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Distribution of the Hölder Mean of P-Values with Applications to Multiple Testing

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Abstract

We study the null distribution of the Hölder mean with a scalar parameter $m \in (-\infty, +\infty)$ of i.i.d. P-values for testing $n \geq 2$ null hypotheses subject to the familywise error rate (FWER) control. We find the exact critical values for $n = 2, 3$ and the asymptotic critical values for $n > 3$ for selected values of m. We use them in a closed multiple testing procedure (MTP) which we illustrate by a numerical example. We compare the powers of the tests of the intersection hypothesis $H_0 = \cap_{i=1}^{n} H_i$ for $n = 2$ and 3 using the Hölder means with different values of m to find the best choice. Asymptotic critical values are not very accurate (are generally too conservative) and so power comparisons are not performed for larger n.

Key words: Arithmetic mean; Closed procedure; Distribution theory; Familywise error rate; Geometric mean; Harmonic mean; Hölder mean; Power comparison.

AMS Subject Classifications: 62E99

1. Introduction

In Gou and Tamhane (2024) we studied the null distribution of the harmonic mean of the P-values with application to multiple testing. We compared the resulting multiple testing procedure (MTP) with the commonly used P-value based MTPs of Holm (1979), Hochberg (1988) and Hommel (1988) and found it to be generally more powerful.

The arithmetic, geometric and harmonic means are special cases of Hölder mean, so it is natural to ask whether in the class of all the Hölder mean based MTPs, if there is some subclass that is more powerful under certain non-null configurations of interest. However, we must first derive the null distribution of the Hölder mean and obtain its critical values. This is the main focus of the present paper. In Section 6 we give a closed MTP (Marcus *et al.*, 1976) that uses the Hölder means for testing multiple hypotheses.

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Consider testing $n \geq 2$ hypotheses, $H_1, \ldots, H_n$, subject to the strong familywise error rate (FWER) control requirement (Hochberg and Tamhane, 1987):

$$\text{FWER} = \Pr\{\text{Reject at least one true } H_i\} \leq \alpha \tag{1}$$

where $\alpha \in (0, 1)$ is prespecified. Let $P_1, \ldots, P_n$ denote the P-values associated with the hypotheses $H_1, \ldots, H_n$. The overall null hypothesis is denoted by $H_0 = \cap_{i=1}^{n} H_i$. We will assume that under H_0, the P_i's are independent and identically distributed (i.i.d.) uniform random variables over $[0, 1]$. This assumption is relaxed to allow for de[endent P-values in simulations reported in Section 8.

Consider a given real-valued parameter $m \in (-\infty, \infty)$ and weights

$$w_i > 0 \ (i = 1, \ldots, n) \ \text{ such that } \ \sum_{i=1}^{n} w_i = 1. \tag{2}$$

Then the weighted Hölder mean of the P-values $P_1, \ldots, P_n$ with parameter m is defined as

$$\bar{P}_n(m, w) = \left(\frac{1}{n} \sum_{i=1}^{n} w_i P_i^m \right)^{1/m}. \tag{3}$$

The unweighted Hölder mean corresponds to $w_1 = \cdots = w_n = 1/n$ and is denoted simply by $\bar{P}_n(m)$, dropping w in the notation. The arithmetic, geometric and harmonic means are special cases of the Hölder mean for $m = 1, 0$ and -1, respectively. The Hölder means for selected values of m have been previously considered by Vovk and Wang (2020) and by Tian *et al.* (2023). Here we study them in more detail with focus on their exact null distributions for $n = 2$ and their asymptomatic null distributions for $n > 2$.

The outline of the paper is as follows. Section 2 gives expressions for the c.d.f. of the unweighted Hölder mean for general m and $n = 2$. Section 3 gives expressions for the cumulative distribution function (c.d.f.) of the weighted Hölder mean for selected values of m and the expressions for their lower α critical values for $n = 2$. Section 4 gives the critical values for $n = 3$. Section 5 derives the asymptotic null distributions of the unweighted Hölder mean. Section 6 gives the closed MTP based on the Hölder means. Section 7 gives a numerical example to illustrate this MTP for harmonic, geometric and arithmetic means. Section 8 gives a numerical type I error and power comparisons for testing $H_0 = \cap_{i=1}^{n}$ for $n = 2$ and as well as for type I error for selected $n \geq 10$. Finally Section 9 gives concluding remarks. Derivations of all analytical results and proofs of theorems are presented in the Appendix.

