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Frequentist Predictive Inference for Wind Direction Data Under *l*-modal Circular Normal Model Through Sufficiency Approach

Sahana Bhattacharjee and Kishore Kumar Das
Department of Statistics, Gauhati University, Guwahati-781014, Assam, India

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Abstract

In this paper, the estimation of the future density of wind direction conditioned on the past and present wind direction data using the Sufficiency Approach of Predictive Inference under the *l*-modal Circular Normal model, followed by the equal tail area predictive interval estimation has been done. Point predictive estimator of future observation, termed as the frequentist predictive point estimator under the circular loss function has been obtained Finally, some basic properties of the estimator have been explored.

Keywords: Sufficiency approach of predictive inference; *l*-modal circular normal model; Equal tail area predictive interval estimation; Circular loss function; frequentist predictive point estimator.

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1. Introduction

The prediction of the pattern of future occurrences, based on the occurrences in past, is an important aim of statistics and according to some authors; it is the sole aim of this subject. The object of interest to be predicted can be a single value, a set of values or a function of these. The literature boasts of a number of techniques to obtain predictive likelihoods and density functions. Cox and Hinkley (1974) had initially conditioned the data on a minimal sufficient statistic of the parameter and it was Butler (1986) who later on incorporated the future observation in the data and suggested the expression for conditional predictive likelihood based on the minimal sufficient statistic. This constitutes the Sufficiency Approach. In the Bayesian Approach of predictive density estimation, the conditional distribution of the future observation(s) given the past data is obtained simply by marginalization of the joint distribution of the future observations and the population parameter(s) with respect to the parameter. For this purpose, at the outset, the prior densities of the parameters are assumed to be known. The Profile or Maximum Likelihood method consists in predicting the density of future observation from the maximum likelihood function based on the maximum likelihood estimate of both the given and future observation.

After having predicted the future observations(s), one might be interested in carrying out both the point and interval estimation based on the predictive density, followed by evaluating the error or loss incurred in predicting the true value of the observation by its estimator. The loss incurred can be quantified using a loss function. In prediction problems, as stated by Hennig and Kutlukaya (2007), the quality of a predictor is judged with the help

Corresponding Author: Sahana Bhattacharjee
sahana.bhattacharjee@hotmail.com

of loss function, which depends on the observed value and predicted value of the observation. Jammalamadaka and Sen Gupta (1998) had derived the predictive density estimation of the future observation given present and previous data and then carried out the predictive Highest Posterior Density (HPD) interval estimation under von Mises model in circular case and von Mises-Fisher model in spherical case. In predictive analysis concerning circular data, the usual linear loss functions are not well-defined. Gelfand and Ghosh (1998) had proposed the Squared Predicted Errors (SPE) loss function and had used it to choose the best fitting model to circular data by minimizing the posterior predictive loss. Ravindran and Ghosh (2012) had proposed the Absolute Predicted Errors (APE) loss function and used it to choose the best fitting circular model by minimizing this loss with respect to the posterior predictive density. Under the circular loss defined by Sen Gupta and Maitra (1998), the same authors had studied the best equivariance and admissibility property of the maximum likelihood estimator of the mean direction for a single von Mises distribution and also for that of the several independently distributed circular normal distributions. In the linear statistics literature, the Bayes estimator of the parameters of different distributions is obtained by minimizing the posterior loss/predictive loss under different loss functions. Several properties of these loss functions have also been explored. However, in the circular statistics predictive inference literature, the estimation of parameters by minimization of predictive density has not been attempted yet. Another interesting prospect that still remains unexplored is studying the properties of these estimators. Keeping in view these points, the objectives of this paper have been decided upon.

This paper attempts to predict the future density of wind direction conditioned on the past and present wind direction data using the Sufficiency Approach of Predictive Inference under the *l*-modal Circular Normal model and then carry out the equal tail area predictive interval estimation. Further, the predictive point estimation of the future observation under the circular loss function and for the same model has been carried out. Finally, some basic properties of the estimators obtained under the circular loss function are studied.

For achieving the objectives of this paper, daily data on wind direction for Dibrugarh Meteorological station located in Assam, measured during morning for the Monsoon season (June-September) during the years 2012 and 2013 has been procured from the Regional Meteorological Center, Guwahati.

