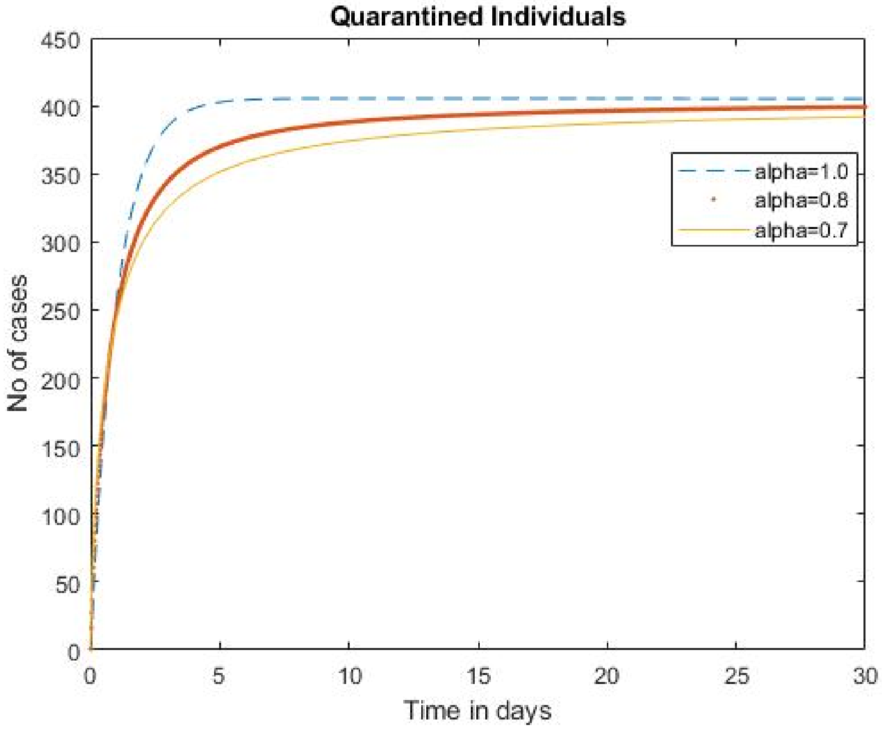


Figure 2: Graph for quarantined population

5. Conclusion

This research studied the mathematical model for the spread of COVID-19 disease using fractional derivatives successfully. The results regarding positivity, boundedness of

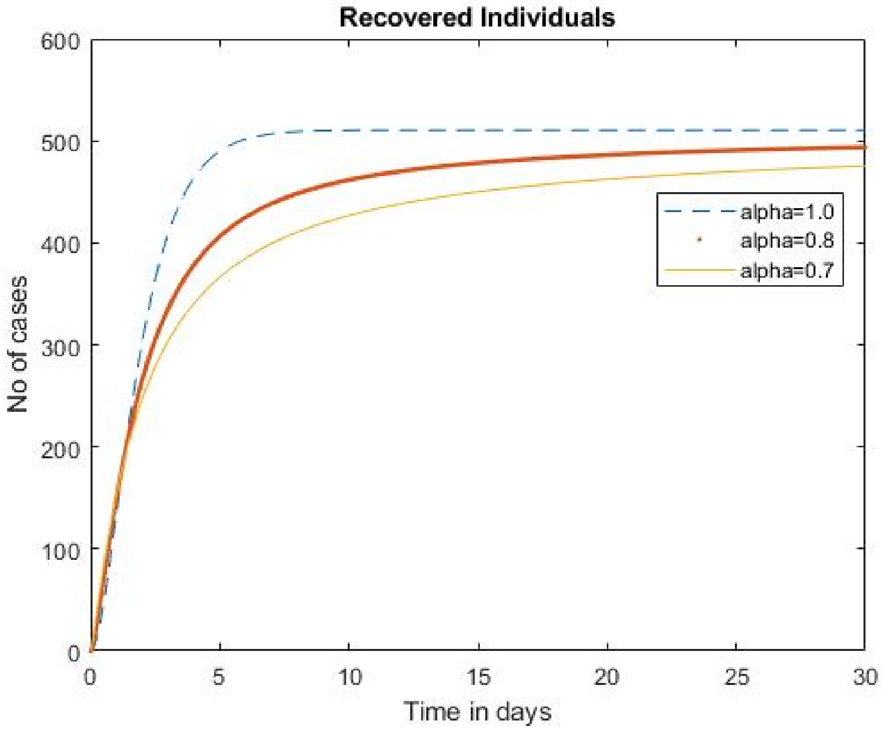


Figure 3: Graph for recovered population

the solution, and stability at equilibrium were obtained and analysed. The mathematical model's results demonstrate better performance compared to the conventional SIR model.

Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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