2. Null distribution of unweighted Hölder mean for general m and $n = 2$

Before we state the main theorem of this section about the null distribution of $\bar{P}_2(m)$, we show in Figure 1 how the rejection boundaries in the (P_1, P_2) space change with m for selected values of $m = -\infty, -1, 0, 1, \infty$ for fixed $\alpha = 0.25$. (A large value of α is chosen so that the plotted rejection boundaries are distinguishable from each other.) The rejection boundaries also change with α but their relative behavior with respect to m remains the same.

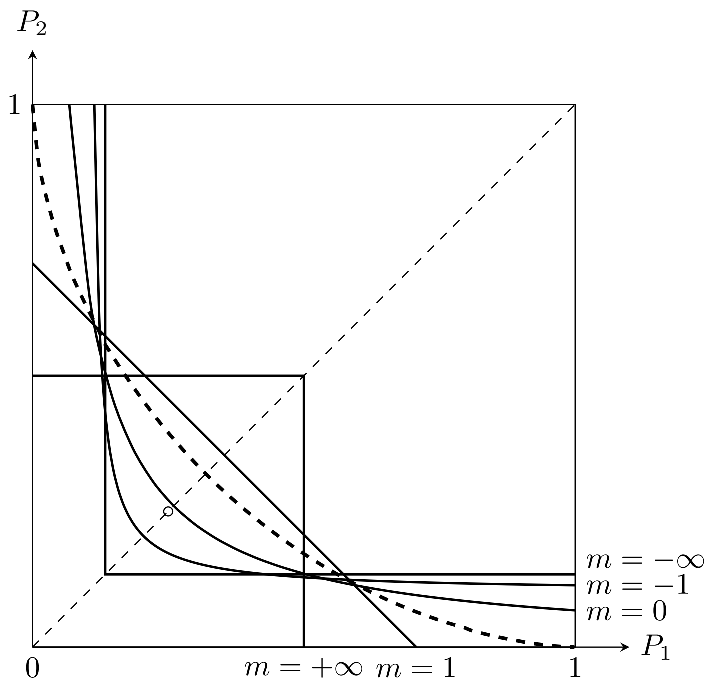

Figure 1: Rejection boundaries for selected values of m

Notice from this figure that the rejection boundaries for $m = 0, -1$ and $-\infty$ go from the top edge of the square to the right edge, while for $m = 1$ and ∞ they go from the bottom edge to the left edge. By the continuity in m and symmetry in P_1 and P_2, it follows that there exists an $m = m^* \in (-\infty, \infty)$ and an associated critical value $c = c^* \in (0, 1)$, for which the rejection boundary connects the top left corner $(P_1, P_2) = (0, 1)$ to the bottom right corner $(P_1, P_2) = (1, 0)$. This rejection boundary is shown by a dotted line in the figure and we refer to it as the *critical boundary*.

Given that the rejection boundary is defined by $P_1^m + P_2^m = 2c^m$ and the critical boundary passes through the points $(0, 1)$ and $(1, 0)$, it follows that for the critical boundary we have $2(c^*)^{m^*} = 1$ or $c^* = (1/2)^{1/m^*}$.

The following numerical example illustrates the calculation of m^* and c^* for $\alpha = 0.05$. First note that

$$\alpha = \Pr\{\bar{P}_2(m) \le c\} = \Pr\{P_1 \le (2c^m - P_2^m)^{1/m}\} = \int_0^1 (2c^m - x^m)^{1/m} dx.$$

Substitute $m^* = 1/3$ and $2(c^*)^{m^*} = 1$ in the above integral, which then becomes

$$\alpha = \int_0^1 (1 - x^{1/3})^3 dx.$$

Now put $1 - x^{1/3} = y$. Then $dx = 3(1 - y)^2 dy$. So we get

$$\alpha = 3 \int_0^1 y^3 (1 - y)^2 dy = \frac{3}{60} = 0.05.$$

Thus $m^* = 1/3$ and $c^* = (1/2)^3 = 0.125$ gives $\alpha = 0.05$. This pair of (m^*, c^*) values is shown in Table 1 along with other pairs of values for selected α values computed using MATLAB function `fsolve()`.