2. Predictive Density Estimation of the Future Observation Through Sufficiency Approach Under the *l*-modal Circular Normal Model

Suppose that $\theta_1, \theta_2, \ldots, \theta_n$ is a sample from *l*-modal Circular Normal distribution, whose p.d.f is given by

$$f(\alpha; \mu, \kappa) = \frac{1}{2\pi I_0(\kappa)} exp\{\kappa \cos l(\theta - \mu)\} \qquad 0 < \theta, \mu < 2\pi;\ \mu < \frac{2\pi}{l}$$

Here, μ and κ represent the mean direction and concentration parameter of the population respectively and the parameter *l* stands for the number of modes of the distribution (Rao and Sengupta, 2001).

Upon computation, while fixing the value of *l* as the number of modes in the sample as it appears in the corresponding histogram, the maximum likelihood estimators of μ and κ have been found to be

$\hat{\mu} = \frac{1}{l} arctan \left\{ \frac{\sum_{i=1}^{n} \sin(l\theta_i)}{\sum_{i=1}^{n} \cos(l\theta_i)} \right\}$ and $\hat{\kappa} = A^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} \cos l(\theta - \mu)\right]$

$A^{-1}(.)$ being the inverse function of the ratio of the first and zeroth order Bessel functions of the first kind, both of which are evaluated at a specified non-negative real number.

Now, for a known l, $\left(C_{l,n}, S_{l,n}\right)$ is a minimal sufficient statistic for (μ_0, κ) (Rao and SenGupta, 2001, pp.209), where $C_{l,n} = \sum_{i=1}^{n} \cos l\theta_i$ and $S_{l,n} = \sum_{i=1}^{n} \sin l\theta_i$. Based on the $(n+1)$ observations, for which $\left(C_{l,n+1}, S_{l,n+1}\right)$ is the sufficient statistic, the conditional probability of the future observation θ_{n+1} given $\theta_1, \theta_2, \dots, \theta_n$ is given by

$$Pr(\theta_{n+1}|\theta_1, \theta_2, \dots, \theta_n) = \frac{Pr(\theta_1, \theta_2, \dots, \theta_n, \ \theta_{n+1})}{Pr\left(C_{l,n+1}, S_{l,n+1}\right)}$$
$$= \frac{\left[\frac{1}{2\pi I_0(\kappa)} exp\{\kappa \cos l(\theta_i - \mu)\}\right]^{n+1}}{\left[\frac{1}{2\pi I_0(\kappa)} exp\left\{\kappa\left(C_{l,n} \sin \mu + S_{l,n} \cos \mu\right)\right\}\right] l\psi_n(r_l)}$$
$$= \frac{1}{(2\pi)^n \psi_{n+1}\left(\sqrt{{C_{l,n+1}}^2 + {S_{l,n+1}}^2}\right)} \tag{1}$$

where $C_{l,n+1} = \sum_{i=1}^{n+1} \cos l\theta_i$, $S_{l,n+1} = \sum_{i=1}^{n+1} \sin l\theta_i$ and

$$\psi_n(r) = \int_0^\infty J_0(rt) J_0^n(t) t dt, \qquad 0 \le r \le n$$

$J_0(z)$ being the Bessel function of zeroth order.

Again,

$C_{l,n+1} = C_{l,n} + \cos\theta_{l,n+1}$, $S_{l,n+1} = S_{l,n} + \sin\theta_{l,n+1}$, $C_{l,n} = R_{l,n} \cos\bar{\theta}_{l,n}$, $S_{l,n} = R_{l,n} \sin\bar{\theta}_{l,n}$, so that $R_{l,n} = \sqrt{{C_{l,n}}^2 + {S_{l,n}}^2}$ and $\bar{\theta}_{l,n} = \arctan\left(\frac{S_{l,n}}{C_{l,n}}\right)$.

Therefore, it follows from equation (1) that the predictive density of θ_{n+1} given $\theta_1, \theta_2, \dots, \theta_n$ is

$$g(\theta_{n+1}|\theta_1, \theta_2, \dots, \theta_n) = \frac{1}{(2\pi)^n \psi_{n+1}\left(\sqrt{{R_{l,n}}^2 + 1 + 2R_{l,n} \cos\left(\theta_{l,n+1} - \bar{\theta}_{l,n}\right)}\right)}$$
$$\propto \frac{1}{\psi_{n+1}\left(\sqrt{{R_{l,n}}^2 + 1 + 2R_{l,n} \cos\left(\theta_{l,n+1} - \bar{\theta}_{l,n}\right)}\right)}$$

(Rao and Sen Gupta, 2001, pp. 209).

