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On the Status of Variance Balanced Block Designs in the Presence of Both-sided Neighbour Effects: Two Examples

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Abstract

The research work presented in this paper is geared towards analysis of variance balanced [VB] block designs in the presence of both-sided neighbour effects. There is a vast literature on VB designs of which the BIBDs are simplest examples. We shall take up two such designs and examine their behaviours in respect of (i) estimates of treatment contrasts, (ii) estimates of block contrasts, and (iii) linear error functions - in the presence of both-sided neighbour-effects. We shall assume a circular model. Estimability issues regarding treatment effects contrasts and block effects contrasts point towards discouraging notes.

Key words: Block designs; Variance balance; Neighbour-effects; Estimability issues; Linear model; ANOVA.

1. Introduction

Variance balance and efficiency balance are two choice-based criteria for selection of designs in many contexts. Block designs, row-column designs and higher dimensional designs have been extensively studied with respect to these two criteria. Combinatorial designs have been characterized utilizing these requirements and this, undoubtedly, forms a fascinating area of research. Some of the works are Hedayat and Stufken (1989), Mishra (2016), Morgan and Uddin (1995), Khatri(1982), Raghavarao (1971), Sinha, Jones and Kageyama (1997).

On the other hand, neighbour-designs, incorporating neighbour-effects, have been studied at length and the concept of balancing has also been introduced. However, though combinatorial balance has been introduced and studied, it seems that there is a gap in this kind of study. From data analysis point of view, no serious attention seems to have been paid for understanding the nature of (i) error functions, (ii) estimable treatment- and block-contrasts, in the presence of NEffects [both Left-sided and Right-sided]. We attempt to fill up this gap. We shall take up two variance-balanced block designs and carefully examine their status with respect to the above - mentioned features.

2. BIBD(7, 7, 3, 3, 1) and Neighbour-Effects

The blocks of the design are obtained by starting with the initial block (1, 2,4) and expanding it, modulo (7). The blocks are (1,2,4); (2,3,5); (3,4,6); (4,5,7); (5,6,1); (6,7,2);

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(7,1,3). Equivalently, the blocks can be represented in the form of a 7 × 3 matrix as shown in Table 1 below:

Table 1

Block	Col1	Col2	Col3
1	1	2	4
2	2	3	5
3	3	4	6
4	4	5	7
5	5	6	1
6	6	7	2
7	7	1	3

In the absence of any Neighbour Effects [NEs], a complete set of all the eight linearly independent error functions is easily identifiable and each one is shown as difference of two Terms in Table 2 below:

Table 2: Error functions : Term 1 and Term 2

Error Function 1: Term 1(=Column Sum 1) – Term 2(=Column Sum 2)
Error Function 2: Term 1(=Column Sum 1) – Term 2 (=Column Sum 3)
Error Function 3: Term 1 = y(1,1) – y(1,2); Term 2 = [y(7,2) – y(7,3)] + [y(2,2) – y(2,1)]
Error Function 4: Term 1 = y(7,2) – y(7,3); Term 2 = [y(1,1) – y(1,3)] + [y(3,2) – y(3,1)]
Error Function 5: Term 1 = y(1,1) – y(1,3); Term 2 = [y(5,3) – y(5,1)] + [y(4,2) – y(4,1)]
Error Function 6: Term 1 = y(5,3) – y(5,1); Term 2 = [y(7,2) – y(7,3)] + [y(2,2) – y(2,3)]
Error Function7: Term 1 = y(5,3) – y(5,2);Term 2 = [y(7,2) – y(7,3)] + [y(3,1) – y(3,3)]
Error Function 8: Term 1 = y(7,2) – y(7,1); Term 2 = [y(5,3) – y(5,1)] + [y(4,2) – y(4,3)]

It is further verified that the 8 × 21 matrix of the coefficients in these error functions, which is shown below in Table 3, has rank 8.

Remark 1: Thus far we have found out a set of 8 linearly independent error functions which correspond to the error df in the model. This holds under the assumption that there are no neighbour-effects of the plots in the blocks. Below we embark on the problem of examining the status of these error functions in the presence of both the left-and right-sided neighbour effects [LNEs and RNEs].

Table 3: Matrix L of coefficients in linear error functions

EF1	1	−1	0	1	−1	0	1	−1	0	1	−1	0	1	−1	0	1	−1	0	1	−1	0
EF 2	1	0	−1	1	0	−1	1	0	−1	1	0	−1	1	0	−1	1	0	−1	1	0	−1
EF 3	1	−1	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	1
EF 4	−1	0	1	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	−1
EF 5	1	0	−1	0	0	0	0	0	0	1	−1	0	1	0	−1	0	0	0	0	0	0
EF 6	0	0	0	0	−1	1	0	0	0	0	0	0	−1	0	1	0	0	0	0	−1	1
EF 7	0	0	0	0	0	0	−1	0	1	0	0	0	0	−1	1	0	0	0	0	−1	1
EF 8	0	0	0	0	0	0	0	0	0	0	−1	1	1	0	−1	0	0	0	−1	1	0

Note: EF – Error Function

2.1. Nature of error functions in the presence of LN Effects and RN Effects

Towards understanding the status of error functions in the presence of LN- and RN-Effects, it is almost immediate to realize that error functions 1 and 2 are free from these effects. For the rest, we need to carry out the exercises. These are shown below in terms of their expectations under the model with NEs :

Error 3= Term1 – Term 2

Term 1 = y(1,1) – y(1,2)
$=(\tau_1 + \text{LN4} + \text{RN2}) - (\tau_2 + \text{LN1} + \text{RN4}) = \tau_1 - \tau_2 + \text{LN4} - \text{LN1} + \text{RN2} - \text{RN4}$

