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Statistical Model for Brand Loyalty and Switching

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Abstract

Consumer's decision on purchase of an item or brand depends on many factors. Some stochastic distributions and markov chain models were used to analyze the consumers' purchase behavior. In this paper, a statistical linear model is constructed to study the repeated purchase behavior based on the performance measures of various brands. The proposed model will generate transformation matrix, which is used to estimate the repeat purchase of brands depending on polarized index and market share values. The model is also illustrated with a suitable example.

Key words: Brand loyalty; Dirichlet model; Least squares estimate; Markov model; Repeat rate.

1. Introduction

Scientifically, the customer choice behaviors can be broadly classified as the state of decisions and actions that influence the purchase pattern. The buying pattern of consumers begins coherently with factors attitudes towards the brands in a particular product category, made up of various components attracting the preferred brand over all other brands. The study reflects how changes are in consumer attitudes incorporate the purchase behavior; changes in consumer's attitudes with the various factors can be viewed in terms of probabilities.

Lipstein (1965) constructed a statistical analytical model to study the consumer behavior on advertising effect in marketplace. Colombo and Morrison (1989) focused on marketing strategies for brand switching model with classes of consumers: hardcore loyal and potential switchers. Dirichlet probabilistic model used to study the behavior of consumer purchasing pattern of brand choice and purchase incidence in Abel *et al*. (1980), Bound (2009), and Rungie and Goodhardt (2004).

2. Statistical model for repeated purchase

Definition 1 (Repeat rate): Let the purchase occasion is the event when a shopper makes a purchase from any one of the brand categories. The proportion of buyers of a particular brand at the last purchase occasion and also buys the same brand in the next purchase occasion. It is an intuitive measure of loyalty and it records how much a brand hangs onto its buyers.

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Definition 2 (Market share): The market share of a brand is the proportion of total purchases of that particular brand to total purchases of all brands.

Definition 3 (Polarized index): The brand Polarized index of a particular brand is defined as $\phi_i = (\rho_i - \mu_i)/ (1 - \mu_i)$.

Definition 4 (Dirichlet model): The Dirichlet model is a probability distribution function that describes the distribution of consumer purchases of each brand within a product category over time. The data set is multivariate, contains numerous brands, and the model counts the number of transactions and is discrete and instructive, *i.e.*, counts are integer type, non-negative because purchases are whole numbers. As a result, it cannot be non-negative.

Notations:

Total no. of purchases of i^{th} brand is α_i.

Total no. of purchases of all brands $S = \Sigma_i \alpha_i$.

Market share of i^{th} brand is $\mu_i = \alpha_i / S$.

Repeat rate is ρ_i.

Loyalty polarized index of brand 'i' is ϕ_i.

Inter relationships / properties of the Dirichlet model:

1. $\mu_i = \alpha_i / S$
2. $0 \le \mu_i \le 1$
3. $\Sigma_i \mu_i = 1$.
4. $\phi_i = (\rho_i - \mu_i)/ (1 - \mu_i)$ and
5. $0 \le \phi_i \le 1$

To analyze consumer switching behaviors across various brands in a specific product category, a basic comprehensive intuitive model is necessary. The brand performance measures such as penetration, repeat rate, market share, and polarize index *etc*. are estimated using likelihood theory.

Let us assume there are 'k' brands in the competitive environment for an item. Let p_{ij} be the transition probability indicates the loyalty / disloyalty of customers. When $i = j$, the p_{ii} denotes the loyalty probability for i^{th} brand, *i.e.*, who, after being convinced by market share pressure, sticks with the same brand and when $i \neq j$, the p_{ij} denotes disloyal customers who switch the brand i to j. Let the brand loyalty polarized index for the i^{th} brand be ϕ_i and market share of the brand be μ_i, where $0 \le \phi_i, \mu_i \le 1$, and $\Sigma_i \mu_i = 1$ $(i = 1, 2, \ldots, k)$. The transition probability matrix $P = ((p_{ij}))$ be the transition probability matrix defined in terms of polarized index ϕ_i and market share μ_j as

$$\left.\begin{array}{ll} p_{ii} = \phi_i + (1 - \phi_i)\mu_j & \text{for } i = j \\ p_{ij} = (1 - \phi_i)\mu_j & \text{for } i \neq j. \end{array}\right\} \tag{1}$$

The repeated stationary purchase probabilities can be estimated using Chapman-Kolmogorov equation which is presented below.