By Rayleigh's approximation for large *n* of the length of the sample resultant length (Lord Rayleigh, 1880), it can be seen that

$$\psi_n(r) \approx \frac{2}{n} exp\left(-\frac{r^2}{n}\right)$$

Then it follows that

$$g(\,\theta_{n+1}|\theta_1,\theta_2,\dots,\theta_n) \propto \frac{1}{exp\left(\left(-\frac{2R_{l,n}}{n+1}\right)\cos\left(\theta_{l,n+1}-\bar{\theta}_{l,n}\right)\right)}$$
$$\propto\ exp\left(\left(\frac{2R_{l,n}}{n+1}\right)\cos\left(\theta_{l,n+1}-\bar{\theta}_{l,n}\right)\right)$$

which is the p.d.f of a von Mises distribution with center at $\bar{\theta}_{l,n}$ and concentration parameter $\frac{2R_{l,n}}{n+1}$, *i.e.*, $\hat{\mu}_1 = \bar{\theta}_{l,n}$ and $\hat{\kappa}_1 = \frac{2R_{l,n}}{n+1}$.

Thus,

$$g(\,\theta_{n+1}|\theta_1,\theta_2,\dots,\theta_n)\sim VM\left(\bar{\theta}_{l,n},\frac{2R_{l,n}}{n+1}\right)$$

We see that the predictive distribution is symmetric and unimodal in nature, the mode being at $\bar{\theta}_{l,n}$. In the following section, we discuss the predictive interval estimation.

3. Predictive Interval Estimation

Let $f(\theta,\mu,\kappa)$ be the predictive density of θ_{n+1} given $\theta_1,\theta_2,\dots,\theta_n$. A $100(1-\alpha)\%$ Predictive Interval for θ is given by $[\theta_L,\theta_U]$ where θ_L and θ_U are such that

$$\int_{\theta_L}^{\theta_U} f(\theta,\mu,\kappa)d\theta = 1-\ \alpha \tag{2}$$

In addition to (2), if the area under the predictive density to the left of θ_L is equal to the area under the predictive density to the right of θ_U, *i.e.* if

$$\int_0^{\theta_L} f(\theta,\mu,\kappa)d\theta = \int_{\theta_U}^{2\pi} f(\theta,\mu,\kappa)d\theta = \frac{\alpha}{2} \tag{3}$$

then the corresponding predictive interval is termed as $100(1-\alpha)\%$ equal tail area predictive interval.

It can further be seen that the $100(1-\alpha)\%$ equal tail area predictive lower and upper limits, *viz.* θ_L and θ_U are nothing but the $\left(\frac{100\alpha}{2}\right)$th and $\left(100-\frac{100\alpha}{2}\right)$th percentiles of the predictive distribution respectively, since the density is symmetrical.

4. Predictive Risk Function and Predictive Loss in Predictive Density Estimation of Circular Random Variable

Analogous to the posterior expected loss in the Bayesian parametric inference literature; we have the concept of induced loss in the Predictive inference literature.
Suppose $L(y,a)$ is the loss function associated with predicting the true value $y \in Y$ of a future observation (or set of observations) by $a \in Y,$ where Y is the set of future

observation(s). Further assume that $g(y|\tilde{x})$ is the predictive density conditioned on the past and present observations $\tilde{x}$.
Then the expected loss with respect to the predictive density $g(y|\tilde{x})$ is given by

$$L(a) = \int_y L(y,a) g(y|\tilde{x}) dy$$

In Bayesian predictive context, Aitchison and Dunsmore (1975) had termed this expected predictive loss as the "Induced Loss" and the value of a that minimized the induced loss had been called as the Bayes point predictor of the future observation y.
In the Frequentist predictive inference literature, the expected loss w.r.t the predictive density may be termed as the predictive risk function and our aim would consist in minimizing the predictive risk function or equivalently, to find an optimum value of a for which the predictive risk function will be the minimum. This predictor may be termed as the frequentist predictive point estimator.

Here, we are dealing with the predictive density estimation of a circular random variable which lies in the range $(0,2\pi)$. So, special loss functions need to be designed which consider the periodicity property of the circular r.v. The loss function is essentially a non-negative function as the loss incurred is positive if the predicted value is different from the true value and zero, otherwise. In other words, the loss function should be an increasing function of the absolute difference between the true value and its predicted value. The circular loss function is hereby considered and the frequentist predictive point estimator of the future observation y under this loss function for the *l*-modal Circular Normal predictive density has been worked out.

