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Construction of Nested Partially Balanced Incomplete Block Designs

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Abstract

Some new methods of construction of 2- and 3-associate class nested partially balanced incomplete block (NPBIB) designs have been given. Catalogues of NPBIB designs with number of treatments (v) ≤ 30 and number of replications (r) ≤ 15 have also been given. Some results on non-existence of NPBIB designs are also given.

Key words: Nested partially balanced incomplete block designs; Partially balanced incomplete block designs; Rectangular designs.

1 Introduction

Nested block designs are the designs in which one system of blocks is nested within another system of blocks. Here the blocks with larger size are called bigger blocks and sub-blocks are nested within these bigger blocks. Preece (1967) introduced nested balanced incomplete block (NBIB) designs and gave methods of construction of NBIB designs. Jimbo and Kuriki (1983), Dey *et al.* (1986), Saha *et al.* (1998) and Morgan *et al.* (2001) gave some systematic methods of construction of NBIB designs. All NBIB designs for v (number of treatments) ≤ 16, r (replication number) ≤ 30 are catalogued by Morgan *et al.* (2001). An NBIB design may not always exist or even if it exists may require a large number of replications, which the experimenter may not be able to afford. To deal with such situations, Homel and Robinson (1975) defined nested partially balanced incomplete block (NPBIB) designs.

Definition 1.1: An NPBIB design based on m (≥ 2)-class association scheme defined in v symbols, is an arrangement of v symbols into b_2 sub-blocks of size k_2 nested within b_1 (= b_2 / q , q is an integer) blocks of size k_1 (= $qk_2 < v$) such that

(i) every symbol occurs at most once in a block;
(ii) every symbol appears at most r times in the design;
(iii) if two symbols, say α and β, are i^{th} associates, then they occur together in λ_{1i} blocks and λ_{2i} sub-blocks, the numbers λ_{1i}, λ_{2i} being independent of the particular pair of i^{th} associates α and β, i = 1, 2, ..., m.

The numbers $v, b_1, b_2, r, k_1, k_2, \lambda_{1i}, \lambda_{2i}$ (i = 1, 2, ..., m) are called parameters of the design. If $\lambda_{1i} = \lambda_1$ and $\lambda_{2i} = \lambda_2$; $\forall$ i = 1, 2, ..., m, then an NPBIB design reduces to NBIB design.

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Several methods of construction of NPBIB designs are available in the literature. Banerjee and Kageyama (1990, 1993), Kageyama *et al.* (1995), Philip *et al.* (1997); Saha *et al.* (1998) and Satpati and Parsad (2004) obtained some methods of construction of NPBIB designs. Satpati and Parsad (2004) presented a catalogue of two and three associate class NPBIB designs for $v \le 30, r \le 15$. In the present investigation, some new methods of construction of NPBIB designs have been obtained and are given in Section 3 along with catalogue of NPBIB designs for $v \le 30,\ r \le 15$ obtainable from these methods of construction. Some conditions of non-existence of NPBIB designs are given in Section 3.

2 Methods of Construction of NPBIB Designs

In this section, we give some new methods of constructions of NPBIB designs based on 2- and 3- class association schemes.

Method 2.1: This is a generalization of Method 2.1 given by Satpati and Parsad (2004). Let $v = s^2$ symbols be defined on an L_p-association scheme. Take all possible combinations of m ($\le s - 1$) rows of the association scheme. Consider the treatments in m rows to form a block and the treatments from the same row within the block as sub-blocks. This process yields $\binom{s}{m}$ blocks each of size ms, there being m sub-blocks each of size s nested within each block. Repeating the same procedure for columns, we get another set of $\binom{s}{m}$ blocks.

Consider the treatments appearing in the positions of the same alphabet in one of the (p – 2) Latin squares as rows or columns. Repeat this process for each of the (p – 2) Latin squares. Union of all the blocks so obtained gives an NPBIB design based on L_p-association scheme with parameters $v = s^2$, $b_1 = p\binom{s}{m}$, $b_2 = pm\binom{s}{m}$, $r = p\binom{s-1}{m-1}$, $k_1 = ms$, $k_2 = s$,

$\lambda_{11} = \binom{s-1}{m-1} + (p-1)\binom{s-2}{m-2}$, $\lambda_{12} = p\binom{s-2}{m-2}$, $\lambda_{21} = \binom{s-1}{m-1}$, $\lambda_{22} = 0$.