Table 1: m^* and c^* values for selected α for $n = 2$

α	m^*	c^*
0.010	0.2336	0.0515
0.025	0.2812	0.0850
0.050	0.3333	0.1250
0.100	0.4113	0.1854

In the following theorem we give expressions for the c.d.f. of $\bar{P}_2(m)$ for general m. First let $F_2(x; m) = \Pr\{\bar{P}_2(m) \le x\}$ denote the c.d.f. of $\bar{P}_2(m)$. Also let $B_p(a, b)$ denote the incomplete beta function defined as

$$B_p(a,b) = \int_0^p x^{a-1}(1-x)^{b-1}dx,$$

where $p \le 1$. When $p = 1$ we have the complete beta function denoted by $B_1(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$.

Theorem 1: The c.d.f. of $\bar{P}_2(m)$ is given by

$$F_2(x,m) = \begin{cases} (2x^m-1)^{1/m} + \frac{4^{1/m}x^2}{m}\left[B_{1/2x^m}\left(\frac{1}{m},\frac{1}{m}+1\right)\right. \\ \quad \left. -B_{1-1/2x^m}\left(\frac{1}{m},\frac{1}{m}+1\right)\right], \quad \text{when } 0 \le x \le 1, \quad m \le m^*, m \neq 0 \\ \frac{4^{1/m}x^2}{m}B_1\left(\frac{1}{m},\frac{1}{m}+1\right), \quad \text{when } 0 \le x \le 2^{-1/m}, \quad m > m^*. \end{cases} \qquad \square$$

The case $m = 0$ is covered in Part 5 of Theorem 2 and hence is not included here. We don't need to compute the c.d.f. for $x > 2^{-1/m}$ when $m > m^*$ because the corresponding α values are too large to be practically useful.

3. Exact null distribution of weighted Hölder mean for selected values of m and $n = 2$

In this section we obtain the c.d.f. of the weighted Hölder mean, denoted by $F_2(x; m, w)$ for selected m values. These results are given in the following theorem. The lower α critical values in each case can be found by solving the equation $F_2(x; m, w) = \alpha$ for x. Explicit expressions for the critical values are given where available. We denote these critical values by $c_2(m, \alpha)$.

Theorem 2: This theorem has nine parts corresponding to the nine selected m values, $m = -\infty, -2, -1, -0.5, 0, 0.5, 1, 2$ and $+\infty$.

Part 1 ($m = -\infty$):
As $m \to -\infty$, $\bar{P}_n(m, w) \to P_{\min}$ for any choice of weights. Assuming $P_{\min}$ is unique, its c.d.f. and lower α critical value are given by

$$F_n(x; -\infty) = 1 - (1-x)^n \quad \text{and} \quad c_n(-\infty, \alpha) = 1 - (1-\alpha)^{1/n}.$$

Part 2 ($m = -2$):
For $m = -2$ the c.d.f. of

$$\bar{P}_2(-2, w) = \left(\frac{w_1}{P_1^2} + \frac{w_2}{P_2^2}\right)^{-1/2}$$

is given by

$$F_2(x; -2, w) = x\left[\sqrt{w_1(1 - w_2x^2)} + \sqrt{w_2(1 - w_1x^2)}\right]. \tag{4}$$

For equal weights this simplifies to $F_2(x; -2) = x\sqrt{2 - x^2}$. The lower α critical value for equal weights is $c_2(-2, \alpha) = \sqrt{1 - \sqrt{1 - \alpha}}$.

Part 3 ($m = -1$):
For $m = -1$ the c.d.f. of the weighted harmonic mean,

$$\bar{P}_2(-1, w) = \left(\frac{w_1}{P_1} + \frac{w_2}{P_2}\right)^{-1},$$

is given by

$$F_2(x; -1, w) = x + w_1w_2x^2 \ln\left[1 + \frac{1 - x}{w_1w_2x^2}\right].$$

For equal weights this simplifies to

$$F_2(x; -1) = x + \frac{x^2}{4}\ln\left[1 + \frac{4(1 - x)}{x^2}\right]. \tag{5}$$

There is no closed form solution to the equation $F_2(x; -1) = \alpha$.