Theorem 1: The stationary probability for j^{th} brand is $\pi_j = R_j / \sum R_j$ where $R_j = \mu_j / (1 - \phi_j)$, ϕ_j is the polarized index and μ_j is the market share of j^{th} brand satisfying (1).

Proof: Let ϕ_i be the loyalty polarization index and let μ_i be the market share of i^{th} brand. Let $\boldsymbol{P}$ be the transition probability matrix constructed using (1). Let $\pi_0 = \mu$ be the initial probabilities for the brands.

For $j = 2$ brands, we have the Chapman-Kolmogorov equation $\pi_j = \pi_{j-1}.\, P$

$$\begin{bmatrix}\pi_1 & \pi_2\end{bmatrix} = \begin{bmatrix}\pi_1 & \pi_2\end{bmatrix}\begin{bmatrix}\phi_1 + (1-\phi_1)\mu_1 & (1-\phi_1)\mu_2 \\ (1-\phi_2)\mu_1 & \phi_2 + (1-\phi_2)\mu_2\end{bmatrix} \tag{2}$$

It provides

$$\pi_1 = \pi_1[\phi_1 + (1-\phi_1).\mu_1] + \pi_2[(1-\phi_2).\mu_1]$$

$$\Rightarrow \{\pi_1(1-\phi_1) + \pi_2(1-\phi_2)\}\mu_1 = \pi_1(1-\phi_1) \tag{i}$$

Similarly,

$$\pi_2 = \pi_1[(1-\phi_1).\mu_2] + \pi_2[\phi_2 + (1-\phi_2).\mu_2$$

$$\Rightarrow \{\pi_1(1-\phi_1) + \pi_2(1-\phi_2)\}\mu_2 = \pi_2(1-\phi_2) \tag{ii}$$

from the equations, (i)/(ii)

$$\frac{\mu_1}{\mu_2} = \frac{\pi_1(1-\phi_1)}{\pi_2(1-\phi_2)}$$

$$\Rightarrow \frac{\pi_1}{\pi_2} = \frac{\dfrac{\mu_1}{(1-\phi_1)}}{\dfrac{\mu_2}{(1-\phi_2)}}$$

therefore,

$$\pi_1 = \frac{\left(\dfrac{\mu_1}{(1-\phi_1)}\right)}{\left(\dfrac{\mu_1}{(1-\phi_1)} + \dfrac{\mu_2}{(1-\phi_2)}\right)}$$

and

$$\pi_2 = \frac{\left(\dfrac{\mu_2}{(1-\phi_2)}\right)}{\left(\dfrac{\mu_1}{(1-\phi_1)} + \dfrac{\mu_2}{(1-\phi_2)}\right)}$$

for $j = 3$ brands we have,

$$\begin{bmatrix}\pi_1 & \pi_2 & \pi_3\end{bmatrix} = \begin{bmatrix}\pi_1 & \pi_2 & \pi_3\end{bmatrix}\begin{bmatrix}\phi_1 + (1-\phi_1)\mu_1 & (1-\phi_1)\mu_2 & (1-\phi_1)\mu_3 \\ (1-\phi_2)\mu_1 & \phi_2 + (1-\phi_2)\mu_2 & (1-\phi_2)\mu_3 \\ (1-\phi_3)\mu_1 & (1-\phi_3)\mu_2 & \phi_3 + (1-\phi_3)\mu_3\end{bmatrix}$$

$$\pi_1 = \pi_1[\phi_1 + (1-\phi_1).\mu_1] + \pi_2[(1-\phi_2).\mu_1] + \pi_3[(1-\phi_3).\mu_1]$$

$$\pi_1[(1-\phi_1)(1-\mu_1)] = [\pi_2(1-\phi_2) + \pi_3(1-\phi_3)]\mu_1$$

$$\pi_1[(1-\phi_1)(1-\mu_1)] = [(1-\pi_1-\pi_3)(1-\phi_2) + (1-\pi_1-\pi_2)(1-\phi_3)]\mu_1$$

