Statistics and Application {ISSN 2454-7395 (online)}
Volume 21, No. 2, 2023 (New Series), pp 17-36
http://www.ssca.org.in/journal

Estimating Finite Population Mean using Multiple Parameters of an Ancillary Variable

Deepak Singh[1], Rohini Yadav[2] and Rajesh Singh[3]

[1]*ICAR-Indian Agricultural Statistics Research Institute, New Delhi, India.*
[2]*Department of Statistics, University of Lucknow, Lucknow, India.*
[3]*Department of Statistics, Institute of Science, Banaras Hindu University, Varanasi, India*

Received: 20 June 2022; Revised: 25 August 2022; Accepted: 09 September 2022

Abstract

This study deals with an improved class of estimators for estimating the unknown finite population mean of the study variable using auxiliary information. It has been developed by using the power transformation in Singh and Yadav (2017) family of estimators. The expression for bias and mean squared error of the proposed estimator is derived under large sample approximation. The conditions have been derived for the suggested class of estimators under which it performs better than the estimators considered in this study. The theoretical results are supported by numerical illustration. Two phase sampling version of the proposed family of estimators is suggested and its properties are also studied.

Keywords: Study variable; Auxiliary variable; Bias; Mean squared error; Ratio-product-ratio type estimator; Simple random sampling; Double sampling.

Mathematics Subject Classification Code: 62D05.

1. Introduction and notations used

In survey sampling, it is well recognized that the use of auxiliary information results in substantial gain in efficiency over the estimators which do not utilize such information. When the auxiliary variable is available, the ratio, product and regression methods of estimation are the classical examples, which uses auxiliary information and are better than usual mean estimator.

Let there be a finite population $U=\{U_1,U_2,...U_N\}$ of N units and $(y,\ x)$ be the study and auxiliary variables assuming real non-negative values of the finite population U. The population means of the study and auxiliary variables are denoted by $\bar{Y}=\sum_{i=1}^{N} Y_i / N$, $\bar{X}=\sum_{i=1}^{N} X_i / N$; respectively, and sample means by $\bar{y}=\sum_{i=1}^{n} y_i / n$, $\bar{x}=\sum_{i=1}^{n} x_i / n$ respectively.

Corresponding Author: Rohini Yadav
Email: rohiniyadav.ism@gmail.com

Some common notations used in this paper are-

The population variance of the study variable y: $S_y^2 = \left\{1/\left(N-1\right)\right\}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)^2$

The population variance of the auxiliary variable x: $S_x^2 = \left\{1/\left(N-1\right)\right\}\sum_{i=1}^{N}\left(x_i - \bar{X}\right)^2$

The population covariance: $S_{xy} = \left\{1/\left(N-1\right)\right\}\sum_{i=1}^{N}\left(x_i - \bar{X}\right)\left(y_i - \bar{Y}\right)$

The coefficient of variation of x: $C_x = S_x / \bar{X}$,

The coefficient of variation of y: $C_y = S_y / \bar{Y}$

The population correlation coefficient of x and y: $\rho = S_{xy} / S_x S_y$

and $C = \rho C_y / C_x$.

To estimate the unknown population mean of the study variable $\bar{Y}$, let n pairs of sample observations $(y_i, x_i), i = 1, 2, ...n$ are drawn using simple sampling without replacement (SRSWOR) from the population U for the study and auxiliary variables respectively. In case no auxiliary information is available, the mean squared error of usual unbiased estimator for population mean under SRSWOR is given by

$$\text{MSE}\left(\bar{y}\right) = \eta \bar{Y}^2 C_y^2 \tag{1}$$

where, $\eta = n^{-1}\left(1-f\right)$, $f = n/N$ (sample fraction).

It is assumed that the population mean of the auxiliary variable $\bar{X}$ is known. The classical ratio estimator $\bar{y}_R = \bar{y}\left(\bar{X}/\bar{x}\right)$ suggested by Cochran (1940) is useful when the study variable and auxiliary variable are positively correlated but when study variable and auxiliary variable are negatively correlated, product estimator $\bar{y}_P = \bar{y}\left(\bar{x}/\bar{X}\right)$ given by Murthy (1964) is more appropriate. The expression for biases and mean squared errors for ratio and product estimators are respectively given by-

$$\text{Bias}\left(\bar{y}_R\right) = \eta \bar{Y} C_x^2 \left(1-C\right) \tag{2}$$

$$\text{MSE}\left(\bar{y}_R\right) = \eta \bar{Y}^2 \left[C_y^2 + C_x^2\left(1-2C\right)\right] \tag{3}$$

$$\text{Bias}\left(\bar{y}_P\right) = \eta \bar{Y} C_x^2 C \tag{4}$$

$$\text{MSE}\left(\bar{y}_P\right) = \eta \bar{Y}^2 \left[C_y^2 + C_x^2\left(1+2C\right)\right] \tag{5}$$

Improved estimators for estimating unknown population mean of the study variable utilizing auxiliary are studied by various authors viz. Searls (1964), Upadhyaya *et al.* (1985), Upadhyaya and Singh (1999), Singh and Ruiz Espejo (2003), Upadhyaya *et al.* (2011), Yadav *et al.* (2012), Yadav *et al.* (2013) etc. and the references cited therein.

Chami *et al.* (2012) proposed two-parameter ratio-product-ratio estimator for estimating unknown population mean of the study variable is given by

$$T_{\alpha,1-\alpha,\beta}^{1,1} = \bar{y}\left[\alpha\left\{\frac{(1-\beta)\bar{x}+\beta\bar{X})}{\beta\bar{x}+(1-\beta)\bar{X}}\right\} + (1-\alpha)\left\{\frac{\beta\bar{x}+(1-\beta)\bar{X}}{(1-\beta)\bar{x}+\beta\bar{X}}\right\}\right] \tag{6}$$

Following Chami *et al.* (2012), Singh and Yadav (2017) proposed a ratio-product-ratio family of estimators given by

$$T_{\alpha_1,\alpha_2,\beta}^{\delta,1} = \bar{y}\left[\alpha_1\left\{\frac{(1-\beta)\bar{x}+\beta\bar{X})}{\beta\bar{x}+(1-\beta)\bar{X}}\right\}^{\delta} + \alpha_2\left\{\frac{\beta\bar{x}+(1-\beta)\bar{X}}{(1-\beta)\bar{x}+\beta\bar{X}}\right\}\right] \tag{7}$$

In this paper, a generalized family of ratio-product-ratio type estimators for estimating the population mean of study variable y is proposed which generalizes the earlier works of Chami *et al*. (2012) and Singh and Yadav (2017). It is assumed throughout the paper that the population size N is very large so that the finite population correction term is ignored and $(N-1)\cong N$.

2. The proposed family of estimators

Motivated by Singh and Yadav (2017), we have proposed the following five-parameter ratio-product-ratio type estimator for estimating the population mean $\bar{Y}$ as follows

$$T_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma} = \bar{y}\left[\alpha_1\left\{\frac{(1-\beta)\bar{x}+\beta\bar{X})}{\beta\bar{x}+(1-\beta)\bar{X}}\right\}^{\delta} + \alpha_2\left\{\frac{\beta\bar{x}+(1-\beta)\bar{X}}{(1-\beta)\bar{x}+\beta\bar{X}}\right\}^{\gamma}\right] \tag{8}$$

where α_1, α_2 are constants to be determined such that MSE of the generalized class is minimum, and δ, γ are constants which take finite values for designing the different estimators and β can take any values of the known parameters like coefficient of variation, coefficient of skewness, coefficient of kurtosis and the correlation coefficient (see Singh and Kumar (2011) and Singh and Solanki (2012)). Introducing power transformation in the product type part of the Singh and Yadav (2017) family of estimators $T_{\alpha_1,\alpha_2,\beta}^{\delta,1}$ in the form of γ substantially improves the efficiency of the Singh and Yadav (2017) estimator.