Term 2 = [y(7,2) – y(7,3)] + [y(2,2) – y(2,1)]
$=(\tau_1 + \text{LN7} + \text{RN3}) - (\tau_3 + \text{LN1} + \text{RN7}) + (\tau_3 + \text{LN2} + \text{RN5}) - (\tau_2 + \text{LN5} + \text{RN3})$
$= \tau_1 - \tau_2 + (\text{LN7} + \text{LN2} - \text{LN1} - \text{LN5}) + (\text{RN3} + \text{RN5} - \text{RN7} - \text{RN3})$

Error 3 = Term1 –Term 2 = (LN4 + LN5 – LN7 – LN2) + (RN2 + RN7 – RN4 – RN5)

Error 4 = Term 1 – Term2

Term 1 = y(7,2) – y(7,3)
$= (\tau_1 + \text{LN7} + \text{RN3}) - (\tau_3 + \text{LN1} + \text{RN7})$
$= \tau_1 - \tau_3 + \text{LN7} - \text{LN1} + \text{RN3} - \text{RN7}$

Term 2 = [y(1,1) – y(1,3)] + [y(3,2) – y(3,1)]
$= (\tau_1 + \text{LN4} + \text{RN2}) - (\tau_4 + \text{LN2} + \text{RN1}) + (\tau_4 + \text{LN3} + \text{RN6}) - (\tau_3 + \text{LN6} + \text{RN4})$
$= \tau_1 - \tau_3 + \text{LN4} - \text{LN2} + \text{LN3} - \text{LN6} + \text{RN2} - \text{RN1} + \text{RN6} - \text{RN4}$

Error 4 = Term 1 – Term2

= (LN2 + LN6 + LN7 – LN1 – LN3 – LN4) + (RN1 + RN3 + RN4 – RN2 – RN6 – RN7)

Error 5 = Term1 –Term 2

Term 1 = y(1,1) – y(1,3)
$= (\tau_1 + \text{LN4} + \text{RN2}) - (\tau_4 + \text{LN2} + \text{RN1})$
$= \tau_1 - \tau_4 + \text{LN4} - \text{LN2} + \text{RN2} - \text{RN1}$

Term 2 = [y(5,3) – y(5,1)] + [y(4,2) – y(4,1)]
$= (\tau_1 + \text{LN6} + \text{RN5}) - (\tau_5 + \text{LN1} + \text{RN6}) + (\tau_5 + \text{LN4} + \text{RN7}) - (\tau_4 + \text{LN7} + \text{RN5})$
$= \tau_1 - \tau_4 + (\text{LN6} + \text{LN4}) - (\text{LN1} + \text{LN7}) + (\text{RN7} - \text{RN6})$

Error 5 = Term1 – Term 2 = (LN1 + LN7 – LN2 – LN6) + (RN2 + RN6 – RN1 – RN7)

Error 6 = Term1 – Term 2

Term 1 = y(5,3) – y(5,1)

$= (\tau_1 + \text{LN6} + \text{RN5}) - (\tau_5 + \text{LN1} + \text{RN6}) = \tau_1 - \tau_5 + \text{LN6} - \text{LN1} + \text{RN5} - \text{RN6}$

Term 2 = [y(7,2) – y(7,3)] + [y(2,2) – y(2,3)]
$= (\tau_1 + \text{LN7} + \text{RN3}) - (\tau_3 + \text{LN1} + \text{RN7}) + (\tau_3 + \text{LN2} + \text{RN5}) - (\tau_5 + \text{LN3} + \text{RN2})$
$= \tau_1 - \tau_5 + \text{LN7} + \text{LN2} - \text{LN1} - \text{LN3} + \text{RN3} + \text{RN5} - \text{RN2} - \text{RN7}$

Error 6 = Term1 – Term 2 = (LN3 + LN6 – LN2 – LN7) + (RN2 + RN7 – RN3 – RN6)

Error 7 = Term1 – Term 2

Term 1 = y(5,3) – y(5,2)
$= (\tau_1 + \text{LN6} + \text{RN5}) - (\tau_6 + \text{LN5} + \text{RN1})$
$= \tau_1 - \tau_6 + \text{LN6} - \text{LN5} + \text{RN5} - \text{RN1}$
Term 2 = [y(7,2) – y(7,3)] + [y(3,1) – y(3,3)]

$= (\tau_1 + \text{LN7} + \text{RN3}) - (\tau_3 + \text{LN1} + \text{RN7}) + (\tau_3 + \text{LN6} + \text{RN4}) - (\tau_6 + \text{LN4} + \text{RN3})$

$= \tau_1 - \tau_6 + \text{LN7} + \text{LN6} - \text{LN1} - \text{LN4} + \text{RN3} + \text{RN4} - \text{RN7} - \text{RN3}$

Error 7 = Term1 – Term 2 = (LN1 + LN4 – LN5 – LN7) + (RN5 + RN7 – RN1 – RN4)

Error 8 = Term1 – Term 2

Term 1 = y(7,2) – y(7,1)

$= (\tau_1 + \text{LN7} + \text{RN3}) - (\tau_7 + \text{LN3} + \text{RN1}) = \tau_1 - \tau_7 + \text{LN7} - \text{LN3} + \text{RN3} - \text{RN1}$

Term 2 = [y(5,3) – y(5,1)] + [y(4,2) – y(4,3)]

$= (\tau_1 + \text{LN6} + \text{RN5}) - (\tau_5 + \text{LN1} + \text{RN6}) + (\tau_5 + \text{LN4} + \text{RN7}) - (\tau_7 + \text{LN5} + \text{RN4})$

$= \tau_1 - \tau_7 + \text{LN6} - \text{LN1} + \text{LN4} - \text{LN5} + \text{RN5} - \text{RN6} + \text{RN7} - \text{RN4})$