The circular loss function is defined in the literature as follows:

$$L(y,a) = 1 - \cos(a-y); 0 < a, y < 2\pi$$

The circular loss function can be seen to be a mapping from the set $[0,\pi]$ to $[0,2]$.

The predictive risk function of θ_{n+1} under circular loss function is found to be

$$1 - A(\hat{\kappa}_1)\cos\left(a - \hat{\mu}_1\right)$$

Solution of the equation $\frac{d}{da} L(a) = 0$ yields the stationary value of a to be

$$a = n\pi + \hat{\mu}_1 \quad , \qquad n = 0,1$$

We further see that the value a_0 of a for which $\frac{d^2}{da^2} L(a)|_{a=a_0} > 0$ is attained and hence, becomes the frequentist predictive point estimator of θ_{n+1} given $\theta_1, \theta_2, \dots, \theta_n$ under the circular loss function is found to be

$$n\pi + \hat{\mu}_1 \quad , \qquad n = 0$$

or

$$\hat{\mu}_1 = \bar{\theta}_{l,n} \tag{4}$$

We, thus, see that the frequentist predictive point estimate under the circular loss function coincides with the mean direction of the observations $l\theta_1, l\theta_2, \ldots, l\theta_n$.

5. Properties of the frequentist predictive point estimate under the circular loss function

The frequentist predictive point estimates under the circular loss function is equal to the sample circular mean direction of the observations $l\theta_1, l\theta_2, \ldots, l\theta_n$. It has the following properties:

a) The conditional sampling distribution of $\bar{\theta}_{l,n}$ given the resultant length $R = r_{l,n}$, is given by

$$f(\bar{\theta}_{l,n} \,|R = r_{l,n}) \sim VM(l\mu, \kappa r_{l,n})$$

b) $\bar{\theta}_{l,n}$ is an unbiased estimate of $l\mu$.

The proofs of both these properties are deferred to the Appendix A.

6. Result and Analysis

6.1. *l*-modal circular normal distribution as density of the past data on wind direction

Figure 1 displays the histogram of the daily wind direction data collected from Dibrugarh Meteorological station measured during morning for the Monsoon season (June-September) during the years 2012 and 2013:

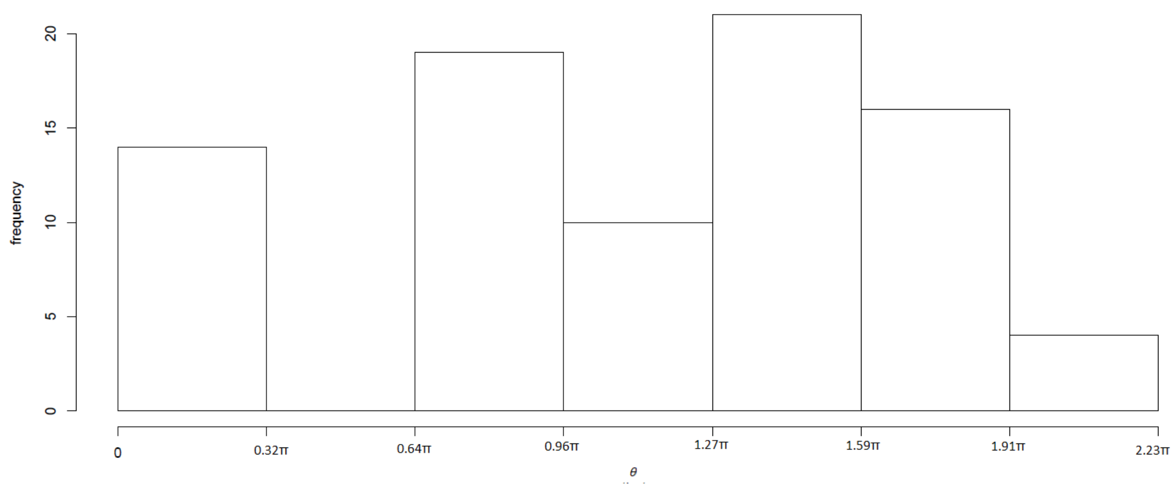

Figure 1: Histogram of the wind direction data collected from Dibrugarh Meteorological station measured during morning for the monsoon season during the years 2012 and 2013

The histogram of the wind direction data under consideration is showing the data to have 3 equidistant modes. The maximum likelihood estimates of the parameters of the *l*-modal Circular Normal distribution are

$\hat{\mu} = 0.1745$ (measured in radians), $\hat{\kappa} = 0.0127$, $\hat{l} = 3$.