2.1. First-degree approximation to the bias and mean squared error

To obtain the bias and mean squared error (MSE) up to first-degree approximation, we define the following relative error terms

$$\bar{y} = \bar{Y}(1+e_o) \text{ and } \bar{x} = \bar{X}(1+e_1)$$

such that

$E(e_o)=E(e_1)=0,\ E(e_o^2)=\eta C_y^2,\ E(e_1^2)=\eta C_x^2$ and $E(e_o e_1)=\eta\rho C_y C_x=\eta C C_x^2$

We assume that the sample size n is large enough such that contributions from $E(e_o^i),\ E(e_1^i)$ when $i>2$ and $E(e_o^i e_1^j)$ when $(i+j)>2$ are negligible. Expressing the equation (8) in error terms $(e_i\,'s)$, we get

$$T_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}=\bar{y}(1+e_o)\left[\alpha_1\{1+(1-\beta)e_1\}^{\delta}(1+\beta e_1)^{-\delta}+\alpha_2(1+\beta e_1)^{\gamma}\{1+(1-\beta)e_1\}^{-\gamma}\right] \tag{9}$$

Expanding $\{1+(1-\beta)e_1\}^{\delta}$, $(1+\beta e_1)^{-\delta}$, $(1+\beta e_1)^{\gamma}$ and $\{1+(1-\beta)e_1\}^{-\gamma}$ as a series in powers of e_1, and assuming $|e_1|<\min.\{1/|\beta|,1/|(1-\beta)|\}$, keeping series up to $O(e_1^2)$ and neglecting higher orders, the bias of $T_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}$ to order $O(n^{-1})$ is obtained as

$$B\left(T_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}\right)=\bar{Y}\begin{bmatrix}\alpha_1\left\{1+\dfrac{\delta(1-2\beta)}{2}\eta C_X^2\left(\delta(1-2\beta)+2C-1\right)\right\}\\ +\alpha_2\left\{1+\dfrac{\gamma(1-2\beta)}{2}\eta C_X^2\left(\gamma(1-2\beta)-2C+1\right)\right\}-1\end{bmatrix} \tag{10}$$

The bias tends to zero when n tends to N and $\alpha_1+\alpha_2=1$. The $MSE(T_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma})$ of the suggested family of estimators to the first degree of approximation is given by

$$MSE\left(T_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}\right)=\bar{Y}^2\left(1+\alpha_1^2 Z_1+\alpha_2^2 Z_2+2\alpha_1\alpha_2 Z_3-2\alpha_1 Z_4-2\alpha_2 Z_5\right) \tag{11}$$

where,

$$Z_1=1+\eta\left[C_y^2+C_x^2\delta(1-2\beta)\{2\delta(1-2\beta)+4C-1\}\right]$$
$$Z_2=1+\eta\left[C_y^2+C_x^2\gamma(1-2\beta)\{2\gamma(1-2\beta)-4C+1\}\right]$$
$$Z_3=1+\eta\left[C_y^2+C_x^2\frac{(1-2\beta)(\delta-\gamma)}{2}\{(1-2\beta)(\delta-\gamma)+4C-1\}\right]$$
$$Z_4=1+\eta C_x^2\frac{\delta(1-2\beta)}{2}\{\delta(1-2\beta)+2C-1\}$$
$$Z_5=1+\eta C_x^2\frac{\gamma(1-2\beta)}{2}\{\gamma(1-2\beta)-2C+1\}$$

Differentiating the $MSE\left(T_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}\right)$ at equation (11) with respect to α_1 and α_2 and equating them to zero, we get

$$\begin{bmatrix}Z_1 & Z_3\\ Z_3 & Z_2\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}Z_4\\ Z_5\end{bmatrix} \tag{12}$$

Solving equation (12), we get the optimum values of α_1 and α_2 respectively as

$$\begin{aligned} \alpha_1 &= \left(\frac{Z_2Z_4 - Z_3Z_5}{Z_1Z_2 - Z_3^2}\right) = \alpha_1' \\ \alpha_2 &= \left(\frac{Z_1Z_5 - Z_3Z_4}{Z_1Z_2 - Z_3^2}\right) = \alpha_2' \end{aligned} \tag{13}$$

Putting the optimum values α_1' and α_2' in place of α_1 and α_2 in equation (11), the minimum MSE of the suggested estimator is given by

$$MSE_{\min}\left(T_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}\right) = \bar{Y}^2\left\{1 - \frac{\left(Z_2Z_4^2 + Z_1Z_5^2 - 2Z_3Z_4Z_5\right)}{Z_1Z_2 - Z_3^2}\right\} \tag{14}$$

The equation (14) provides the minimum value of the MSE of the proposed family of estimator $T_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}$.

3. A four-parameter ratio-product-ratio estimator

Putting $\alpha_1 = \alpha$ and $\alpha_2 = 1-\alpha$, the four parameters ratio-product-ratio estimator $T_{\alpha,1-\alpha,\beta}^{\delta,\gamma}$ is given by

$$T_{\alpha,1-\alpha,\beta}^{\delta,\gamma} = \bar{y}\left[\alpha\left(\frac{(1\text{-}\beta)\bar{x} + \beta\bar{X})}{\beta\bar{x} + (1\text{-}\beta)\bar{X}}\right)^{\delta} + (1-\alpha)\left(\frac{\beta\bar{x} + (1\text{-}\beta)\bar{X}}{(1\text{-}\beta)\bar{x} + \beta\bar{X}}\right)^{\gamma}\right] \tag{15}$$

where $\alpha_1 + \alpha_2 = 1$. The bias and MSE of the estimator upto the first degree of approximation are respectively given by

$$B\left(T_{\alpha,1-\alpha,\beta}^{\delta,\gamma}\right) = \bar{Y}\frac{(1-2\beta)}{2}\eta C_X^2\left[(1-2\beta)\left\{\alpha\left(\delta^2-\gamma^2\right)+\gamma^2\right\}+(2C-1)\left\{\alpha(\delta+\gamma)-\gamma\right\}\right] \tag{16}$$

$$\left.\begin{aligned} &MSE\left(T_{\alpha,1-\alpha,\beta}^{\delta,\gamma}\right) = \bar{Y}^2\eta\left[C_y^2 + C_x^2(1-2\beta)\left\{\gamma-\alpha(\delta+\gamma)\right\}\left[(1-2\beta)\left\{\gamma-\alpha(\delta+\gamma)\right\}-2C\right]\right] \\ &\text{or} \\ &MSE\left(T_{\alpha,1-\alpha,\beta}^{\delta,\gamma}\right) = \bar{Y}^2\left[(1+Z_2-2Z_5)+\alpha^2(Z_1+Z_2-2Z_3)-2\alpha(Z_2-Z_3+Z_4-Z_5)\right] \end{aligned}\right\} \tag{17}$$

$MSE\left(T_{\alpha,1-\alpha,\beta}^{\delta,\gamma}\right)$ is minimum when $\alpha = \dfrac{(Z_2-Z_3+Z_4-Z_5)}{(Z_1+Z_2-2Z_3)} = \alpha_{opt}$

and is given by

$$MSE_{\min}\left(T_{\alpha,1-\alpha,\beta}^{\delta,\gamma}\right) = \bar{Y}^2\left[1+Z_2-2Z_5-\frac{2(Z_2-Z_3+Z_4-Z_5)^2}{Z_1+Z_2-2Z_3}\right] = \eta S_Y^2\left(1-\rho^2\right) = MSE\left(\bar{y}_{lr}\right) \tag{18}$$

In equation (18), $MSE(\bar{y}_{lr})$ indicates the mean square error of the linear regression estimator $\bar{y}_{lr} = \bar{y} + \beta(\bar{x} - \bar{X})$. So, $T_{\alpha,1-\alpha,\beta}^{\delta,\gamma}$ is equally efficient to the regression estimator.