$$\Rightarrow \{\pi_1(1-\phi_1)+\pi_2(1-\phi_2)+\pi_3(1-\phi_3)\}\mu_1 = \pi_1(1-\phi_1) \quad \text{(a)}$$

similarly,
$$\{\pi_1(1-\phi_1)+\pi_2(1-\phi_2)+\pi_3(1-\phi_3)\}\mu_2 = \pi_2(1-\phi_2) \quad \text{(b)}$$

$$\{\pi_1(1-\phi_1)+\pi_2(1-\phi_2)+\pi_3(1-\phi_3)\}\mu_3 = \pi_3(1-\phi_3) \quad \text{(c)}$$

we have

$$(a)/(b) \Rightarrow \frac{\pi_1}{\pi_2} = \frac{\mu_1}{(1-\phi_1)} \div \frac{\mu_2}{(1-\phi_2)} \quad \text{(*a)}$$

$$(b)/(c) \Rightarrow \frac{\pi_2}{\pi_3} = \frac{\mu_2}{(1-\phi_2)} \div \frac{\mu_3}{(1-\phi_3)} \quad \text{(*b)}$$

and
$$(c)/(a) \Rightarrow \frac{\pi_3}{\pi_1} = \frac{\mu_3}{(1-\phi_3)} \div \frac{\mu_1}{(1-\phi_1)} \quad \text{(*c)}$$

From any of the two equations above, $\pi_1 : \pi_2 : \pi_3 = \frac{\mu_1}{(1-\phi_1)} : \frac{\mu_2}{(1-\phi_2)} : \frac{\mu_3}{(1-\phi_3)}$

therefore, $\pi_1 = \dfrac{\left(\dfrac{\mu_1}{(1-\phi_1)}\right)}{\left(\dfrac{\mu_1}{(1-\phi_1)}+\dfrac{\mu_2}{(1-\phi_2)}+\dfrac{\mu_3}{(1-\phi_3)}\right)}$, $\pi_2 = \dfrac{\left(\dfrac{\mu_2}{(1-\phi_2)}\right)}{\left(\dfrac{\mu_1}{(1-\phi_1)}+\dfrac{\mu_2}{(1-\phi_2)}+\dfrac{\mu_3}{(1-\phi_3)}\right)}$ and

$$\pi_3 = \frac{\left(\dfrac{\mu_3}{(1-\phi_3)}\right)}{\left(\dfrac{\mu_1}{(1-\phi_1)}+\dfrac{\mu_2}{(1-\phi_2)}+\dfrac{\mu_3}{(1-\phi_3)}\right)}.$$

In general, it can be expressed for $t = k$ brands

$$\pi_j = \frac{R_j}{\sum_{j=1}^{k} R_j}; \text{ where } R_j = \frac{\mu_j}{(1-\phi_j)}. \quad (3)$$

3. Analysis for the repeated purchase model

1. The repeated rate is an intuitive measure for brand loyalty, the higher repeated rate indicates larger loyal customers. The polarization index is also a measure of loyalty and where the repeat rate is standardized for market share.
2. When it is extended to 'k' brands, the stationary probabilities for j^{th} brand are

$$\pi_j = \frac{\left(\dfrac{\mu_j}{(1-\phi_j)}\right)}{\sum_{j=1}^{k}\left(\dfrac{\mu_j}{(1-\phi_j)}\right)}; \; j = 1,2,\ldots,k \quad (4)$$

3. The empirical evidence indicated that brand loyalty transmits relatively slow and the market-share may change from purchase-to-purchase scenario. It can be noted that that

brand loyalty is assumed to be constant over some time horizon and market share is time-dependent.