Part 4 ($m = -0.5$):
For $m = -0.5$, for equal weights the c.d.f. of

$$\bar{P}_2(-0.5) = \left[\frac{1}{2}\left(\frac{1}{\sqrt{P_1}} + \frac{1}{\sqrt{P_2}}\right)\right]^{-2}$$

is given by

$$\begin{aligned} &F_2(x, -0.5) \\ &= \frac{x}{(2 - \sqrt{x})^2} + \frac{x}{8}\left(6\sqrt{x} - x - \frac{x(4 + \sqrt{x})}{2 - \sqrt{x}} + 3x\ln\left(\frac{(2 - \sqrt{x})^2}{x}\right) + 2 - \frac{2x}{(2 - \sqrt{x})^2}\right). \end{aligned} \tag{6}$$

There is no closed form solution to the equation $F_2(x; -0.5) = \alpha$.

Part 5 ($m = 0$):
For $m = 0$ the c.d.f. of the weighted geometric mean

$$\bar{P}_2(0, w) = P_1^{w_1}P_2^{w_2}$$

is given by

$$F_2(x; 0, w) = \left(1 - \frac{w_2}{w_1}\right)x^{1/w_1} + \left(1 - \frac{w_1}{w_2}\right)x^{1/w_2} \quad w_1 \neq w_2.$$

For equal weights the c.d.f. is given by

$$F_2(x;0) = \Pr\left\{\chi^2_4 > -4\ln x\right\} = x^2(1-2\ln x) \tag{7}$$

and its lower α critical value equals

$$c_2(0,\alpha) = \exp\left(-\frac{1}{4}\chi^2_{4,\alpha}\right), \tag{8}$$

where $\chi^2_{4,\alpha}$ is the upper α critical point of the χ^2_4 distribution.

Part 6 ($m = 0.5$):
For $m = -0.5$, the c.d.f. of

$$\bar{P}_2(0.5, w) = \left(w_1\sqrt{P_1} + w_2\sqrt{P_2}\right)^2.$$

for equal weights is given by

$$F_2(x, -0.5) = \frac{8x^2}{3}, \tag{9}$$

The lower α critical value equals

$$c_2(0.5,\alpha) = \sqrt{\frac{3\alpha}{8}} \quad \text{for} \quad \alpha \le \frac{1}{6}.$$

Part 7 ($m = 1$):
For $m = 1$ the c.d.f. of the weighted arithmetic mean, assuming $w_1 \le w_2$, is given by

$$F_2(x;1,w) = \begin{cases} \frac{x^2}{2w_1w_2}, & 0 \le x \le w_1, \\ \frac{2x-w_1}{2w_2}, & w_1 \le x \le w_2, \\ 1 - \frac{(1-x)^2}{2w_1w_2}, & w_2 < x \le 1. \end{cases}$$

For equal weights this simplifies to

$$F_2(x,1) = \begin{cases} 2x^2, & 0 \le x \le 1/2, \\ 1-2(1-x)^2, & 1/2 < x \le 1. \end{cases} \tag{10}$$

The lower α critical value for equal weights is given by $c_2(1,\alpha) = \sqrt{\alpha/2}$ if $\alpha \le 1/2$.

Part 8 ($m = 2$):
For $m = 2$, assuming that $w_1 \le w_2$, the c.d.f. of

$$\bar{P}_2(2,w) = (w_1P_1^2 + w_2P_2^2)^{1/2}$$

is given by

$$F_2(x;2,w) = \begin{cases} \frac{\pi x^2}{4\sqrt{w_1w_2}}, & x \le \sqrt{w_1}, \\ \frac{\sqrt{w_1(x^2-w_1)} + x^2\tan^{-1}\left(\sqrt{\frac{w_1}{x^2-w_1}}\right)}{2\sqrt{w_1w_2}}, & \sqrt{w_1} < x \le \sqrt{w_2}, \\ \frac{1}{2}\left(\sqrt{\frac{x^2-w_2}{w_1}} + \sqrt{\frac{x^2-w_1}{w_2}}\right) & \\ \quad + \frac{x^2}{2\sqrt{w_1w_2}}\left(\tan^{-1}\left(\sqrt{\frac{w_1}{x^2-w_2}}\right) - \tan^{-1}\left(\sqrt{\frac{x^2-w_1}{w_1}}\right)\right), & x > \sqrt{w_2}. \end{cases}$$