For $m = 2$, the method reduces to Method 2.1 of Satpati and Parsad (2004).

Example 2.1: Consider $v = 16$ treatments defined on L_3-association scheme. First associates of a particular treatment are the treatments appearing in the same rows, same columns and with the same symbols on one of the 3 (4 × 4) mutually orthogonal Latin squares. For example,

$$\begin{bmatrix} 1A & 2B & 3C & 4D \\ 5B & 6C & 7D & 8A \\ 9C & 10D & 11A & 12B \\ 13D & 14A & 15B & 16C \end{bmatrix}.$$

Following the procedure of Method 2.1 for $m = 3$, we get an NPBIB design based on L_3-association scheme with the blocks

[(1,2,3,4); (5,6,7,8); (9,10,11,12)]; [(1,2,3,4); (5,6,7,8); (13,14,15,16)];
[(1,2,3,4); (9,10,11,12); (13,14,15,16)]; [(5,6,7,8); (9,10,11,12); (13,14,15,16)];

[(1,5,9,13); (2,6,10,14); (3,7,11,15)]; [(1,5,9,13); (2,6,10,14); (4,8,12,16)];
[(1,5,9,13); (3,7,11,15); (4,8,12,16)]; [(2,6,10,14); (3,7,11,15); (4,8,12,16)];
[(1,8,11,14); (2,5,12,15); (3,6,9,16)]; [(1,8,11,14); (2,5,12,15); (4,7,10,13)];
[(1,8,11,14); (3,6,9,16); (4,7,10,13)]; [(2,5,12,15); (3,6,9,16); (4,7,10,13)].

The parameters of the above design are $v = s^2 = 16$, $b_1 = 12$, $b_2 = 36$, $r = 9$, $k_1 = 12$, $k_2 = 4$, $\lambda_{11} = 7$, $\lambda_{12} = 6$, $\lambda_{21} = 3$, $\lambda_{22} = 0$.

The designs generated by Method 2.1 for $v \le 30$, $r \le 15$ are catalogued in Table 2.1 given in the Appendix. The design considering sub-blocks as blocks at serial numbers 1, 2 and 5 are PBIB designs LS32, LS 33 and LS 59 listed in Clatworthy (1973).

Method 2.2: This method is also a generalization of Method 2.3 of Satpati and Parsad (2004). Consider a rectangular association scheme with $v = mn$, treatments arranged in a rectangular array of m rows and n columns. As per definition of rectangular association scheme the treatment symbols, in the same row, are 1st associates, treatment symbols in the same column are 2nd associates and the remaining are 3rd associates. For $m = n + 1$, an NPBIB design based on rectangular association scheme may be constructed using the following procedure.

Step 2.2.1: Take all the treatments appearing in the i^{th} row of the association scheme into one block, say B_{1i}.

Step 2.2.2: Write n-sub-blocks each of size $n = (m - 1)$ by taking treatment symbols in the columns (except the treatment symbols in i^{th} row) as sub-blocks. Number these sub-blocks as $B_{21(i)}$, $B_{22(i)}$, ..., $B_{2n(i)}$.

Step 2.2.3: Take all possible combination of α ($2 \le \alpha \le n$) blocks from such n-blocks to form $\binom{n}{\alpha}$-blocks like the following manner

$$\left[(B_{1i});(B_{21(i)});(B_{22(i)});\cdots;(B_{2\alpha(i)})\right];\ \left[(B_{1i});(B_{21(i)});\cdots;(B_{2(\alpha-1)(i)})(B_{2(\alpha+1)(i)})\right]$$
$$\cdots;\left[(B_{1i});(B_{2(n-\alpha+1)(i)});\cdots;(B_{2n(i)})\right].$$

Step 2.2.4: Repeat Steps 2.2.1 to 2.2.3 for all the rows $i = 1, 2, ..., m$.

This procedure yields an NPBIB design based on rectangular association scheme with following parameters

$v = mn$, $b_1 = m\binom{n}{\alpha}$, $b_2 = m(\alpha+1)\binom{n}{\alpha}$, $r = \left(\frac{n}{\alpha}+n\right)\binom{n-1}{\alpha-1}$, $k_1 = n(\alpha+1)$, $k_2 = n$, $\lambda_{11} = n\binom{n-2}{\alpha-2} + \binom{n}{\alpha}$, $\lambda_{12} = m\binom{n-1}{\alpha-1}$, $\lambda_{13} = (\alpha+1)\binom{n-1}{\alpha-1}$, $\lambda_{21} = \binom{n}{\alpha}$, $\lambda_{22} = (n-1)\binom{n-1}{\alpha-1}$, $\lambda_{23} = 0$.