The goodness-of-fit test that has been employed here is based on Watson's U^2 test of circular uniformity (Mardia and Jupp, 2000). The critical value of the test statistic at 1% level of significance has been found to be 0.267 whereas the observed value is 0.2397. Thus, the *l*-modal Circular Normal distribution is found to be a good fit to the data on wind direction for Dibrugarh Meteorological station measured during morning for the Monsoon season (June-September) during the years 2012 and 2013. We, therefore, carry out the predictive density estimation of the future observation through sufficiency approach under the *l*-modal Circular Normal model.

6.2. Predictive density estimation

From the data, we have found

$$\bar{\theta}_{l,n} = 0.5235 \text{ and } \frac{2R_{l,n}}{n+1} = 0.0126$$

Thus,

$$g(\theta_{n+1}|\theta_1, \theta_2, \ldots, \theta_n) \sim VM(0.5235, 0.0126)$$

i.e., the distribution of θ_{n+1} given $\theta_1, \theta_2, \ldots, \theta_n$ is von Mises with parameters 0.5235 and 0.0126.

6.3. Predictive Interval Estimation

A 95% equal tail area predictive interval for θ_{n+1} is given by $[\theta_L, \theta_U]$ where θ_L=2.5th Percentile and θ_U=97.5th Percentile of $VM(0.5235, 0.0126)$ distribution.

Solving (3) for $\alpha = 0.05$ yields

$$\theta_L = 0.157 \text{ and } \theta_U = 6.126$$

Thus, the 95% equal tail area predictive interval for θ_{n+1} is given by $[0.157, 6.126]$. The interpretation of the above statement is "There is 95% chance that the future observation θ_{n+1} would lie between 0.157 and 6.126".

Similarly, a 90% equal tail area predictive interval for θ_{n+1}, is represented by $[\theta_L', \theta_U']$ where θ_L'=5th Percentile and θ_U'=95th Percentile of $VM(0.5235, 0.0126)$ distribution.

Solving (3) for $\alpha = 0.10$ gives

$$\theta_L' = 0.282 \text{ and } \theta_U' = 6.001$$

Thus, the 99% equal tail area predictive interval for θ_{n+1} is given by $[0.282, 6.001]$. This means there is a 99% chance that the future observation θ_{n+1} would lie within the values 0.282 and 6.001.

6.4. Predictive risk function and frequentist predictive point estimator of the future observation under the circular loss function

In this section, we determine the frequentist predictive point estimator of the future observation under the circular loss function.

It can be seen from expression (4) that the frequentist predictive point estimator of the future observation θ_{n+1} under the Circular loss function (measured in radians) is

$$\hat{\theta}_{n+1} = 0.5235$$

7. Discussion

Through this paper, the future density of wind direction prevailing at Dibrugarh Meteorological station located in Assam, measured during morning for the Monsoon season (June-September) during the years 2012 and 2013, conditioned on the past and present wind direction data using the Sufficiency Approach of Predictive Inference under the *l*-modal Circular Normal model has been derived and then the equal tail area predictive interval estimation of the future observation has been carried out. The predictive point estimator of the future observation under circular loss function has been obtained, which has been termed as the frequentist predictive point estimator. Lastly, the properties of the frequentist predictive point estimator have been explored and it has been found that it follows von Mises or Circular Normal distribution.

As a future scope of the present study, the frequentist predictive point estimator under the different circular distributions can be studied assuming several loss functions and compare their relative efficiencies. Having obtained these estimators, one can then attempt to explore the properties of these estimators.