4. Efficiency comparison

From equations (1), (3), (5) and (18), we get

$$MSE(\bar{y}) - MSE_{\min}(T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}) = \eta S_Y^2 \rho^2 > 0 \tag{19}$$

$$MSE(\bar{y}_R) - MSE_{\min}(T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}) = \eta \bar{Y}^2 C_x^2 (1-C)^2 > 0 \tag{20}$$

$$MSE(\bar{y}_p) - MSE_{\min}(T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}) = \eta \bar{Y}^2 C_x^2 (1+C)^2 > 0 \tag{21}$$

Hence, the $T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$ is more efficient than sample mean $\bar{y}$, ratio $\bar{y}_R$ and product $\bar{y}_P$ estimator. The minimum MSE of proposed family of estimators $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ is compared with that of four-parameter sub-family of estimators $MSE\left(T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right)$ as

$$MSE_{\min}\left(T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right) - MSE_{\min}\left(T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}\right) = \bar{Y}^2 \frac{\left[Z_1\left(Z_2 - Z_5\right) + Z_4\left(Z_3 - Z_2\right) + Z_3\left(Z_5 - Z_3\right)\right]^2}{\left(Z_1 + Z_2 - 2Z_3\right)\left(Z_1 Z_2 - Z_3^2\right)} > 0 \tag{22}$$

In case of $T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$ subfamily of the estimators i.e. $T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$, due to the restriction on α_1 and α_2 $\left(\alpha_1 + \alpha_2 = 1\right)$, both the α and $1-\alpha$ coefficients of ratio and product type part of family of estimators are interdependent to each other that leads to obtain the minimum mean square error of $T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$ under $\alpha_1 + \alpha_2 = 1$ restriction at the optimum value of α i.e. $\alpha_{opt} = \frac{\left(Z_2 - Z_3 + Z_4 - Z_5\right)}{\left(Z_1 + Z_2 - 2Z_3\right)}$. For the proposed family of estimators $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ there is no restriction on α_1 and α_2 constants, therefore, the ratio and product type part of family of estimators are independent to each other, which leads to obtain the optimum values of α_1 and α_2 separately i.e. $\alpha_1 = \left(\frac{Z_2 Z_4 - Z_3 Z_5}{Z_1 Z_2 - Z_3^2}\right) = \alpha_1'$ and $\alpha_2 = \left(\frac{Z_1 Z_5 - Z_3 Z_4}{Z_1 Z_2 - Z_3^2}\right) = \alpha_2'$. Since, the ratio and product part of the proposed family of estimators are optimized separately, the minimum mean square error of the proposed family of estimators $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ will be always lesser than its subfamily of estimators $T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$. From equation 22, it is inferred that proposed family of estimators is more efficient than its subfamily of estimators $\left(T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right)$. Therefore, the suggested family of estimators $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ is more efficient in comparison to the sample mean, ratio, product, regression, and Chami *et al*. (2012) estimator.

Comparing MSE of the proposed subfamily of estimators $T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$ and Singh and Yadav (2017) subfamily of estimators $T^{\delta,1}_{\alpha,1-\alpha,\beta}$, we get

$$MSE\left(T^{\delta,1}_{\alpha,1-\alpha,\beta}\right) - MSE\left(T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right) = \eta C_x^2 \bar{Y}^2 \begin{bmatrix} (1-2\beta)(1-\gamma) \\ \left[(1-2\beta)(1+\gamma) - 2\alpha\left\{(1+\delta) - 2C\right\}\right] \end{bmatrix} > 0 \tag{23}$$

Therefore, we get the following conditions for efficiency for $T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$ to be better performer than $T^{\delta,1}_{\alpha,1-\alpha,\beta}$,

A. $\gamma < 1,\ \beta < 1/2,\ C > (1-2\beta)\left[(1+\gamma)-2\alpha(1+\delta)\right]/2$

B. $\gamma > 1,\ \beta < 1/2,\ C < (1-2\beta)\left[(1+\gamma)-2\alpha(1+\delta)\right]/2$

C. $\gamma < 1,\ \beta > 1/2,\ C < (1-2\beta)\left[(1+\gamma)-2\alpha(1+\delta)\right]/2$

D. $\gamma > 1,\ \beta > 1/2,\ C > (1-2\beta)\left[(1+\gamma)-2\alpha(1+\delta)\right]/2$

Some known members of the proposed family of estimators $\boldsymbol{T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}}$ as well as sub-family of estimators $\boldsymbol{T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}}$ and some new members of the proposed family of estimators along with their corresponding members of Singh and Yadav (2017) estimator are given in the Table 1, Table 2 and Table 3 (see appendix) respectively.

5. Empirical study

To illustrate the relative performance of the members of the proposed family of estimators with other estimators considered in this article, the three natural populations from literature are considered whose descriptions are given below:

Population I [Source: Chami *et. al.* (2012)]

y: Maximum daily values (in feet) of groundwater for the period of October 2009 to September 2010 collected at site number 02290829501 located in Florida.

x: Maximum daily values (in feet) of groundwater for the period of October 2008 to September 2009 collected at site number 02290829501 located in Florida

$$N=365,\ n=112,\ \bar{Y}=0.5832,\ \bar{X}=0.6277,\ C_x=1.1504,\ C_y=0.7681,\ \rho=0.9125.$$

Population II [Source: Steel and Torrie (1960), pp. 282]

y: Log of leaf burn in sack

x: Chlorine percentage

$$N=30,\ n=6,\ \bar{Y}=0.6860,\ \bar{X}=0.8077,\ C_y=0.7001,\ C_x=0.7493,\ \rho=0.4996.$$

Population-III [Source: Murthy (1967), pp. 399]

y: Area under wheat in 1964

x: Area under wheat in 1963

$$N=34,\ \bar{Y}=199.4411,\ \bar{X}=208.8823,\ C_y=0.753193,\ C_x=0.720486,\ \rho=0.9801.$$

The percent relative efficiencies (PREs) of the suggested members of family of estimators

$T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ and $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ are compared with the linear regression estimator $\bar{y}_{lr}$ by using the following formula:

$$PRE(t,\bar{y}_{lr}) = \frac{Var(\bar{y}_{lr})}{MSE_{\min}(t)}*100 \text{ ; where } t = T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta} \text{ and } T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$$

The percentage gain in PRE due to the effect of optimized γ in proposed estimator (power transformation in Singh and Yadav (2017) estimator) is given by

$$\%\, gain\ in\ PRE = \{(B-A)/A\}*100$$

where A is Singh and Yadav (2017) estimator and B is our proposed estimator. The average percentage gain in PRE due to the effect of optimized γ (proposed estimator) in the Singh and Yadav (2017) estimator over three populations considered for study is given by

$$Avg\ \%\ \text{gain}\ in\ PRE = \frac{1}{3}\sum_{i=1}^{3}\left\{\frac{(B_i - A_i)}{A_i}\right\}*100,\ (i\text{=1 to 3})$$

In Table 4 (see appendix), the PRE of proposed $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ is more to corresponding $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ estimator except for four estimators *w.r.t.* population III where the efficiencies are equal. Therefore, it may be concluded that the $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ are either more or equally efficient to $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ with respect to almost all the estimators when $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$ which is indicated by average % gain in PRE. Comparing Table 4 (see appendix), it is concluded that all the $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ estimators are more efficient than linear regression estimator.