Error 8 = Term1 – Term 2

= (LN1 + LN5 + LN7 – LN3 – LN4 – LN6) + (RN3 + RN4 + RN6 – RN1 – RN5 – RN7)

Now we consider the LN effects only and develop all these 6 equations, that is, from Error 3 to Error 8 into a matrix. Further, we append the row vector $(1,1,...,1)'$ to make it a square matrix of order 7. This is shown in Table 4 below:

Table 4: Coefficients of LN Effects in expectations of observational contrasts for errors

Error Sl. No.	Co-efficient of LNE						
Special row	1	1	1	1	1	1	1
3	0	–1	0	1	1	0	–1
4	–1	1	–1	–1	0	1	1
5	1	–1	0	0	0	–1	1
6	0	–1	1	0	0	1	–1
7	1	0	0	1	–1	0	–1
8	1	0	–1	–1	1	–1	1

It follows that this matrix is of full rank. Moreover, the matrix underlying R-sided NEs is obtainable from the above by simply changing the signs of elements in each row. Consequently, error df will remain intact at 8 df if and only if all the Left-sided NEs are equal and at the same time all the Right-sided NEs are also equal. Unless this is satisfied, we cannot go for the usual ANOVA Table-based data analysis. Once this is satisfied, we see no effect whatsoever of these LN and RN effects on the analysis of data. Further to this, we also find that there are 2 df for error - no matter what happens to the NEs. Thus ANOVA F-tests can be carried out for estimable treatment contrasts and estimable block contrasts - even in the presence of NEs - provided such estimable treatment/block contrasts are available.

2.2. Estimable treatment contrasts in the BIBD with NEffects

We list below in Table 5 simple-minded unbiased estimators of a set of elementary treatment contrasts based on the observations underlying the BIBD - in the absence of LNEs and RNEs. For later use, we have also indicated model expectations of these observational contrasts in the table - assuming the presence of NEs - both Left-sided and Right-sided. Suppose now that Error 3 is a valid error *i.e.*, E[Error3] = 0. That means

LN4 + LN5 = LN2 + LN7.....(LNC1)

and, at the same time,

RN4 + RN5 = RN2 + RN7.....(RNC1)

We now demonstrate that under the condition (LNC1), there is an observational contrast whose expectation is free from LNEs and at the same time, it involves one treatment contrast. We re-write (LNC1) as :

Table 5: Expectations of observational contrasts in terms of treatment contrasts

$$E[\mathrm{y}(1,1) - \mathrm{y}(1,2)] = \tau_1 - \tau_2 + \mathrm{LN4} - \mathrm{LN1} + \mathrm{RN2} - \mathrm{RN4}$$
$$E[\mathrm{y}(7,2) - \mathrm{y}(7,3)] = \tau_1 - \tau_3 + \mathrm{LN7} - \mathrm{LN1} + \mathrm{RN3} - \mathrm{RN7}$$
$$E[\mathrm{y}(1,1) - \mathrm{y}(1,3)] = \tau_1 - \tau_4 + \mathrm{LN4} - \mathrm{LN2} + \mathrm{RN2} - \mathrm{RN1}$$
$$E[\mathrm{y}(5,3) - \mathrm{y}(5,1)] = \tau_1 - \tau_5 + \mathrm{LN6} - \mathrm{LN1} + \mathrm{RN5} - \mathrm{RN6}$$
$$E[\mathrm{y}(5,3) - \mathrm{y}(5,2)] = \tau_1 - \tau_6 + \mathrm{LN6} - \mathrm{LN5} + \mathrm{RN5} - \mathrm{RN1}$$
$$E[\mathrm{y}(7,2) - \mathrm{y}(7,1)] = \tau_1 - \tau_7 + \mathrm{LN7} - \mathrm{LN3} + \mathrm{RN3} - \mathrm{RN1}$$

LN4 – LN2 = LN7 – LN5

or, LN4 – LN2 = (LN7 – LN1) + (LN1 – LN6) + (LN6 – LN5).

We now examine both sides of the expression, which are expressed in terms of LNEffects contrasts. We refer to the Table 5 of treatment contrasts. This yields:

$(\tau_1 - \tau_4) = (\tau_1 - \tau_3) + (\tau_5 - \tau_1) + (\tau_1 - \tau_6)$.

This leads to

$(\tau_3 - \tau_4 - \tau_5 + \tau_6)$.............(TC1).

The message is clear. Under the condition that Error 3 is a valid error, there is an observational contrast, *viz*.,

[y(1,1) − y(1,3)] − [y(7,2) − y(7,3)] + [y(5,3) − y(5,1)] − [y(5,3) − y(5,2)]

whose expectation is free from LNEs and, moreover, it involves the treatment contrast (TC1). We have yet to verify the status of this observational contrast in the presence of the RN Effects. It is easy to check that the RN Effects contrast [RN2 + RN7 – RN3 – RN6] remains present along with the treatment contrast. The condition (RNC1) is different from this and hence, *E*[Error 3] = 0 alone does not provide any positive result towards estimability of any treatment contrast. The condition

RN2 + RN7 = RN3 + RN6.......................(RNC2)

is also needed. That means: (LNC1), (RNC1) and (RNC2) together ensure estimability of (TC1) along with existence of a valid error *viz*., Error 3 [This holds, besides the errors: Error 1 and Error 2].

We now analyse (RNC2) and readily observe that *E*[Error 6] = 0 whenever (RNC2) holds in addition to LN2 + LN7 = LN3 + LN6..........(LNC2).

Further to this,

$\tau_3 + \tau_6 - \tau_2 - \tau_7$.............................(TC2)

becomes estimable and an unbiased estimator is given by [y(1,1) − y(1,2)] + [y(7,3) − y(7,1)] + [y(5,2) − y(5,3)].