4. The repeated purchase probabilities for the j^{th} brand at time t = 1, 2, 3, …, M is

$$y_{j,t} = \phi_j y_{j,t-1} + \mu_{j,t}(1-\phi_j)\, y_{j,t-1} + \varepsilon_{j,t} \tag{5}$$

when all the brands are having equal loyalty $\phi_j = \phi$ Then the least squares estimate of ϕ can be obtained as

$$\varphi = \frac{\sum_{j=1}^{K}\sum_{t=2}^{M}\left(y_{jt-1}-\mu_{jt}\right)\left(y_{jt}-\mu_{jt}\right)}{\sum_{j=1}^{K}\sum_{t=2}^{M}\left(y_{jt-1}-\mu_{jt}\right)^2} \tag{6}$$

It can be noted that, when $\phi = 0$, there is no loyalty *i.e.*, when all the consumers switch frequently then $y_{j,t} = \mu_{j,t}$, and when $\phi = 1$, there is complete loyalty *i.e.*, when all consumers repeatedly purchase the same brand then $y_{j,t} = y_{j,t-1}$.

5. Consider the two brand cases *i.e.*, k = 2 in a competitive market environment. Then

$$S = \sum_{t=2}^{M}\left\{y_{1,t} - \varphi_1 y_{1,t} - \mu_{1,t} + \mu_{1,t}(\varphi_1 y_{1,t-1} + \varphi_2 y_{2,t-1})\right\}^2 +$$

$$\sum_{t=2}^{M}\left\{y_{2,t} - \varphi_2 y_{2,t} - \mu_{2,t} + \mu_{2,t}(\varphi_1 y_{1,t-1} + \varphi_2 y_{2,t-1})\right\}^2$$

The resulting normal equations are,

$$\frac{\partial S}{\partial \varphi_1} = 0 \Rightarrow \varphi_1 \sum_{t=2}^{M} y_{1,t-1}^2\left[\mu_{1,t}^2 + \mu_{2,t}^2 - 2\mu_{1,t} + 1\right] + \varphi_2 \sum_{t=2}^{M} y_{1,t-1} y_{2,t-1}\left[\mu_{1,t}^2 + \mu_{2,t}^2 - \mu_{1,t} - \mu_{2,t}\right]$$

$$= \sum_{t=2}^{M} y_{1,t-1}\left[\mu_{1,t}\left(\mu_{1,t} - y_{1,t}\right) + \mu_{2,t}\left(\mu_{2,t} - y_{2,t}\right) - \left(\mu_{1,t} - y_{1,t}\right)\right]$$

$$\frac{\partial S}{\partial \varphi_2} = 0 \Rightarrow \varphi_2 \sum_{t=2}^{M} y_{2,t-1}^2\left[\mu_{1,t}^2 + \mu_{2,t}^2 - 2\mu_{2,t} + 1\right] + \varphi_1 \sum_{t=2}^{M} y_{1,t-1} y_{2,t-1}\left[\mu_{1,t}^2 + \mu_{2,t}^2 - \mu_{1,t} - \mu_{2,t}\right]$$

$$= \sum_{t=2}^{M} y_{2,t-1}\left[\mu_{1,t}\left(\mu_{1,t} - y_{1,t}\right) + \mu_{2,t}\left(\mu_{2,t} - y_{2,t}\right) - \left(\mu_{2,t} - y_{2,t}\right)\right]$$

The resulting solution to the normal equations $C\phi = B$, and $\phi = C^{-1} B$, where

$$C = \begin{bmatrix} \sum_{t=2}^{M} y_{1,t-1}^2(\mu_{1,t}^2 + \mu_{2,t}^2 - 2\mu_{1,t} + 1) & \sum_{t=2}^{M} y_{1,t-1} y_{2,t-1}(\mu_{1,t}^2 + \mu_{2,t}^2 - \mu_{1,t} - \mu_{2,t}) \\ \sum_{t=2}^{M} y_{1,t-1} y_{2,t-1}(\mu_{1,t}^2 + \mu_{2,t}^2 - \mu_{1,t} - \mu_{2,t}) & \sum_{t=2}^{M} y_{2,t-1}^2(\mu_{1,t}^2 + \mu_{2,t}^2 - 2\mu_{2,t} + 1) \end{bmatrix}, \text{and}$$

$$B=\begin{bmatrix} \sum_{t=2}^{M} y_{1,t-1}\left[\mu_{1,t}\left(\mu_{1,t}-y_{1,t}\right)+\mu_{2,t}\left(\mu_{2,t}-y_{2,t}\right)-\left(\mu_{1,t}-y_{1,t}\right)\right] \\ \\ \sum_{t=2}^{M} y_{2,t-1}\left[\mu_{1,t}\left(\mu_{1,t}-y_{1,t}\right)+\mu_{2,t}\left(\mu_{2,t}-y_{2t,}\right)-\left(\mu_{2,t}-y_{2,t}\right)\right] \end{bmatrix}$$