To compare the performance of $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ with $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$, different combinations have been developed for $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}(5\beta \text{ x } 4\delta \text{ x } 4\gamma = 80$ combinations) and $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}(5\beta \text{ x } 4\delta = 20$ combinations) for δ and γ taking values = 1, 1.5, 2, 2.5 for $\beta = 1, \rho, C_y, C_x$ and C. In table 5 (see appendix), out of 80 combinations of $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ and 20 combinations of $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ the best performing estimator for same value of β of $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ and $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ with respect to population I has been retained and presented.

It is clear from the Table 5 (see appendix) that at same value of β the % gain in efficiency ranges from 0 to 168.07% which concludes that at same value of β, the $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ are either equally or more efficient to $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ at $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$.

In Table 6 (see appendix), out of 80 combinations of $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ and 20 combinations of $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ the top performing estimator with respect to each population has been retained and presented.

From Table 6 (see appendix) , it is clear that the proposed family of estimators is more efficient than Singh and Yadav (2017) family of estimators when dealing with practical and real-world problems where the % gain in efficiency ranges from 20.26 to 564.64 with respect to three populations.

From the empirical study, it is concluded that $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ should be preferred over $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ Singh and Yadav (2017) family of estimators. Thus, we recommend the use of proposed family of estimators in practice.

6. The proposed family of estimators in double (two-phase) sampling

In some practical situations, the value of the population mean of the auxiliary variable is unavailable. In such situations, double sampling (two-phase sampling) is used to estimate the population mean $\bar{X}$, from a large sample of size n' drawn from population. A second sample of size $n\ (<n')$ is drawn from this preliminary large sample to observes the study variable y.

Let $\bar{x}' = \sum_{i=1}^{n} x_i \big/ n'$, $\bar{y} = \sum_{i=1}^{n} y_i \big/ n$, and $\bar{x} = \sum_{i=1}^{n} x_i \big/ n$. The usual ratio estimator, product estimator and regression estimators of population mean of study variable y in double sampling are respectively defined as

$$\bar{y}_{Rd} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right) \tag{24}$$

$$\bar{y}_{Pd} = \bar{y}\left(\frac{\bar{x}}{\bar{x}'}\right) \tag{25}$$

$$\bar{y}_{lrd} = \bar{y} + \hat{\beta}\left(\bar{x}' - \bar{x}\right) \tag{26}$$

where $\hat{\beta} = \left(s_{xy} / s_{x^2}\right)$ is the sample regression coefficient.

The double sampling version of suggested generalized family of estimators $D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ is defined as

$$D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta} = \bar{y}\left[\alpha_1\left\{\frac{(1-\beta)\bar{x} + \beta\bar{x}'}{\beta\bar{x}' + (1-\beta)\bar{x}}\right\}^{\delta} + \alpha_2\left\{\frac{\beta\bar{x} + (1-\beta)\bar{x}'}{(1-\beta)\bar{x} + \beta\bar{x}'}\right\}^{\gamma}\right] \tag{27}$$

where δ, γ are real constants and β can take values of known parameters like coefficient of variation, coefficient of skewness, coefficient of kurtosis and the correlation coefficient along with real constants and $(\alpha_1,\ \alpha_2)$ are suitably chosen constants such that the mean squared error (MSE) of the developed estimator is minimal. To obtain the bias and mean squared error (MSE) up to first-degree approximation, we define the following relative error terms

$\bar{y} = \bar{Y}\left(1+e_o\right)$, $\bar{x} = \bar{X}\left(1+e_1\right)$ and $\bar{x}' = \bar{X}\left(1+e_1'\right)$

such that $E(e_o) = E(e_1) = E(e_1') = 0$, $E(e_o^2) = \eta C_y^2 = V_{(0,2)}$, $E(e_1^2) = \eta C_x^2 = V_{(2,0)}$, $E(e_1'^2) = \eta' C_x^2 = V'_{(2,0)}$, $E(e_o e_1) = \eta\rho C_y C_x = \eta C C_x^2 = V_{(0,1)}$, $E(e_1 e_1') = \eta' C_x^2 = V'_{(1,1)}$ and $E(e_o e_1') = \eta'\rho C_y C_x = \eta' C C_x^2 = V'_{(0,1)}$

We consider the following notations for getting the expression of bias and MSE of the proposed estimator

$$\alpha_1\delta-\alpha_2\gamma=K$$

$$\alpha_1\delta^2-\alpha_2\gamma^2=L$$

$$\alpha_1\delta\left\{\delta\left(1-2\beta\right)-1\right\}+\alpha_2\gamma\left\{\gamma\left(1-2\beta\right)+1\right\}=S$$

$$\alpha_1\delta\left\{\delta\left(1-2\beta\right)+1\right\}+\alpha_2\gamma\left\{\gamma\left(1-2\beta\right)-1\right\}=T$$

$$\psi=V_{(2,0)}+V'_{(2,0)}-2V'_{(1,1)}$$

$$\omega=\gamma\left(1-2\beta\right)\left[V_{(2,0)}\left\{\gamma\left(1-2\beta\right)+1\right\}+V'_{(2,0)}\left\{\gamma\left(1-2\beta\right)-1\right\}-2V_{(0,1)}+2V'_{(0,1)}-2V'_{(1,1)}\gamma\left(1-2\beta\right)\right]$$

$$\mu=\delta\left(1-2\beta\right)\left[V_{(2,0)}\left\{\delta\left(1-2\beta\right)-1\right\}+V'_{(2,0)}\left\{\delta\left(1-2\beta\right)+1\right\}+2V_{(0,1)}-2V'_{(0,1)}-2V'_{(1,1)}\delta\left(1-2\beta\right)\right]$$

$$\tau=\delta\left(1-2\beta\right)\left[\frac{V_{(2,0)}}{2}\left\{\delta\left(1-2\beta\right)-1\right\}+\frac{V'_{(2,0)}}{2}\left\{\delta\left(1-2\beta\right)+1\right\}+2V_{(0,1)}-2V'_{(0,1)}-V'_{(1,1)}\delta\left(1-2\beta\right)\right]$$

$$\phi=\gamma\left(1-2\beta\right)\left[\frac{V_{(2,0)}}{2}\left\{\gamma\left(1-2\beta\right)+1\right\}+\frac{V'_{(2,0)}}{2}\left\{\gamma\left(1-2\beta\right)-1\right\}-2V_{(0,1)}+2V'_{(0,1)}-V'_{(1,1)}\gamma\left(1-2\beta\right)\right]$$

$$\xi=1+V_{(0,2)}$$

We assume that the sample size n is large enough such that contributions from $E\left(e_o^i\right)$, $E\left(e_1'^i\right)$ when $i>2$ and $E\left(e_o^ie_1^j\right)$ when $(i+j)>2$ are negligible. Expressing the equation (27) in error terms $(e_i\text{'}s)$, we get

$$D_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}=\bar{Y}\left(1+e_o\right)=\bar{y}\left[\alpha_1\left\{\frac{1+\beta e_1'+(1-\beta)e_1}{1+\beta e_1+(1-\beta)e_1'}\right\}^{\delta}+\alpha_2\left\{\frac{1+\beta e_1+(1-\beta)e'}{1+\beta e_1'+(1-\beta)e_1}\right\}^{\gamma}\right] \tag{28}$$