Combining the results, we have the following:

Based on the assumptions (LNC1), (LNC2), (RNC1) and (RNC2), there are two valid errors [Error 3 and Error 6] and also there are two estimable treatment contrasts (TC1) and (TC2).

Likewise, we made an attempt to identify a pair of error functions, which together would produce similar result on a different pair of treatment contrasts. However, we are partially successful. Assuming E[Error 4] = E[Error 7] = 0, we end up with the conditions:

(a) LN2 + LN6 = LN3 + LN5 and RN2 + RN6 = RN3 + RN5,

(b) LN1 + LN4 = LN5 + LN7 and RN1 + RN4 = RN5 + RN7.

In view of (a) and (b), it turns out that

$$E[[y(7,2) - y(7,3)] - [y(1,1) - y(1,2)] + [y(5,3) - y(5,1)] - [y(5,3) - y(5,2)]]$$
$$= \tau_2 + \tau_6 - \tau_3 - \tau_5.$$

We failed to find out another estimable treatment contrast based on (a) and (b). At this stage, we did not make any further attempt with the last two error functions *viz*., Error 5 and Error 8.

Remark 2: It is interesting to observe that the errors [Error 3 to Error 8]remain as valid errors even in the presence of RNEs and LNEs provided RNE of every treatment is the same as the corresponding LNE. However, this does not ensure estimability of treatment contrasts / block contrasts without further unusual conditions, as indicated above [for treatment contrasts].

2.3. Estimable block contrasts in the BIBD with NEffects

It is well-known that a treatment-connected block design is also automatically block-connected. However, when the NEs are present, we have to analyse the block contrasts separately. At first, we display in Table 6 elementary block contrasts and their simple-minded estimates under the assumption of absence of LNEs and RNEs. These are shown in columns 1 and 2. Further, assuming that the LNEs and RNEs are present, we show in columns 3 and 4 of the same table their effects on the chosen observational contrasts. We start with Error 3 which is a valid error whenever (LNC1) and (RNC1) both hold simultaneously. Upon re-writing (LNC1) as: LN4 − LN7 = LN2 − LN5, we find that the LHS corresponds to $\beta_1 - \beta_7$. To find a 'matching' for the RHS, we re-write it as (LN2 − LN7) + (LN7 − LN1) + (LN1 − LN5) which is again expressed as $(\beta_1 - \beta_4) + (\beta_6 - \beta_1) + (\beta_1 - \beta_2)$ and this simplifies to $(\beta_1 + \beta_6 - \beta_2 - \beta_4)$. Therefore, combining the two, we infer that for the block contrast given by $\beta_2 + \beta_4 - \beta_6 - \beta_7$, there is an observational contrast *viz*.,
[y(1,1) − y(7,2)] − [y(1,3) − y(4,1)] + y(1,2) − y(6,3)] − [y(1,2) − y(2,1)]

Table 6: Expectations of observational contrasts in terms of block contrasts

Block contrast	Observational contrasts	LNE(+)(−)	RNE(+)(−)
$\beta_1 - \beta_2$	y(1,2) − y(2,1)	1,5	4,3
$\beta_1 - \beta_3$	y(1,3) − y(3,2)	2,3	1,6

$\beta_1 - \beta_4$	y(1,3) − y(4,1)	2,7	1,5
$\beta_1 - \beta_5$	y(1,1) − y(5,3)	4,6	2,5
$\beta_1 - \beta_6$	y(1,2) − y(6,3)	1,7	4,6
$\beta_1 - \beta_7$	y(1,1) − y(7,2)	4,7	2,3

which eliminates the LN Effects in expectation. This is so far as LN Effects are concerned. We now examine the nature of involvement of RN Effects. An analysis similar to the case of estimation of treatment contrasts suggests:

Under the conditions required for the validity of Error 3 *i.e.*, under (LNC1) and (RNC1):

There is a block contrast *viz.*, $\beta_2 + \beta_4 - \beta_6 - \beta_7$, which is estimable provided further that

RN1 − RN6 = RN2 − RN5.............(RNC3)

holds. When this holds, another condition

LN2 − LN4 = LN3 − LN6.............................(LNC3)

ensures estimation of a second block contrast *viz.*, $\beta_3 - \beta_5$ and an estimator is given by [y(1,1) − y(5,3)] − [y(1,3) − y(3,2)]. These are uninteresting conditions on the LN Effects and RN Effects. We do not pursue the matter anymore. To summarize, in the presence of LN Effects and RN Effects, only under certain conditions [like (LNC1) and (RNC1)], we can identify error function(s) on the top of the basic two errors [Error 1 and Error 2]. However, we need further conditions on the LN Effects and RN Effects to provide estimable treatment contrasts and estimable block contrasts - that too - only 1 or 2. There is no substantial promise for the analysis of VB Designs (also possibly for EB Designs) - in the presence of LN Effects and RN Effects.

3. VB Design With Unequal Replications

Mishra and Sarvate (2019, Private Communication) studied a block design which is variance-balanced but based on unequal replication numbers. The blocks of the design are given as:

B1 = (1,1,2,3); B2 = (1,1,4,5); B3 = (1,1,6,7); B4 = (1,2,4,6); B5 = (1,3,5,7); B6 = (2,3,5,6); B7 = (2,3,4,7); B8 = (2,4,5,7); B9 = (2,5,6,7); B10 = (3,4,5,6); B11 = (3,4,6,7).

Here, the total df = 44−1= 43 is decomposed as: 10 df for blocks, 6 df for treatments and 27 df for error. Below in Table 7 we present a full set of 27 linearly independent error functions. Errors 1 to 3 are readily derived as observational contrasts within blocks whereas Errors 4 to 27 are identified as differences of two terms based on observational contrasts.