Example 1: Let B_1, B_2, B_3 and B_4 are four competitive brands for an item in the market with loyalties, market shares and with transition probability matrix P are

Table 1: Loyalty and market shares for brands

	B_1	B_2	B_3	B_4
Polarized Index (ϕ_i)	0.30	0.20	0.50	0.60
Market Share (μ_i)	0.30	0.10	0.40	0.20

The repeated purchase stationary probabilities are evaluated as

$$\pi_j = [\mu_j / (1-\phi_j)] / \Sigma [\mu_j / (1-\phi_j)].$$

Table 2: Evaluation of repeated purchase stationary probabilities

	B_1	B_2	B_3	B_4
Polarized Index (ϕ_i)	0.30	0.20	0.50	0.60
Market Share (μ_i)	0.30	0.10	0.40	0.20
$R_j = \mu_j / (1-\phi_j)$	0.42857	0.125	0.8	0.5
$\pi_j = R_j / \sum R_j$	0.2312	0.0675	0.4316	0.2697

The loyalty transition probabilities p_{ij} can be evaluated

$p_{11} = 0.30 + (1\text{-}0.30).(0.30) = 0.51$; $p_{22} = 0.20 + (1\text{-}0.20).(0.10) = 0.28$;
$p_{33} = 0.50 + (1\text{-}0.50).(0.40) = 0.70$; $p_{44} = 0.60 + (1\text{-}0.60).(0.20) = 0.68$;

The disloyalty transition probabilities p_{ij} can be evaluated

$p_{12} = (1\text{-}0.30).(0.10)=0.07$; $p_{13} = (1\text{-}0.30).(0.40)=0.28$; $p_{14} = (1\text{-}0.30).(0.20) = 0.14$;
$p_{21} = (1\text{-}0.20).(0.30)=0.24$; $p_{23} = (1\text{-}0.20).(0.40)=0.32$; $p_{24} = (1\text{-}0.20).(0.20) = 0.16$;
$p_{31} = (1\text{-}0.50).(0.30)=0.15$; $p_{32} = (1\text{-}0.50).(0.10)=0.05$; $p_{34} = (1\text{-}0.50).(0.20) = 0.1$;
$p_{41} = (1\text{-}0.60).(0.30)=0.12$; $p_{42} = (1\text{-}0.60).(0.10)=0.04$; $p_{43} = (1\text{-}0.60).(0.40) = 0.16$;

The resulting transition matrix is

$$P = \begin{bmatrix} 0.51 & 0.07 & 0.28 & 0.14 \\ 0.24 & 0.28 & 0.32 & 0.16 \\ 0.15 & 0.05 & 0.70 & 0.10 \\ 0.12 & 0.04 & 0.16 & 0.68 \end{bmatrix}$$