Expanding $\left\{1+\beta e_1'+(1-\beta)e_1\right\}^{\delta}$, $\left\{1+\beta e_1+(1-\beta)e'\right\}^{-\delta}$, $\left\{1+\beta e_1+(1-\beta)e'\right\}^{\gamma}$ and $\left\{1+\beta e_1'+(1-\beta)e_1\right\}^{-\gamma}$ as a series in powers of e_1 and e_1', it is assumed that $|e_1|<\min.\left\{\frac{1}{|\beta|},\frac{1}{|(1-\beta)|}\right\}$. Keeping series up to $O\left(e_1^2\right)$ and neglecting higher orders, the bias of $D_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}$ to order $O(n^{-1})$ is obtained as

$$B\left(D_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}\right)=\bar{Y}\left[\alpha_1\left(1+\frac{\mu}{2}\right)+\alpha_2\left(1+\frac{\omega}{2}\right)-1\right] \tag{29}$$

The MSE of the proposed estimator is given by

$$MSE\left(D_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}\right)=\bar{Y}^2\left(1+\alpha_1^2Z_1+\alpha_2^2Z_2+2\alpha_1\alpha_2Z_3-2\alpha_1Z_4-2\alpha_2Z_5\right) \tag{30}$$

where,

$$Z_1 = \xi + 2\tau + \delta^2\psi(1-2\beta)^2$$

$$Z_2 = \xi + 2\phi + \gamma^2\psi(1-2\beta)^2$$

$$Z_3 = \xi + \phi + \tau - \delta\gamma\psi(1-2\beta)^2$$

$$Z_4 = \frac{\mu+2}{2}$$

$$Z_5 = \frac{\omega+2}{2}$$

Differentiating the $MSE\left(D_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}\right)$ with respect to α_1 and α_2 and equating them to zero, we have

$$\begin{bmatrix} Z_1 & Z_3 \\ Z_3 & Z_2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} Z_4 \\ Z_5 \end{bmatrix} \tag{31}$$

Solving equation (31), we get the optimum values of α_1 and α_2 as

$$\begin{aligned} \alpha_1 &= \left(\frac{Z_2Z_4 - Z_3Z_5}{Z_1Z_2 - Z_3^2}\right) = \alpha_1' \\ \alpha_2 &= \left(\frac{Z_1Z_5 - Z_3Z_4}{Z_1Z_2 - Z_3^2}\right) = \alpha_2' \end{aligned} \tag{32}$$

Putting the optimum values α_1' and α_2' in place of α_1 and α_2 in equation (29) and (30), the optimum bias and the minimum mean square error of $D_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}$ is obtained as

$$B_{opt}\left(D_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}\right) = -\bar{Y}\left\{1 - \frac{\left(Z_2Z_4^2 + Z_1Z_5^2 - 2Z_3Z_4Z_5\right)}{Z_1Z_2 - Z_3^2}\right\} \tag{33}$$

$$MSE_{\min}\left(D_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}\right) = \bar{Y}^2\left\{1 - \frac{\left(Z_2Z_4^2 + Z_1Z_5^2 - 2Z_3Z_4Z_5\right)}{Z_1Z_2 - Z_3^2}\right\} \tag{34}$$

The equation (34) provides the minimum value of the MSE of the proposed family of estimator $D_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}$.

6.1. Particular case in two-phase sampling

For $\alpha_1 + \alpha_2 = 1$ the suggested family reduces to the following family of estimators

$$D_{\alpha,1-\alpha,\beta}^{\delta,\gamma} = \bar{y}\left[\alpha\left(\frac{(1-\beta)\bar{x} + \beta\bar{X})}{\beta\bar{x} + (1-\beta)\bar{X}}\right)^{\delta} + (1-\alpha)\left(\frac{\beta\bar{x} + (1-\beta)\bar{X}}{(1-\beta)\bar{x} + \beta\bar{X}}\right)^{\gamma}\right] \tag{35}$$

The bias and MSE of the estimator to the first degree of approximation are derived as

$$B\left(D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}\right)=\frac{\bar{Y}}{2}\left[\omega+\alpha\left(\mu-\omega\right)\right] \tag{36}$$

$$MSE\left(D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right)=\bar{Y}^2\left[\left(1+Z_2-2Z_5\right)+\alpha^2\left(Z_1+Z_2-2Z_3\right)-2\alpha\left(Z_2-Z_3+Z_4-Z_5\right)\right] \tag{37}$$

For minima, we took gradient $\bar{V}=\left(\frac{\partial}{\partial\alpha}\right)$ of $MSE\left(D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right)$ and equating it to zero, we get $\beta=\left(1/2\right)$ and $C=\left(1-2\beta\right)\left\{\gamma-\alpha\left(\gamma+\delta\right)\right\}$.

When $\beta=\left(1/2\right)$ is used in $D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$, we get the usual unbiased estimator of $\bar{y}$ and the MSE expression becomes

$$MSE\left(D^{\delta,\gamma}_{\frac{\gamma}{\gamma+\delta},\,1-\frac{\gamma}{\gamma+\delta},\,\frac{1}{2}}\right)=\eta S_y^2 \tag{38}$$

and when $C=\left(1-2\beta\right)\left\{\gamma-\alpha\left(\gamma+\delta\right)\right\}$ is used in $D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$, we get the asymptotically optimum estimator (AOE) $D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$ and the MSE expression transforms into

$$MSE\left(D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right)=\eta^* S_y^2\left(1-\rho^2\right)+\eta' S_y^2 \tag{39}$$

Also, $MSE\left(D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right)$ is minimum when $\alpha=\frac{\left(Z_2-Z_3+Z_4-Z_5\right)}{\left(Z_1+Z_2-2Z_3\right)}=\alpha_{opt}$ *i.e.*

$$\begin{aligned}MSE_{\min}\left(D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right)&=\bar{Y}^2\left[1+Z_2-2Z_5-\frac{2\left(Z_2-Z_3+Z_4-Z_5\right)^2}{Z_1+Z_2-2Z_3}\right]\\&=\eta^* S_y^2\left(1-\rho^2\right)+\eta' S_y^2=MSE\left(\bar{y}_{lrd}\right)\end{aligned} \tag{40}$$

where $\eta'=\left(N-n'\right)/\left(Nn'\right)^{-1}$ and $\eta^*=\eta-\eta'=\left(n'-n\right)\left(nn'\right)^{-1}$.

Here, $MSE\left(\bar{y}_{lrd}\right)$ is the MSE of the double sampling version of linear regression estimator. Therefore, the estimator $D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$ is equally efficient to double sampling version of linear regression estimator.

7. Efficiency comparison

The suggested class of estimators is compared with $\bar{y}_d$, $\bar{y}_{Rd}$ and $\bar{y}_{Pd}$ in terms of MSE's.

$$MSE(\bar{y}_d)=\eta' S_y^2 \tag{41}$$

$$MSE(\bar{y}_{Rd})=\eta\bar{Y}^2\left[\eta C_y^2+\eta^* C_x^2(1-2C)\right] \tag{42}$$

$$MSE(\bar{y}_{Pd}) = \eta \bar{Y}^2 \left[\eta C_y^2 + \eta^* C_x^2 (1+2C) \right] \tag{43}$$

From equations (34), (40), (41), (42) and (43) we get

$$MSE\left(\bar{y}_d\right) - MSE_{\min}\left(D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right) = \eta^* \bar{Y}^2 C_y^2 \rho^2 \tag{44}$$

$$MSE\left(\bar{y}_{Rd}\right) - MSE_{\min}\left(D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right) = \eta^* \bar{Y}^2 C_x^2 \left(1-C\right)^2 \tag{45}$$

$$MSE\left(\bar{y}_{Rd}\right) - MSE_{\min}\left(D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right) = \eta^* \bar{Y}^2 C_x^2 \left(1-C\right)^2 \tag{46}$$

$$MSE_{\min}\left(D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}\right) - MSE_{\min}\left(D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}\right) = \bar{Y}^2 \frac{\left[Z_1\left(Z_2-Z_5\right)+Z_4\left(Z_3-Z_2\right)+Z_3\left(Z_5-Z_3\right)\right]^2}{\left(Z_1+Z_2-2Z_3\right)\left(Z_1 Z_2 - Z_3^2\right)} > 0 \tag{47}$$

The minimum MSE of $D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$ subfamily of estimators will always be larger than the proposed family of estimators $D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ as the coefficients of ratio and product type part of family of estimators in subfamily are interdependent in $D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$ while they are independent in $D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$.