Table 7: Error functions: Term 1 and Term 2

Error Function 1 : [y(1,1) − y(1,2)]
Error Function 2 : [y(2,1) − y(2,2)]
Error Function 3 : [y(3,1) − y(3,2)]

Error Function 4 : Term 1 = [y(1,2) − y(1,3)]; Term 2 = [y(4,1) − y(4,2)]
Error Function 5 : Term 1 = [y(1,2) − y(1,4)]; Term 2 = [y(5,1) − y(5,2)]
Error Function 6 : Term 1 = [y(2,2) − y(2,3)]; Term 2 = [y(4,1) − y(4,3)]
Error Function 7 : Term 1 = [y(2,1) − y(2,4)]; Term 2 = [y(5,1) − y(5,3)]
Error Function 8 : Term 1 = [y(3,2) − y(3,3)]; Term 2 = [y(4,1) − y(4,4)]
Error Function 9 : Term 1 = [y(3,1) − y(3,4)]; Term 2 = [y(5,1) − y(5,4)]
Error Function 10 : Term 1 = [y(1,3) − y(1,4)]; Term 2 = [y(6,1) − y(6,2)]
Error Function 11 : Term 1 = [y(6,1) − y(6,2)]; Term 2 = [y(7,1) − y(7,2)]
Error Function 12 : Term 1 = [y(4,2) − y(4,3)]; Term 2 = [y(7,1) − y(7,3)]
Error Function 13 : Term 1 = [y(4,2) − y(4,3)]; Term 2 = [y(8,1) − y(8,2)]
Error Function 14 : Term 1 = [y(6,1) − y(6,3)]; Term 2 = [y(8,1) − y(8,3)]
Error Function 15 : Term 1 = [y(6,1) − y(6,3)]; Term 2 = [y(9,1) − y(9,2)]
Error Function 16 : Term 1 = [y(4,2) − y(4,4)]; Term 2 = [y(6,1) − y(6,4)]
Error Function 17 : Term 1 = [y(4,2) − y(4,4)]; Term 2 = [y(9,1) − y(9,3)]
Error Function 18 : Term 1 = [y(7,1) – y(7,4)]; Term 2 = [y(8,1) − y(8,4)]
Error Function 19 : Term 1 = [y(7,1) − y(7,4)]; Term 2 = [y(9,1) − y(9,4)]
Error Function 20: Term 1 = [y(7,2) − y(7,3)]; Term 2 = [y(10,1) − y(10,2)]
Error Function 21: Term 1 = [y(7,2) − y(7,3)]; Term 2 = [y(11,1) − y(11,2)]
Error Function 22: Term 1 = [y(5,2) − y(5,3)]; Term 2 = [y(6,2) − y(6,3)]
Error Function 23: Term 1= [y(5,2) − y(5,3)]; Term 2=[y(10,1) − y(10,3)]
Error Function 24: Term 1 = [y(6,2) − y(6,4)]; Term 2 = [y(10,1) − y(10,4)]
Error Function 25: Term 1 = [y(6,2) − y(6,4)]; Term 2 = [y(11,1) − y(11,3)]
Error Function 26 : Term 1 = [y(7,2) − y(7,4)]; Term 2= [y(5,2) − y(5,4)]
Error Function 27 : Term 1 = [y(7,2) − y(7,4)]; Term 2 = [y(11,1) − y(11,4)]

It is further verified that the 27×44 matrix of the coefficients of error functions has rank 27. This holds under the assumption that there are no NEffects.

3.1. Nature of error functions in the presence of LN and RN Effects

Now we work out expectations of all the 27 Error functions in the presence of left-and right sided neighbor effects in the plots of the blocks, under the assumption that the blocks are circular.

Error 1 = [y(1,1) − y(1,2)]

E[Error 1] = τ_1+ LN3 + RN1 − τ_1 − LN1 − RN2 = LN3 − LN1 + RN1 − RN2

Error 2 = [y(2,1) − y(2,2)]

E[Error 2] = τ_1 + LN5 + RN1 − τ_1 − LN1 − RN4 = LN5 − LN1 + RN1 − RN4

Error 3 = [y(3,1) − y(3,2)]

E[Error 3] = τ_1 + LN7 + RN1 − τ_1 − LN1 − RN6 = LN7 − LN1 + RN1 − RN6

Error 4 = Term 1 − Term 2

Term 1 = [y(1,2) − y(1,3)]

E[Term 1] = τ_1 + LN1 + RN2 − τ_2− LN1 − RN3 = τ_1 − τ_2 + RN2 − RN3

Term 2 = [y(4,1) − y(4,2)]

E[Term 2] = τ_1 + LN6 + RN2 − τ_2 − LN1 − RN4 = τ_1 − τ_2 + LN6 − LN1 + RN2 − RN4

E[Error 4] = LN1 − LN6 + RN4 − RN3

Error 5 = Term 1 − Term 2

Term 1 = [y(1,2) − y(1,4)]

E[Term 1] = τ_1+ LN1 + RN2 − τ_3 − LN2 − RN1 = τ_1 − τ_3 + LN1 − LN2 + RN2 − RN1

Term 2 = [y(5,1) − y(5,2)]

E[Term 2] = τ_1 + LN7 + RN3 − τ_3 − LN1 − RN5 = τ_1 − τ_3 + LN7 − LN1 + RN3 − RN5

E[Error 5] = 2LN1 − LN2 − LN7 + RN2 + RN5 − RN1 − RN3

Error 6 = Term 1 − Term 2

Term 1 = [y(2,2) − y(2,3)]

E[Term 1] = τ_1 + LN1 + RN4 − τ_4 − LN1 − RN5 = τ_1− τ_4 + RN4 − RN5

Term 2 = [y(4,1) − y(4,3)]