For $\alpha = 1$, this is same as Method 2.3 given by Satpati and Parsad (2004).

Example 2.2: Let v = 12 treatments be arranged in m = 4 rows and n = 3 columns as given below:

1	5	9
2	6	10
3	7	11
4	8	12

Consider that the rectangular association scheme is defined on these 12 treatments. Then applying the procedure of Method 2.2 by taking $\alpha = 2$, we get a NPBIB design based on rectangular association scheme with blocks as

[(1, 5, 9); (2,3,4); (6,7,8)]; [(1,5,9); (2,3,4); (10,11,12)]; [(1,5,9); (6,7,8); (10,11,12)];
[(2,6,10); (1,3,4); (5,7,8)]; [(2,6,10); (1,3,4); (9,11,12)]; [(2,6,10); (5,7,8); (9,11,12)];
[(3,7,11); (1,2,4); (5,6,8)]; [(3,7,11); (1,2,4); (9,10,12)]; [(3,7,11); (5,6,8); (9,10,12)];
[(4,8,12); (1,2,3); (5,6,7)]; [(4,8,12); (1,2,3); (9,10,11)]; [(4,8,12); (5,6,7); (9,10,11)].

The parameters of the above NPBIB design are v = 12, b_1 = 12, b_2 = 36, r = 9, k_1 = 9, k_2 = 3, λ_{11} = 6, λ_{12} = 8, λ_{13} = 6, λ_{21} = 3, λ_{22} = 4, λ_{23} = 0.

The NPBIB designs generated by Method 2.2 are catalogued in Table 2.2 given in the Appendix for $v \le 30$, $r \le 15$. The designs marked with aestriks (*) are nested complete block partially balanced incomplete sub-block designs and the remaining are NPBIB designs, *i.e*. partially balanced with respect to both block as well as sub-block classification.

Method 2.3: Let $v = mn$ symbols be defined on group divisible (GD) association scheme. Take the rows as sub-blocks and put a set of x disjoint sub-blocks to form a block of size nx. Repeat this procedure and form $\binom{m}{x}$ blocks from the complete set of such sub-blocks. Now correspondence of each symbol to v different treatments gives an NPBIB design with the following parameters:

$v = mn$, $b_1 = \binom{m}{x}$, $b_2 = x\binom{m}{x}$, $k_1 = nx$, $k_2 = n$, $r = \binom{m-1}{x-1}$, $\lambda_{11} = r$, $\lambda_{12} = \binom{m-2}{x-2}$, $\lambda_{21} = r$, $\lambda_{22} = 0$.

Example 2.3: Let v = 12 treatments be arranged in m = 3 rows and n = 4 columns as given below:

1	5	9
2	6	10
3	7	11
4	8	12

Consider that group divisible (GD) association scheme is defined on these 12 treatments. Then applying the procedure of Method 3.2 by taking x = 3, we get a NPBIB design based on group divisible association scheme with blocks as

[(1, 5, 9); (2, 6, 10); (3, 7, 11)]; [(1, 5, 9); (2, 6,10); (4, 8, 12)];
[(1, 5, 9); (3, 7, 11); (4, 8, 12)]; [(2, 6,10); (3,7,11); (4, 8, 12)].

The parameters of the above NPBIB design are v = 12, b_1 = 4, b_2 = 12, r = 3, k_1 = 9, k_2 = 3, λ_{11} = 3, λ_{12} = 2, λ_{21} = 3, λ_{22} = 0.

The design generated by Method 2.3 is disconnected in the sub-blocks, therefore, these are not catalogued.