From equations (44) to (47), it is clear that the proposed family of estimators $D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ is more efficient than its subfamily of estimators $D^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$, double sampling version of sample mean estimator $\bar{y}_d$, ratio estimator $\bar{y}_{Rd}$, and product estimator $\bar{y}_{Pd}$.

8. Empirical study

To illustrate the relative performance of the members of the proposed family of estimators with other estimators considered in this article, the two natural populations from literature are considered whose descriptions are given below:

Population I [Source: Koyunchu and Kadilar (2009)]

y: Number of teachers teaching in both primary and secondary schools.
x: Number of students studying in both primary and secondary schools.

$N = 923,\ n' = 270,\ n = 180,\ \bar{Y} = 436.43,\ \bar{X} = 11440.50,\ C_x = 1.86,\ C_y = 1.72,\ C = 0.88,$
$\rho = 0.95.$

Population II [Source: Cochran (1977, p.172)]

y: Production of peaches (I bushels).
x: Peach trees in an orchard.

$N = 256,\ n' = 150,\ n = 100,\ \bar{Y} = 56.47,\ \bar{X} = 44.45,\ C_x = 1.40,\ C_y = 1.42,\ C = 0.90,$
$\rho = 0.89.$

The percent relative efficiencies (*PREs)* of the suggested family of estimators $D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ are compared with the double sampling version linear regression estimator $\bar{y}_{lrd}$ by using the following formulas:

$$PRE\left(D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta},\bar{y}_{lrd}\right)=\frac{\left(\eta^* S_y^2\left(1-\rho^2\right)+\eta' S_y^2\right)}{\bar{Y}^2\left[1-\frac{\left(Z_2Z_4^2+Z_1Z_5^2-2Z_3Z_4Z_5\right)}{Z_1Z_2-Z_3^2}\right]}*100 \tag{48}$$

It is observed from the Table 7 (see appendix) that all the members of proposed family of estimators are performing better in comparison to double sampling linear regression estimator, therefore, the members are also efficient to double sampling version ratio and product estimator.

9. Conclusion

We have dealt with the problem of estimating the population mean $\bar{Y}$ of study variable using the auxiliary information in the form of different parameters of the variable x .The proposed family of estimators are very wide and many new estimators can be derived from the suggested class of estimators. It includes all the estimators recently proposed by Singh and Yadav (2017) along with the two parameters ratio-product-ratio estimator proposed by Chami *et al.* (2012).

To judge the performance of the proposed family of estimators with other estimators, an empirical study has been carried out. From the Table 4 (see appendix), it is observed that the suggested family of estimators is efficient to sample mean estimator $\bar{y}$, linear regression estimator $\bar{y}_{lr}$ and Singh and Yadav (2017) estimators. At same value of β, proposed family of estimators should be preferred over Singh and Yadav (2017) estimators (Table 5 (see appendix)) . For identifying the most efficient estimator, the proposed family of estimators should be preferred over Singh and Yadav (2017) family of estimators (Table 6 (see appendix)). All the members of double sampling version of the proposed family of estimators are efficient to double sampling version of ratio, product, and linear regression estimator. Thus, we recommend the use of proposed family of estimators in practice.

References

Cochran, W. G. (1977). *Sampling Techniques*. New-York: John Wiley and Sons.

Chami, S. P., Singh, B. and Thomas, D. (2012). A two-parameter ratio-product-ratio estimator using auxiliary information. *ISRN Probability and Statistics*, 1-15.

Cochran W. G. (1940). The estimation of the yields of the cereal experiments by sampling for the ratio of grain to total produce. *The Journal of Agricultural Science*, **59**, 1225-1226.

Gupta, P. C. and Kothwala, N. H. (1990). A study of second order approximation for some product type estimators. *Journal of the Indian Society of Agricultural Statistics*, **42**, 171-185.

Kadilar, C. and Cingi, H. (2006). An improvement in estimating the population mean by using the correlation coefficient. *Hacettepe Journal of Mathematics and Statistics,* **35**, 103-109.

Murthy, M. N. (1964). Product method of estimation. *Sankhya, The Indian Journal of Statistics -Series A,* **26**, 69-74.

Murthy, M. N. (1967). *Sampling Theory and Methods*. Calcutta: Statistical Publishing Society.

Pandey, G. S. (1980). Product-cum-power estimators. *Calcutta Statistical Association Bulletin,* **29**, 103-108.

Searls, D. T. (1964). The utilization of known coefficient of variation in the estimation procedure. *Journal of the American Statistical Association,* **59**, 1125-1126.

Singh, H. P. and Ruiz-Espejo, M. (2003). On linear regression and ratio-product estimation of a finite population mean. *Statistics,* **52**, 59-67.

Singh, H. P. and Solanki, R. S. (2012). A new procedure for variance estimation in SRS using auxiliary information. Statistical Papers, **54**, 479–497. DOI 10.1007/s00362-012-0445-2.

Singh, H. P. and Yadav, A. (2017). An improved family of estimators of finite population mean using information on an auxiliary variable in sample surveys. *Journal of Applied Mathematics, Statistics and Informatics,* **13**, 77-107.

Singh, R. and Kumar, M. (2011): A note on transformations on auxiliary variable in survey sampling. *Model Assisted Statistics and Applications*, **6**, 17-19.

Srivastava, S. K. (1967). An estimator using auxiliary information in sample surveys. *Calcutta Statistical Association Bulletin,* **16**, 121–132.

Steel, R. G. D. and Torrie, J. H. (1960). *Principles and Procedures of Statistics*. McGraw Hill Book Co.

Sukhatme, P. V. and Sukhatme, B. V. (1970). *Sampling Theory of Surveys with Applications.* Ames, IO: Iowa State University Press.

Swain, A. K. P. C. (2014). On an improved ratio type estimator of finite population mean in sample surveys. *Revista de Investigacion Operacional,* **35**, 49-57.

Upadhyaya, L. N., Singh, H. P., and Vos, J. W. E. (1985). On the estimation of population means and ratios using supplementary information. *Statistica Neerlandica,* **39**, 309-318.

Upadhyaya, L. N. and Singh, H. P. (1999). Use of transformed auxiliary variable in estimating the finite population means. *Biometrical Journal*, **41**, 627-636.

Upadhyaya, L. N., Singh, H. P., Chatterjee, S., and Yadav, R. (2011). A generalized family of transformed ratio-product estimators in sample surveys. *Model Assisted Statistics and Applications,* **6**, 137-150.

Yadav, R., Upadhyaya, L. N., Singh, H. P., and Chatterjee, S. (2012). Almost unbiased ratio and product type exponential estimators. *Statistics in Transition,* **13**, 537-550.