E[Term 2] = τ_1 + LN6 + RN2 −τ_4− LN2 − RN6 = τ_1 − τ_4+ LN6 − LN2 + RN2 − RN6

E[Error 6] = LN2 − LN6 + RN4 + RN6 − RN2 − RN5

Error 7 = Term 1− Term 2

Term 1 = [y(2,1) − y(2,4)]

E[Term 1] = τ_1 + LN5 + RN1 − τ_5 − LN4 − RN1 = τ_1 − τ_5 + LN5 − LN4

Term 2 = [y(5,1) − y(5,3)]

E[Term 2] = τ_1 + LN7 + RN3 − τ_5− LN3 − RN7 = τ_1 − τ_5 + LN7 − LN3 + RN3 − RN7

E[Error 7] = LN5 + LN3 − LN4 − LN7 + RN7 − RN3

Error 8 = Term 1 − Term 2

Term 1 = [y(3,2) − y(3,3)]

E[Term 1] = τ_1 + LN1 + RN6 − τ_6 − LN1 − RN7 = τ_1− τ_6 + RN6 − RN7

Term 2 = [y(4,1) − y(4,4)]

E[Term 2] = τ_1+ LN6 + RN2 − τ_6− LN4 − RN1 = τ_1− τ_6 + LN6 − LN4 + RN2 − RN1

E[Error 8] = LN4 − LN6 + RN6 + RN1 − RN2 − RN7

Error 9 = Term 1− Term 2

Term 1 = [y(3,1) − y(3,4)]

E[Term 1] = τ_1 + LN7 + RN1 − τ_7 − LN6 − RN1 = τ_1− τ_7 + LN7 − LN6

Term 2 = [y(5,1) − y(5,4)]

E[Term 2] = τ_1 + LN7 + RN3 −τ_7− LN5 − RN1 = τ_1 − τ_7 + LN7 − LN5 + RN3 − RN1

E[Error 9] = LN5 − LN6 + RN1 − RN3

Error 10 = Term 1 − Term 2

Term 1 = [y(1,3) − y(1,4)]

E[Term 1] = τ_2+ LN1 + RN3 − τ_3 − LN2 − RN1 = τ_2 − τ_3 + LN1 − LN2 + RN3 − RN1

Term 2 = [y(6,1) − y(6,2)]

E[Term 2] = τ_2 + LN6 + RN3 − τ_3 − LN2 − RN5 = τ_2 − τ_3 + LN6 − LN2 + RN3 − RN5

E[Error 10] = LN1 − LN6 + RN5 − RN1

Error 11 = Term 1− Term 2

Term 1 = [y(6,1) − y(6,2)]

E[Term 1] = τ_2 + LN6 + RN3 − τ_3 − LN2 − RN5 = τ_2 − τ_3 + LN6 − LN2 + RN3 − RN5

Term 2 = [y(7,1) − y(7,2)]

E[Term 2] = τ_2 + LN7 + RN3 − τ_3 − LN2 − RN4 = τ_2 − τ_3 + LN7 − LN2 + RN3 − RN4

E[Error 11] = LN6 − LN7 + RN4 − RN5

Error 12 = Term 1 − Term 2

Term 1 = [y(4,2) − y(4,3)]

E[Term 1] = τ_2 + LN1 + RN4 − τ_4 − LN2 − RN6 = τ_2 − τ_4 + LN1 − LN2 + RN4 − RN6

Term 2 = [y(7,1) − y(7,3)]

E[Term 2] = τ_2 + LN7 + RN3 − τ_4 − LN3 − RN7 = τ_2 − τ_4 + LN7 − LN3 + RN3 − RN7

E[Error 12] = LN1 + LN3 − LN2 − LN7 + RN4 + RN7 − RN3 − RN6

Error 13 = Term 1 − Term 2

Term 1 = [y(4,2) − y(4,3)]

E[Term 1] = τ_2 + LN1 + RN4 − τ_4 − LN2 − RN6 = τ_2 − τ_4 + LN1 − LN2 + RN4 − RN6

Term 2 = [y(8,1) − y(8,2)]

E[Term 2] = τ_2 + LN7 + RN4 − τ_4− LN2 − RN5 = τ_2 − τ_4 + LN7 − LN2 + RN4 − RN5

E[Error 13] = LN1 − LN7 + RN5 − RN6

Error 14 = Term 1 − Term 2

Term 1 = [y(6,1) − y(6,3)]

E[Term 1] = τ_2 + LN6 + RN3 − τ_5 − LN3 − RN6 = τ_2− τ_5 + LN6 − LN3 + RN3 − RN6

Term 2 = [y(8,1) − y(8,3)]

E[Term 2] = τ_2 + LN7 + RN4 − τ_5 − LN4 − RN7 = τ_2 − τ_5 + LN7 − LN4 + RN4 − RN7

E[Error 14] = LN6 + LN4 − LN3 − LN7 + RN3 + RN7 − RN4 − RN6

Error 15 = Term 1− Term 2

Term 1 = [y(6,1) − y(6,3)]

E[Term 1] = τ_2 + LN6 + RN3 − τ_5− LN3 − RN6 = τ_2− τ_5 + LN6 − LN3 + RN3 − RN6

Term 2 = [y(9,1) − y(9,2)]

E[Term 2] = τ_2 + LN7 + RN5 − τ_5 − LN2 − RN6 = τ_2 − τ_5 + LN7 − LN2 + RN5 − RN6

E[Error 15] = LN6 + LN2 − LN3 − LN7 + RN3 − RN5

Error 16 = Term 1 − Term 2

Term 1 = [y(4,2) − y(4,4)]

E[Term 1] = τ_2 + LN1 + RN4 − τ_6− LN4 − RN1 = τ_2 − τ_6 + LN1 − LN4 + RN4 − RN1

Term 2 = [y(6,1) − y(6,4)]