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APPENDIX A

A.1: If $\theta_1, \theta_2, \ldots, \theta_n$ is a random sample from l-modal Circular Normal distribution with mean direction μ and concentration parameter κ, the conditional sampling distribution of $\bar{\theta}_{l,n}$, the mean direction of $l\theta_1, l\theta_2, \ldots, l\theta_n$ given the resultant length $R = r_{l,n}$, is given by

$$f(\bar{\theta}_{l,n} \,|R_{l,n} = r_{l,n}) \sim VM(l\mu, \kappa r_{l,n})$$

Proof: Given the random sample $\theta_1, \theta_2, \ldots, \theta_n$ from l-modal Circular Normal distribution with mean direction μ and concentration parameter κ, the joint density of the observations $(\theta_1, \theta_2, \ldots, \theta_n)$ is given by

$$f_{\kappa,l}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{I_0^n(\kappa)(2\pi)^n} e^{\kappa \sum_{i=1}^n \cos(l\theta_i - l\mu)}$$

$$= \frac{1}{I_0^n(\kappa)(2\pi)^n} e^{\kappa \sum_{i=1}^n (\cos l\theta_i \cos l\mu + \sin l\theta_i \sin l\mu)}$$

$$= \frac{e^{(\kappa \cos l\mu)(\sum_{i=1}^n \cos l\theta_i) + (\kappa \sin l\mu)(\sum_{i=1}^n \sin l\theta_i)}}{I_0^n(\kappa)(2\pi)^n}$$

$$= \left\{\frac{e^{(\kappa \cos l\mu)(\sum_{i=1}^n \cos l\theta_i) + (\kappa \sin l\mu)(\sum_{i=1}^n \sin l\theta_i)}}{I_0^n(\kappa)}\right\} \frac{1}{(2\pi)^n}$$

$$= \left\{\frac{e^{(\kappa \cos l\mu)(\sum_{i=1}^n \cos l\theta_i) + (\kappa \sin l\mu)(\sum_{i=1}^n \sin l\theta_i)}}{I_0^n(\kappa)}\right\} f_0(\theta_1, \theta_2, \ldots, \theta_n) \qquad (5)$$

$f_0(\theta_1, \theta_2, \ldots, \theta_n)$ being the joint density of a random sample from Circular Uniform distribution, whose concentration parameter $\kappa = 0$.

It was established by Kent *et al*. (1979) that for circular uniform samples, the resultant length $R_{l,n}$ and $\bar{\theta}_{l,n}$ and hence, $\left(\bar{\theta}_{l,n} - \mu\right) = \beta$ is independently distributed and $R_{l,n}, \beta$ have the following respective distributions:

$$f_0\left(r_{l,n}\right) = r_{l,n}\psi_n\left(r_{l,n}\right), \quad f_0(\beta) = \frac{1}{2\pi}$$

Again, we know that the sample mean direction is rotationally equivariant. So, $\left(\bar{\theta}_{l,n} - l\mu\right) = \beta'$ (say) is also uniformly distributed in the range $(0,2\pi)$.

Following Rao and Sen Gupta (2001), pp. 71, it can be seen that the pdf of $\cos\beta' = c'$ (say) is

$$f_0(c') = \frac{1}{\pi\sqrt{1 - c'^2}}$$

Consequently, the joint distribution of the resultant length $r_{l,n}$ and c' for a sample from circular uniform distribution is

$$f_0\left(r_{l,n}, c'\right) = \frac{r_{l,n}\psi_n\left(r_{l,n}\right)}{\pi\sqrt{1 - c'^2}}$$

The joint density of the resultant length $R_{l,n}$ and the quantity $\cos\left(\bar{\theta}_{l,n} - l\mu\right) = c'$ for the l-modal Circular Normal (μ, κ) distribution can be obtained from the joint density in expression (5) by integrating over the samples that have given values of $R_{l,n}$ and c'.

Let $\mathcal{A} = \left\{(\theta_1, \theta_2, \dots, \theta_n): R_{l,n} = \sqrt{(\sum_{i=1}^{n}\cos\theta_i)^2 + (\sum_{i=1}^{n}\sin\theta_i)^2} = r_{l,n}, c' = \cos\beta'\right\}$.

Thus, the joint density of $\left(r_{l,n}, c'\right)$ for the l-modal Circular Normal distribution is given by

$$f_{\kappa,l}\left(r_{l,n}, c'\right) = \int_{\mathcal{A}} f_{\kappa,l}(\theta_1, \theta_2, \dots, \theta_n) d\theta_1 d\theta_2 \dots d\theta_n$$

$$= \frac{e^{(\kappa\cos l\mu)(\sum_{i=1}^{n}\cos l\theta_i) + (\kappa\sin l\mu)(\sum_{i=1}^{n}\sin l\theta_i)}}{I_0^n(\kappa)} \int_{\mathcal{A}} f_0(\theta_1, \theta_2, \dots, \theta_n) d\theta_1 d\theta_2 \dots d\theta_n$$