Yadav, R., Upadhyaya, L. N., Singh, H. P. and Chatterjee, S. (2013). A generalized family of transformed ratio-product estimators for variance in sample surveys. *Communications in Statistics - Theory and Methods,* **42**, 1839-1850.

Appendix

Table 1: Some known members of the proposed family of estimators $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$

S.No.	Value of constants $(\alpha_1, \alpha_2, \beta, \delta, \gamma)$	Estimator	S.No.	Value of constants $(\alpha_1, \alpha_2, \beta, \delta, \gamma)$	Estimator
1	$(\alpha_1, \alpha_2, 1, 0, 1)$ Upadhyaya *et al.*(1985) estimator	$T^{0,1}_{\alpha_1,\alpha_2,1}$	2.	$(\alpha_1, \alpha_2, 1, 1, 0)$ Upadhyaya *et al.* (1985) estimator	$T^{1,0}_{\alpha_1,\alpha_2,1}$

Table 2: Some known members of the $T^{\delta,\gamma}_{\alpha,1-\alpha,\beta}$ sub-family of estimators

S.No.	Value of constants $(\alpha_1, \alpha_2, \beta, \delta, \gamma)$	Estimator	S.No.	Value of constants $(\alpha_1, \alpha_2, \beta, \delta, \gamma)$	Estimator
1.	$(\alpha, 1\text{-}\alpha, 1, 1, 1)$ Singh & Ruiz Espejo (2003)	$T^{1,1}_{\alpha,1-\alpha,1}$	4.	$(\alpha, 1\text{-}\alpha, 1, \delta, 1)$ Pandey (1980)	$T^{\delta,1}_{\alpha,1-\alpha,1}$
2.	$(\alpha, 1\text{-}\alpha, \beta, 1, 1)$ Chami *et al*. (2012)	$T^{1,1}_{\alpha,1-\alpha,\beta}$	5.	$(1, 0, 1, 1/2, *)$ Swain (2014)	$T^{1/2,*}_{1,0,1}$
3.	$(1, 0, 1, 2, *)$ Kadilar and Cingi (2006)	$T^{2,*}_{1,0.1}$	6.	$(1, 0, 1, \delta, *)$ Srivastava (1967)	$T^{\delta,*}_{1,0,1}$

Table 3: Some members of the proposed family of estimators along with their corresponding members of Singh and Yadav (2017) estimator

Sl. No.	Members of estimator $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$		Corresponding members of Singh and Yadav (2017) Estimator $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ at $\gamma=1$	
	Value of constants $(\alpha_1,\alpha_2,\beta,\delta,\gamma)$	Estimator	Value of constants $(\alpha_1,\alpha_2,\beta,\delta,\gamma)$	Estimator
1.	$(\alpha_1,\ \alpha_2,\ 1,\ 0,\ \gamma)$	$T^{0,\gamma}_{\alpha_1,\alpha_2,1}$	$(\alpha_1,\ \alpha_2,\ 1,\ 0,\ 1)$	$T^{0,1}_{\alpha_1,\alpha_2,1}$
2.	$(\alpha_1,\ \alpha_2,\ 0,\ 0,\ \gamma)$	$T^{0,\gamma}_{\alpha_1,\alpha_2,0}$	$(\alpha_1,\ \alpha_2,\ 0,\ 0,\ 1)$	$T^{0,1}_{\alpha_1,\alpha_2,0}$
3.	$(\alpha_1,\ \alpha_2,\ 1,\ 1/2,\ \gamma)$	$T^{1/2,\gamma}_{\alpha_1,\alpha_2,1}$	$(\alpha_1,\ \alpha_2,\ 1,\ 1/2,\ 1)$	$T^{1/2,1}_{\alpha_1,\alpha_2,1}$
4.	$(\alpha_1,\ \alpha_2,\ 1,\ 2,\ \gamma)$	$T^{2,\gamma}_{\alpha_1,\alpha_2,1}$	$(\alpha_1,\ \alpha_2,\ 1,\ 2,\ 1)$	$T^{2,1}_{\alpha_1,\alpha_2,1}$
5.	$(\alpha_1,\ \alpha_2,\ C_x,\ 1/2,\ \gamma)$	$T^{1/2,\gamma}_{\alpha_1,\alpha_2,C_x}$	$(\alpha_1,\ \alpha_2,\ C_x,\ 1/2,\ 1)$	$T^{1/2,1}_{\alpha_1,\alpha_2,C_x}$
6.	$(\alpha_1,\ \alpha_2,\ C_x,\ 1,\ \gamma)$	$T^{1,\gamma}_{\alpha_1,\alpha_2,C_x}$	$(\alpha_1,\ \alpha_2,\ C_x,\ 1,\ 1)$	$T^{1,1}_{\alpha_1,\alpha_2,C_x}$
7.	$(\alpha_1,\ \alpha_2,\ C_x,\ 2,\ \gamma)$	$T^{2,\gamma}_{\alpha_1,\alpha_2,C_x}$	$(\alpha_1,\ \alpha_2,\ C_x,\ 2,\ 1)$	$T^{2,1}_{\alpha_1,\alpha_2,C_x}$
8.	$(\alpha_1,\ \alpha_2,\ C_x,\ 0,\ \gamma)$	$T^{0,\gamma}_{\alpha_1,\alpha_2,C_x}$	$(\alpha_1,\ \alpha_2,\ C_x,\ 0,\ 1)$	$T^{0,1}_{\alpha_1,\alpha_2,C_x}$
9.	$(\alpha_1,\ \alpha_2,\ \rho,\ 1,\ \gamma)$	$T^{1,\gamma}_{\alpha_1,\alpha_2,\rho}$	$(\alpha_1,\ \alpha_2,\ \rho,\ 1,\ 1)$	$T^{1,1}_{\alpha_1,\alpha_2,\rho}$
10.	$(\alpha_1,\ \alpha_2,\ \rho,\ 2,\ \gamma)$	$T^{2,\gamma}_{\alpha_1,\alpha_2,\rho}$	$(\alpha_1,\ \alpha_2,\ \rho,\ 2,\ 1)$	$T^{2,1}_{\alpha_1,\alpha_2,\rho}$
11.	$(\alpha_1,\ \alpha_2,\ C_y,\ 1,\ \gamma)$	$T^{1,\gamma}_{\alpha_1,\alpha_2,C_y}$	$(\alpha_1,\ \alpha_2,\ C_y,\ 1,\ 1)$	$T^{1,1}_{\alpha_1,\alpha_2,C_y}$
12.	$(\alpha_1,\ \alpha_2,\ C_y,\ 2,\ \gamma)$	$T^{2,\gamma}_{\alpha_1,\alpha_2,C_y}$	$(\alpha_1,\ \alpha_2,\ C_y,\ 2,\ 1)$	$T^{2,1}_{\alpha_1,\alpha_2,C_y}$
13.	$(0.5,\ 0.5,\ 1,\ 2,\ \gamma)$	$T^{2,\gamma}_{0.5,0.5,1}$	$(0.5,\ 0.5,\ 1,\ 2,\ 1)$	$T^{2,1}_{0.5,0.5,1}$
14.	$(0.5,\ 0.5,\ C_x,\ 2,\ \gamma)$	$T^{2,\gamma}_{0.5,0.5,C_x}$	$(0.5,\ 0.5,\ C_x,\ 2,\ 1)$	$T^{2,1}_{0.5,0.5,C_x}$
15.	$(0.5,\ 0.5,\ \rho,\ 1,\ \gamma)$	$T^{2,\gamma}_{0.5,0.5,\rho}$	$(0.5,\ 0.5,\ \rho,\ 1,\ 1)$	$T^{2,1}_{0.5,0.5,\rho}$
16.	$(0.5,\ 0.5,\ \rho,\ 2,\ \gamma)$	$T^{2,\gamma}_{0.5,0.5,C_y}$	$(0.5,\ 0.5,\ \rho,\ 2,\ 1)$	$T^{2,1}_{0.5,0.5,C_y}$
17.	$(0.5,\ 0.5,\ C_y,\ 2,\ \gamma)$	$T^{2,\gamma}_{0.5,0.5,C_y}$	$(0.5,\ 0.5,\ C_y,\ 2,\ 1)$	$T^{2,1}_{0.5,0.5,C_y}$