E[Term 2] = τ_2 + LN6 + RN3 − τ_6 − LN5 − RN2 = τ_2 − τ_6+ LN6 − LN5 + RN3 − RN2

E[Error 16] = LN1 + LN5 − LN4 − LN6 + RN4 + RN2 − RN1 − RN3

Error 17 = Term 1 − Term 2

Term 1 = [y(4,2) − y(4,4)]

E[Term 1] = τ_2+ LN1 + RN4 − τ_6 − LN4 − RN1 = τ_2− τ_6 + LN1 − LN4 + RN4 − RN1

Term 2 = [y(9,1) − y(9,3)]

E[Term 2] = τ_2 + LN7 + RN5 − τ_6 − LN5 − RN7 = τ_2− τ_6+ LN7 − LN5 + RN5 − RN7

E[Error 17] = LN1 + LN5 − LN4 − LN7 + RN4 + RN7 − RN1 − RN5

Error 18 = Term 1 − Term 2

Term 1 = [y(7,1) − y(7,4)]

E[Term 1] = τ_2 + LN7 + RN3 − τ_7− LN4 − RN2 = τ_2 − τ_7 + LN7 − LN4 + RN3 − RN2

Term 2 = [y(8,1) − y(8,4)]

E[Term 2] = τ_2 + LN7 + RN4 − τ_7 − LN5 − RN2 = τ_2 − τ_7 + LN7 − LN5 + RN4 − RN2

E[Error 18] = LN5 − LN4 − RN3 − RN4

Error 19 = Term 1 − Term 2

Term 1 = [y(7,1) − y(7,4)]

E[Term 1] = τ_2 + LN7 + RN3 − τ_7 − LN4 − RN2 = τ_2 − τ_7 + LN7 − LN4 + RN3 − RN2

Term 2 = [y(9,1) − y(9,4)]

E[Term 2] = τ_2 + LN7 + RN5 − τ_7 − LN6 − RN2 = τ_2− τ_7 + LN7 − LN6 + RN5 − RN2

E[Error 19] = LN6 − LN4 + RN3 − RN5

Error 20 = Term 1 − Term 2

Term 1 = [y(7,2) − y(7,3)]

E[Term 1] = τ_3 + LN2 + RN4 − τ_4 − LN3 − RN7 = τ_3 − τ_4+ LN2 − LN3 + RN4 − RN7

Term 2 = [y(10,1) − y(10,2)]

E[Term 2] = τ_3 + LN6 + RN4 − τ_4 − LN3 − RN5 = τ_3− τ_4 + LN6 − LN3 + RN4 − RN5

E[Error 20] = LN2 − LN6 + RN5 − RN7

Error 21 = Term 1 − Term 2

Term 1 = [y(7,2) − y(7,3)]

E[Term 1] = τ_3+ LN2 + RN4 − τ_4 − LN3 − RN7 = τ_3 − τ_4 + LN2 − LN3 + RN4 − RN7

Term 2 = [y(11,1) − y(11,2)]

E[Term 2] = τ_3+ LN7 + RN4 − τ_4 − LN3 − RN6 = τ_3 − τ_4 + LN7 − LN3 + RN4 − RN6

E[Error 21] = LN2 − LN7 + RN6 − RN7

Error 22 = Term 1− Term 2

Term 1 = [y(5,2) − y(5,3)]

E[Term 1] = τ_3 + LN1 + RN5 − τ_5 − LN3 − RN7 = τ_3 − τ_5+ LN1 − LN3 + RN5 − RN7

Term 2 = [y(6,2) − y(6,3)]

E[Term 2] = τ_3 + LN2 + RN5 − τ_5 − LN3 − RN6 = τ_3− τ_5 + LN2 − LN3 + RN5 − RN6

E[Error 22] = LN1 − LN2 + RN6 − RN7

Error 23 = Term 1 − Term 2

Term 1 = [y(5,2) − y(5,3)]

E[Term 1] = τ_3 + LN1 + RN5 − τ_5 − LN3 − RN7 = τ_3− τ_5 + LN1 − LN3 + RN5 − RN7

Term 2 = [y(10,1) − y(10,3)]

E[Term 2] = τ_3 + LN6 + RN4 − τ_5 − LN4 − RN6 = τ_3− τ_5 + LN6 − LN4 + RN4 − RN6

E[Error 23] = LN1 + LN4 − LN3 − LN6 + RN5 + RN6 − RN4 − RN7

Error 24 = Term 1 − Term 2

Term 1 = [y(6,2) − y(6,4)]

E[Term 1] = τ_3+ LN2 + RN5 − τ_6 − LN5 − RN7 = τ_3− τ_6 + LN2 − LN5 + RN5 − RN2

Term 2 = [y(10,1) − y(10,4)]

E[Term 2] = τ_3+ LN6 + RN4 − τ_6 − LN5 − RN3 = τ_3− τ_6+ LN6 − LN5 + RN4 − RN3

E[Error 24] = LN2 − LN6 + RN5 + RN3 − RN2 − RN4

Error 25 = Term 1 − Term 2

Term 1 = [y(6,2) − y(6,4)]

E[Term 1] = τ_3+ LN2 + RN5 − τ_6 − LN5 − RN2 = τ_3− τ_6 + LN2 − LN5 + RN5 − RN2

Term 2 = [y(11,1) − y(11,3)]

E[Term 2] = τ_3+ LN7 + RN4 − τ_6 − LN4 − RN7 = τ_3− τ_6+ LN7 − LN4 + RN4 − RN7

E[Error 25] = LN2 + LN4 − LN5 − LN7 + RN5 + RN7 − RN2 − RN4

Error 26 = Term 1 − Term 2

Term 1 = [y(7,2) − y(7,4)]

E[Term 1] = τ_3+ LN2 + RN4 − τ_7 − LN4 − RN2 = τ_3− τ_7+ LN2 − LN4 + RN4 − RN2

Term 2 = [y(5,2) − y(5,4)]