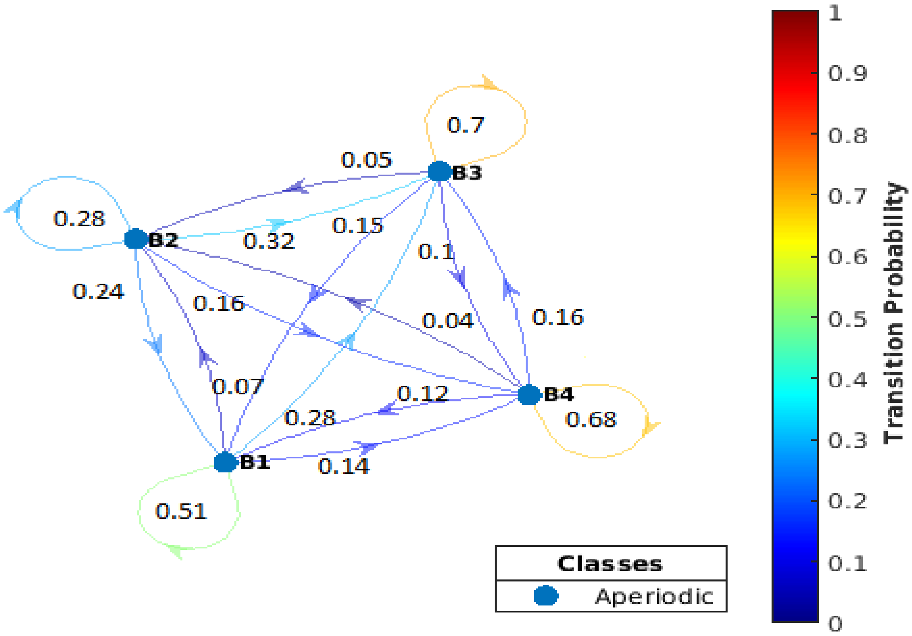

Figure 1: Markov diagram

It can be noted and stated that the Markov property exhibits while switching from one brand to the other, a customer is keeping in his or her memory only the loyalty of the brand he was using just before using the current brand and not keeping in memory the loyalty of all previously used brands, all states (brands) are communicated, implying that brands are essential, that consumers observe repeated purchases and switching among brands, so the transition probability matrix is irreducible. The states of the transition probability matrix are recurrent and have a periodicity of 1. As a consequence, the Markov chain is an ergodic (regular) Markov chain, culminating in a unique stationary distribution. The expected first hitting times (the first arrival from starting one point to the other point after how many transitions or purchase occasion *i.e.*, shifting one brand to the other after how many time transitions) for each state are

Table 3: Expected first hitting times

B_1	B_2	B_3	B_4
-	18.7143	4.0625	8.1964
6.0	-	3.8839	8.0179
6.75	19.2857	-	8.7679
7.25	19.7857	5.1339	-

Let π_0 = [0.30 0.10 0.40 0.20] be the vector of initial brands purchase probabilities.

The subsequent repeat purchase frequency rates can be evaluated as follows: $\pi_j = \pi_{j-1}. P$.

Table 4: The iterative repeated purchase stationary probabilities

Iteration	π_1	π_2	π_3	π_4
1	0.2610	0.0770	0.4280	0.2340
2	0.2439	0.0706	0.4348	0.2508
3	0.2366	0.0686	0.4353	0.2594
4	0.2336	0.0679	0.4344	0.2641
5	0.2323	0.0677	0.4335	0.2666
6	0.2317	0.0675	0.4328	0.2680
7	0.2315	0.0675	0.4323	0.2687
8	0.2313	0.0675	0.4320	0.2692
9	0.2313	0.0675	0.4318	0.2694
10	0.2312	0.0674	0.4317	0.2696
11	0.2312	0.0674	0.4317	0.2696
12	0.2312	0.0674	0.4316	0.2697
13	0.2312	0.0674	0.4316	0.2697
14	0.2312	0.0674	0.4316	0.2697

For the given initial brand shares, polarized Index and market shares of the four brands, and the subsequent stationary brand shares can be evaluated by successive application of Chapman Kolmogorov equation. The consumer's' propensity to choose a loyal brand in a long period of time with purchase probabilities attained with values as the equilibrium states together $\pi = [0.2312, 0.0674, 0.4316, 0.2697]$ (or alternatively the equilibrium purchase probabilities can be obtained from the theorem) *i.e*., the brand B_3 is more likely to be repeatedly purchased with the probability 0.4316 among the other competitive brands in the market. In other words, the concentration of the repeated purchase of the brands is directly proportional to the steady state probabilities. (The higher the steady state probability value indicates more likely to be repeated purchases and vice-versa).

4. Discussion

The brand performance indicators of the Dirichlet model are used to construct a more detailed statistical model for the transition probability matrix. In brand switching and repeat purchase analysis, this model is used to investigate Markovian characteristics. The least square principle is used to estimate the transition model parameters. The stationary probabilities are derived for each state of the Markov chain and analysis is illustrated with an example.

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