Note: *Estimators from S. No. 1 to 12 corresponds to* $\alpha_1=\alpha_1,\ \alpha_2=\alpha_2$*and estimators from S. No. 13 to 17 corresponds for* $\alpha_1=\alpha,\ \alpha_2=1-\alpha$

Table 4: PREs of several members of proposed family of estimators and Singh and Yadav (2017) estimator due to $\gamma = 1$

Estimator	Real values		$T_{\alpha_1,\alpha_2,\beta}^{\delta,1}$	$T_{\alpha_1,\alpha_2,\beta}^{\delta,\gamma}$
	β	δ	PRE (A)	PRE (B)
Population I				
1	1	0	100.88 $(\gamma = 1)$	470.61 $(\gamma = 8)$
2	0	0	100.98 $(\gamma = 1)$	1545.40 $(\gamma = 9)$
3	1	2	122.64 $(\gamma = 1)$	435.12 $(\gamma = 3)$
4	C_x	0.5	100.50 $(\gamma = 1)$	138.31 $(\gamma = 16)$
5	C_x	1	108.47 $(\gamma = 1)$	466.59 $(\gamma = 5)$
6	C_x	2	187.59 $(\gamma = 1)$	478.72 $(\gamma = 1.5)$
7	C_x	0	101.60 $(\gamma = 1)$	388.53 $(\gamma = 6)$
8	ρ	1	101.60 $(\gamma = 1)$	388.53 $(\gamma = 20)$
9	ρ	2	101.24 $(\gamma = 1)$	869.98 $(\gamma = 5)$
10	C_y	2	102.25 $(\gamma = 1)$	1254.19 $(\gamma = 19)$
Population II				
1	1	0	102.23 $(\gamma = 1)$	207.24 $(\gamma = -5)$
2	0	0	109.61 $(\gamma = 1)$	207.24 $(\gamma = 5)$
3	1	2	151.75 $(\gamma = 1)$	1461.06 $(\gamma = 3)$
4	C_x	0.5	105.20 $(\gamma = 1)$	113.62 $(\gamma = -5)$
5	C_x	1	107.24 $(\gamma = 1)$	108.42 $(\gamma = 5)$
6	C_x	2	112.53 $(\gamma = 1)$	132.25 $(\gamma = 5)$
7	C_x	0	103.52 $(\gamma = 1)$	121.42 $(\gamma = -5)$
8	ρ	1	103.52 $(\gamma = 1)$	121.42 $(\gamma = 5)$
9	ρ	2	105.14 $(\gamma = 1)$	105.15 $(\gamma = 5)$
10	C_y	2	109.88 $(\gamma = 1)$	117.19 $(\gamma = 5)$
Population III				
1	1	0	3725.78 $(\gamma = 1)$	3725.78 $(\gamma = 1)$
2	0	0	100.02 $(\gamma = 1)$	902.26 $(\gamma = 3)$

3	1	2	560.56 $(\gamma=1)$	560.56 $(\gamma=1)$
4	C_x	0.5	139.24 $(\gamma=1)$	496.05 $(\gamma=3)$
5	C_x	1	116.27 $(\gamma=1)$	567.12 $(\gamma=5)$
6	C_x	2	100.47 $(\gamma=1)$	116.15 $(\gamma=10)$
7	C_x	0	191.26 $(\gamma=1)$	672.73 $(\gamma=2)$
8	ρ	1	191.26 $(\gamma=1)$	672.73 $(\gamma=10)$
9	ρ	2	100.09 $(\gamma=1)$	353.49 $(\gamma=1)$
10	C_y	2	100.03 $(\gamma=1)$	100.03 $(\gamma=1)$

Note: (1) "A" indicates Singh and Yadav (2017) estimator; (2) "B" indicates proposed estimator.

Table 5: Top performing estimators of $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ and $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ at constant β when $\alpha_1=\alpha_1'$ and $\alpha_2=\alpha_2'$ for Population-I

S.no.	β	$T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$		$T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$			% gain in PRE
		δ	PRE (A)	δ	γ	PRE (B)	
1.	1	2.5	145.85	2.5	2.0	390.98	**168.07**
2.	1.1504	2.5	455.30	1.5	2.5	547.56	20.26
3.	0.9125	2.5	119.60	2.5	2.5	193.41	61.72
4.	0.7681	2.5	104.15	2.5	2.5	109.65	5.28
5.	0.6092	2.5	100.21	2.5	1	100.21	0.00

Table 6: Top performing estimators of $T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ and $T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$ when $\alpha_1=\alpha_1'$ and $\alpha_2=\alpha_2'$ for three populations

Pop.	$T^{\delta,1}_{\alpha_1,\alpha_2,\beta}$			$T^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$				% gain in PRE
	β	δ	PRE (A)	β	δ	γ	PRE (B)	
i.	C_x	2.5	455.30	C_x	1.5	2.5	547.56	20.26
ii.	1	2.5	189.85	1	2.5	2	710.58	274.29
iii.	1	1	560.57	1	0	1	3725.78	**564.64**

Table 7: PREs of $D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ family of estimators when $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$

Estimator	Real values	Population I		Population II	
	β	(δ, γ)	PRE of $D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ w.r.t. $MSE(\bar{y}_{lrd})$	(δ, γ)	PRE of $D^{\delta,\gamma}_{\alpha_1,\alpha_2,\beta}$ w.r.t. $MSE(\bar{y}_{lrd})$
1	1	$(\delta=1, \gamma=8)$	**141465.86**	$(\delta=5, \gamma=5)$	1212.20
2	0	$(\delta=2, \gamma=10)$	1977.39	$(\delta=4, \gamma=6)$	3448.88
3	ρ	$(\delta=1, \gamma=10)$	956.79	$(\delta=2, \gamma=12)$	6151.28
4	C_x	$(\delta=0.5, \gamma=1)$	596.20	$(\delta=0.5, \gamma=6)$	1317.25
5	C_y	$(\delta=0.5, \gamma=1)$	272.94	$(\delta=0.5, \gamma=6)$	**15756.31**
6	C	$(\delta=1, \gamma=20)$	16963.29	$(\delta=2, \gamma=11)$	12634.51
7	f	$(\delta=12, \gamma=6)$	5528.56	$(\delta=0.5, \gamma=2.5)$	445.12
8	$1-f$	$(\delta=2, \gamma=10)$	382.42	$(\delta=4, \gamma=0.5)$	547.72
9	$1/1+f$	$(\delta=2, \gamma=10)$	1360.60	$(\delta=10, \gamma=12)$	1943.52
10	$2f/1+f$	$(\delta=0.5, \gamma=1)$	302.70	$(\delta=5, \gamma=1.5)$	276.38
11	$f/1-f$	$(\delta=1, \gamma=2)$	255.01	$(\delta=1, \gamma=1.5)$	268.89
12	$1-f/1+f$	$(\delta=10, \gamma=15)$	813.12	$(\delta=1, \gamma=5)$	305.58