E[Term 2] = τ_3+ LN1 + RN5 − τ_7 − LN5 − RN1 = τ_3− τ_7 + LN1 − LN5 + RN5 − RN1

E[Error 26] = LN2 + LN5 − LN1 − LN4 + RN4 + RN1− RN2 − RN5

Error 27 = Term 1− Term 2

Term 1 = [y(7,2) − y(7,4)]

E[Term 1] = τ_3 + LN2 + RN4 − τ_7 − LN4 − RN2 = τ_3− τ_7 + LN2 − LN4 + RN4 − RN2

Term 2 = [y(11,1) − y(11,4)]

E[Term 2] = τ_3 + LN7 + RN4 − τ_7 − LN6 − RN3 = τ_3−τ_7+ LN7 − LN6 + RN4 − RN3

E[Error 27] = LN2 + LN6 − LN4 − LN7 + RN3 − RN2

For further analysis, in the above, we consider only the LN Effects and develop the underlying matrix. This is shown in Table 8 below.

Table 8: Coefficients of LN Effects in expectations of observational contrasts for errors

Error Sl. No.							
1	−1	0	1	0	0	0	0
2	−1	0	0	0	1	0	0
3	−1	0	0	0	0	0	1
4	1	0	0	0	0	−1	0
5	1	−1	1	0	0	0	−1
6	0	1	0	0	0	−1	0
7	0	0	1	−1	1	0	−1
8	0	0	0	1	0	−1	0
9	0	0	0	0	1	−1	0
10	1	0	0	0	0	−1	0
11	0	0	0	0	0	1	−1
12	1	−1	1	0	0	0	−1
13	1	0	0	0	0	0	−1
14	0	0	−1	1	0	1	−1
15	0	1	−1	0	0	1	−1
16	1	0	0	−1	1	−1	0
17	1	0	0	−1	1	0	−1
18	0	0	0	−1	1	0	0
19	0	0	0	−1	0	1	0
20	0	1	0	0	0	−1	0
21	0	1	0	0	0	0	−1
22	1	−1	0	0	0	0	0
23	1	0	−1	1	0	−1	0
24	0	1	0	0	0	−1	0
25	0	1	0	1	−1	0	−1
26	−1	1	0	−1	1	0	0
27	0	1	0	−1	0	1	−1

Similarly, we may consider the RN effects only and develop the corresponding matrix of coefficients. This is shown in the Table 9 below.

3.2. Estimability of treatment contrasts in the presence of both-sided neighbor effects

Suppose Error 1 is a valid error, that is, E[Error 1] = 0. Then LN3 − LN1 = 0. Now we can rewrite it as (LN3 − LN7) + (LN7 − LN1) = 0. Using the Table 10 on expectations of some treatment contrasts, it yields $\tau_5 - \tau_1 + \tau_1 - \tau_3 = 0$. This leads to estimability of the treatment contrast $(\tau_3 - \tau_5)$, provided there are no RN Effects. For this we need the condition

Table 9: Coefficients of RN Effects in expectations of observational contrasts for errors

Error Sl. No.							
1	1	−1	0	0	0	0	0
2	1	0	0	−1	0	0	0
3	1	0	0	0	0	−1	0
4	0	0	−1	1	0	0	0
5	0	0	−1	0	1	0	0
6	0	−1	0	1	−1	1	0
7	0	0	−1	0	0	0	1
8	1	−1	0	0	0	1	−1
9	1	0	−1	0	0	0	0
10	−1	0	0	0	1	0	0
11	0	0	0	1	−1	0	0
12	0	0	−1	1	0	−1	1
13	0	0	0	0	1	−1	0
14	0	0	1	−1	0	−1	1
15	0	0	1	0	−1	0	0
16	−1	1	−1	1	0	0	0
17	−1	0	0	1	−1	0	1
18	0	0	1	−1	0	0	0
19	0	0	1	0	0	−1	0
20	0	0	0	0	1	0	−1
21	0	0	0	0	0	1	−1
22	0	0	0	0	0	1	−1
23	0	0	0	−1	1	1	−1
24	0	−1	1	−1	1	0	0
25	0	−1	0	−1	1	0	1
26	1	−1	0	1	−1	0	0
27	0	−1	1	0	0	0	0

RN5 = RN7. When this happens, we identify Error 20 as a valid error when, in addition, we also require: LN2 = LN6. Under this LN-related condition, we find that $\tau_1 - \tau_4$ becomes estimable if, again, we have: RN2 = RN6. The conditions are not encouraging at all. We stop the analysis and observe that presence of LN and RN Effects really poses estimability problem for treatment effects contrasts and similarly, for block effects contrasts as well.

Table 10: Expectations of observational contrasts in terms of basic treatment contrasts

$E[y(4,1) - y(4,2)] = \tau_1 - \tau_2 + LN6 - LN1 + RN2 - RN4$
$E[y(5,1) - y(5,2)] = \tau_1 - \tau_3 + LN7 - LN1 + RN3 - RN5$
$E[y(4,1) - y(4,3)] = \tau_1 - \tau_4 + LN6 - LN2 + RN2 - RN6$
$E[y(5,1) - y(5,3)] = \tau_1 - \tau_5 + LN7 - LN3 + RN3 - RN7$
$E[y(4,1) - y(4,4)] = \tau_1 - \tau_6 + LN6 - LN4 + RN2 - RN1$
$E[y(5,1) - y(5,4)] = \tau_1 - \tau_7 + LN7 - LN5 + RN3 - RN1$

4. Concluding Remarks

These simple examples illustrate the difficulties in carrying out ANOVA Tests in the presence of LNEs and / or RNEs. We should be careful in handling the data analyses issues.

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