$$= \frac{e^{(\kappa\cos l\mu)(\sum_{i=1}^{n}\cos l\theta_i) + (\kappa\sin l\mu)(\sum_{i=1}^{n}\sin l\theta_i)}}{I_0^n(\kappa)} f_0\left(r_{l,n}, c'\right)$$

$$= \frac{e^{(\kappa\cos l\mu)(r_{l,n}\cos\bar{\theta}_{l,n}) + (\kappa\sin l\mu)(r_{l,n}\sin\bar{\theta}_{l,n})}}{I_0^n(\kappa)} \frac{r_{l,n}\psi_n\left(r_{l,n}\right)}{\pi\sqrt{1 - c'^2}}$$

$$= \frac{e^{(\kappa r_{l,n})\cos(\bar{\theta}_{l,n} - l\mu)}}{I_0^n(\kappa)} \frac{r_{l,n}\psi_n\left(r_{l,n}\right)}{\pi\sqrt{1 - c'^2}}$$

$$= \frac{e^{(\kappa r_{l,n})c'}}{I_0^n(\kappa)} \frac{r_{l,n}\psi_n\left(r_{l,n}\right)}{\pi\sqrt{1 - c'^2}} \tag{6}$$

Integrating expression (6) over the range $(-1,1)$ of c', we get the marginal density of the resultant length $R_{l,n}$ of a sample from the l-modal Circular Normal distribution as

$$f_{\kappa,l}(r_{l,n}) = \int_{-1}^{1} f_{\kappa,l}(r_{l,n}, c')dc'$$

$$= \int_{-1}^{1} \frac{e^{(\kappa r_{l,n})c'}}{I_0^n(\kappa)} \frac{r_{l,n}\psi_n(r_{l,n})}{\pi\sqrt{1-c'^2}} dc'$$

$$= \frac{r_{l,n}\psi_n(r_{l,n})}{I_0^n(\kappa)} \int_{-1}^{1} \frac{e^{(\kappa r_{l,n})c'}}{\pi\sqrt{1-c'^2}} dc'$$

$$= \frac{r_{l,n}\psi_n(r_{l,n})}{I_0^n(\kappa)} \int_{0}^{2\pi} \frac{e^{(\kappa r_{l,n})\cos\beta'}}{2\pi}\, d\beta'$$

$$= \frac{I_0(\kappa r_{l,n})}{I_0^n(\kappa)} r_{l,n}\psi_n(r_{l,n}) \tag{7}$$

Again, as $R_{l,n}$ and $\bar{\theta}_{l,n}$ are independently distributed for circular uniform samples, their joint distribution is

$$f_0(r_{l,n}, \bar{\theta}_{l,n}) = r_{l,n}\psi_n(r_{l,n})\frac{1}{2\pi}$$

Suppose $\mathcal{B} = \left\{(\theta_1, \theta_2, \dots, \theta_n): R_{l,n} = \sqrt{(\sum_{i=1}^{n}\cos\theta_i)^2 + (\sum_{i=1}^{n}\sin\theta_i)^2} = r_{l,n}, \bar{\theta}_{l,n} = arctan\frac{\sum_{i=1}^{n}\sin l\theta_i}{\sum_{i=1}^{n}\cos l\theta_i}\right\}$

The joint density of $(r_{l,n}, \bar{\theta}_{l,n})$ for l-modal Circular Normal distribution is

$$f_{\kappa,l}(r_{l,n}, \bar{\theta}_{l,n}) = \int_{\mathcal{B}} f_{\kappa,l}(\theta_1, \theta_2, \dots, \theta_n)d\theta_1 d\theta_2 \dots d\theta_n$$

$$= \frac{e^{(\kappa\cos l\mu)(r_{l,n}\cos\bar{\theta}_{l,n})+(\kappa\sin l\mu)(r_{l,n}\sin\bar{\theta}_{l,n})}}{I_0^n(\kappa)} f_0(r_{l,n}, \bar{\theta}_{l,n})$$

$$= \frac{e^{(\kappa r_{l,n})\cos(\bar{\theta}_{l,n}-l\mu)}}{I_0^n(\kappa)} r_{l,n}\psi_n(r_{l,n})\frac{1}{2\pi} \tag{8}$$