3 Non-Existence of NPBIB Designs Based on Group Divisible Association Scheme

Consider a NPBIB design based on group divisible (GD) association scheme with parameters $v = mn$, b_1, b_2, r, k_1, k_2, λ_{11}, λ_{12}, λ_{21}, λ_{22}, $n_1 = n-1$, $n_2 = n(m-1)$. Here symbols have their usual meaning. Let there be q sub-blocks within a bigger block. Therefore, $vr = b_1k_1 = b_2k_2$, $b_2/b_1 = k_1/k_2 = q$, $n_1\lambda_{11} + n_2\lambda_{12} = r(k_1-1)$, $n_1\lambda_{21} + n_2\lambda_{22} = r(k_2-1)$. Now consider $\lambda_{21} = 0$, *i.e.*, no two treatments that are mutually 1st associates occur together in a sub-block; therefore, among the 1st associates a treatment that occur in a sub-block will occur in any of the remaining (q – 1) sub-blocks nested within a bigger-block. It is clear that possibility of the concurrences of any two mutually 1st associates in bigger block is (q – 1). A treatment i that appears in r blocks may have at most r(q – 1) concurrences with its 1st associates. Treatment i appears with anyone of its 1st associates in λ_{11} bigger blocks. The number of 1st associates of treatment i is n_1; therefore, treatment i can appear with its 1st associates in $n_1\lambda_{11}$ in bigger blocks. Therefore, the NPBIB designs with $\lambda_{21} = 0$, can exist if $n_1\lambda_{11} \le r(q-1)$. Hence, we have the following theorem:

Theorem 3.1: A NPBIB design based on GD association scheme with parameters $v = mn$, b_1, b_2, $r, k_1, k_2, \lambda_{11}, \lambda_{12}, \lambda_{21} = 0, \lambda_{22}, n_1 = n-1, n_2 = n(m-1)$ cannot be constructed if $n_1\lambda_{11} > r(q-1)$.

Example 3.1: NPBIB design based on GD association scheme with $m = 2$, $n = 4$, $v = 8$, $b_1 = 16$ $b_2 = 32$, $r = 8$, $k_1 = 4$, $k_2 = 2$, $\lambda_{11} = 4, \lambda_{12} = 3, \lambda_{21} = 0, \lambda_{22} = 2$, $q = 32/16 = 2$, $n_1 = 3$, $n_2 = 4$ is non-existent because $n_1\lambda_{11} = 3\times 4 = 12 > r(q-1) = 8$.

Theorem 3.2: If $b_1 - 2r + \lambda_{12} = 0$, then b_1 should be a multiple of m provided $v - k_1 \le n$ and/or $b_2 - 2r + \lambda_{22} = 0$, then b_2 should be a multiple of m provided $v - k_2 \le n$. Otherwise design is non-existent. $b_1 - 2r + \lambda_{12} = 0$ or $b_2 - 2r + \lambda_{22} = 0$ implies that the corresponding complementary design is disconnected and for a disconnected design number of blocks has to be multiple of number of rows in the association scheme as per Result 2.2 of Parsad et al. (2007).

4 Discussion

In some of the designs given in Tables 2.1 and 2.2 in the Appendix, $k_1 = v$, these designs are complete block designs with respects to blocks and PBIB designs with respect to sub-blocks. In fact these designs are also taken as resolvable PBIB designs and are called as nested complete block - partially balanced incomplete sub-block design.

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Appendix

Table 2.1: NPBIB Designs based on Latin-square Association Scheme with $v \leq 30$, $r \leq 15$ Using Method 2.1

Sl. No.	v	b_1	b_2	r	k_1	k_2	λ_{11}	λ_{12}	λ_{21}	λ_{22}	m	Association Scheme
1	16	8	24	6	12	4	5	4	3	0	3	L2
2	16	12	36	9	12	4	7	6	3	0	3	L3
3	16	16	48	12	12	4	9	8	3	0	3	L4
4	25	20	60	12	15	5	9	6	6	0	3	L2
5	25	10	40	8	20	5	7	6	4	0	4	L2
6	25	15	60	12	20	5	10	9	4	0	4	L3

Table 2.2: NPBIB Designs based on Rectangular Association scheme with $v \leq 30$, $r \leq 15$ Using Method 2.2

Sl. No.	v	b_1	b_2	r	k_1	k_2	λ_{11}	λ_{12}	λ_{13}	λ_{21}	λ_{22}	λ_{23}
*1	6	3	9	3	6	2	3	3	3	1	1	0
2	12	12	36	9	9	3	6	8	6	3	4	0
*3	12	4	16	4	12	3	4	4	4	1	2	0
*4	20	5	25	5	20	4	5	5	5	1	3	0
*5	30	6	36	6	30	5	6	6	6	1	4	0

* denotes that the design is nested complete block - partially balanced incomplete sub-block design.