Therefore, the conditional distribution of $\bar{\theta}_{l,n}$ given $R_{l,n} = r_{l,n}$ is obtained as follows:

$$f(\bar{\theta}_{l,n} \,|R_{l,n} = r_{l,n}) = \frac{f_{\kappa,l}(r_{l,n}, \bar{\theta}_{l,n})}{f_{\kappa,l}(r_{l,n})}$$

$$= \frac{\frac{e^{(\kappa r_{l,n})\cos(\bar{\theta}_{l,n}-l\mu)}}{I_0^n(\kappa)} r_{l,n}\psi_n(r_{l,n})\frac{1}{2\pi}}{\frac{I_0(\kappa r_{l,n})}{I_0^n(\kappa)} r_{l,n}\psi_n(r_{l,n})}$$

$$= \frac{e^{(\kappa r_{l,n})\cos(\bar{\theta}_{l,n}-l\mu)}}{2\pi I_0(\kappa r_{l,n})}$$

which is the pdf of Von Mises distribution with mean direction $l\mu$ and concentration parameter $\kappa r_{l,n}$.

A.2: The mean direction $\bar{\theta}_{l,n}$ of $l\theta_1, l\theta_2, \ldots, l\theta_n$, where $\theta_1, \theta_2, \ldots, \theta_n$ is a sample from the *l*-modal Circular Normal distribution with mean direction μ and concentration parameter κ is an unbiased estimator of $l\mu$.

Proof: In the context of circular statistics, an estimate t taking values on the unit circle is said to be unbiased for a parameter α of a circular probability distribution (Mardia and Jupp, 2000, pp.83) if

$$\frac{E(\cos t, \sin t)}{\| E(\cos t, \sin t) \|} = (\cos\alpha, \sin\alpha)$$

It follows from proof (1) that the for samples from *l*-modal Circular Normal distribution,

$$\bar{\theta}_{l,n} \sim VM(l\mu, \kappa r_{l,n})$$

Thus,

$$E(\cos\bar{\theta}_{l,n}, \sin\bar{\theta}_{l,n}) = \left(E(\cos\bar{\theta}_{l,n}), E(\sin\bar{\theta}_{l,n})\right)$$

$$= \left(\frac{I_1(\kappa r_{l,n})}{I_0(\kappa r_{l,n})}\cos l\mu, \frac{I_1(\kappa r_{l,n})}{I_0(\kappa r_{l,n})}\sin l\mu\right)$$

$$= \left(A(\kappa r_{l,n})\cos l\mu, A(\kappa r_{l,n})\sin l\mu\right)$$

and

$$\| E(\cos\bar{\theta}_{l,n}, \sin\bar{\theta}_{l,n}) \| = \sqrt{\left(A(\kappa r_{l,n})\cos l\mu\right)^2 + \left(A(\kappa r_{l,n})\sin l\mu\right)^2}$$

$$= \sqrt{\left(A(\kappa r_{l,n})\right)^2}$$

$$= A(\kappa r_{l,n})$$

Finally,

$$\frac{E\left(\cos\bar{\theta}_{l,n}, \sin\bar{\theta}_{l,n}\right)}{\parallel E\left(\cos\bar{\theta}_{l,n}, \sin\bar{\theta}_{l,n}\right)\parallel} = \frac{\left(A\left(\kappa r_{l,n}\right)\cos l\mu, A\left(\kappa r_{l,n}\right)\sin l\mu\right)}{A\left(\kappa r_{l,n}\right)}$$

$$= (\cos l\mu, \sin l\mu)$$

or $\bar{\theta}_{l,n}$ is an unbiased estimator of $l\mu$

APPENDIX B

Table B.1: Data set on daily wind direction for Dibrugarh Meteorological station located in Assam, measured (in degrees) during morning for the Monsoon season (June-September) during the years 2012 and 2013

Wind direction measured in degrees			
50	230	150	230
320	230	150	140
150	20	270	320
360	50	230	270
320	180	180	150
140	150	50	50
150	320	180	50
230	320	150	320
20	270	320	320
210	180	270	230
20	270	50	360
150	210	140	20
320	230	20	270
230	150	320	360
150	50	150	150
360	320	320	180
210	270	50	230
210	150	320	320
150	20	150	230
210	150	230	
230	320	230	

Source: Regional Meteorological Center, Guwahati, Assam

For the remaining days, no wind flow was detected and so, the measure of wind direction corresponding to those days were reported as NIL (and are, hence, excluded from the data set).