For equal weights this simplifies to

$$F_2(x;2) = \begin{cases} \frac{\pi x^2}{2} & x \leq \sqrt{1/2}, \\ \sqrt{2x^2-1} + x^2 \tan^{-1}\left(\frac{1-x^2}{\sqrt{2x^2-1}}\right) & x > \sqrt{1/2}. \end{cases} \tag{11}$$

For $\alpha > \pi/4$, there is no closed form solution to the equation $F_2(x;2) = \alpha$. For $\alpha \leq \pi/4$, we have

$$c_2(2,\alpha) = \sqrt{\frac{2\alpha}{\pi}}.$$

Part 9 ($m = \infty$):
As $m \to +\infty$, $\bar{P}_n(m,w) \to P_{\max}$ for any choice of weights. Assuming $P_{\max}$ is unique, its c.d.f. and lower α critical value are given by

$$F_n(x;\infty) = x^n \quad \text{and} \quad c_n(\infty,\alpha) = \alpha^{1/n}. \qquad \square$$

Table 2 summarizes the formulae for finding the critical values $c_2(m,\alpha)$ for the nine selected values of m and $\alpha = 0.05$. From this table we see that the critical value increases with m. This is true in general for any $n \geq 2$ as stated in Theorem 3.

Table 2: Critical values $c_2(m,\alpha)$ for selected $m, n = 2$ and $\alpha = 0.05$

m	Formula for $c_2(m,\alpha)$	$c_2(m,\alpha)$	m	Formula for $c_2(m,\alpha)$	$c_2(m,\alpha)$
$-\infty$	$c_2(-\infty,\alpha) = 1-\sqrt{1-\alpha}$	0.0253	0.5	$c_2(0.5,\alpha) = \sqrt{\frac{3\alpha}{8}}$ if $\alpha \leq 1/6$	0.1369
-2	$c_2(-2,\alpha) = \sqrt{1-\sqrt{1-\alpha}}$	0.0354	1	$c_2(1,\alpha) = \sqrt{\frac{\alpha}{2}}$ if $\alpha \leq 1/2$	0.1581
-1	Solve $x + \frac{x^2}{2}\ln\left[1+\frac{4(1-x)}{x^2}\right] = \alpha$	0.0460	2	$c_2(2,\alpha) = \sqrt{\frac{2\alpha}{\pi}}$ if $\alpha \leq \pi/4$	0.1784
-0.5	Solve Eqn. (6) $= \alpha$	0.0616	∞	$c_2(\infty,\alpha) = \sqrt{\alpha}$	0.2236
0	Solve $x^2(1-2\ln x) = \alpha$	0.0933			

Theorem 3: For any fixed $\alpha \in (0,1)$ and $n \geq 2$ the critical value $c_n(m,\alpha)$ is an increasing function of m. $\square$

4. Exact critical values for $n = 3$

The rejection region for $n = 3$ is defined by

$$\left(\frac{P_1^m + P_2^m + P_3^m}{3}\right)^{1/m} \leq c,$$

where $c \in (0,1)$ is a critical constant depending on α and m. Just as the critical bound for $n = 2$ passes through the points $(1,0)$ and $(0,1)$ in the (P_1,P_2) space, the critical surface for $n = 3$ passes through the points $(1,0,0), (0,1,0)$ and $(0,0,1)$. The corresponding critical (m^*,c^*) thus satisfy $c^* = (1/3)^{1/m^*}$. The type I error probability α for testing $H_0 = H_1 \cap H_2 \cap H_3$ is given by

$$\int_0^{3^{1/m}c} \int_0^{(3c^m - p_3^m)^{1/m}} \int_0^{(3c^m - p_2^m - p_3^m)^{1/m}} dp_1 \; dp_2 \, dp_3 = \alpha.$$

Table 3: Critical values $c_3(m, .05)$ for selected m values and $\alpha = 0.05$

m	$c_3(m, 0.05)$
$-\infty$	0.0170
-2	0.0289
-1	0.0443
-0.5	0.0691
0	0.1226
0.5	0.1839
0.6848	0.2010
1	0.2231
2	0.2639
$+\infty$	0.3684

Setting $3c^m = 1$ for the critical surface, the above equation reduces to

$$\int_0^1 \int_0^{(1-p_3^m)^{1/m}} (1 - p_2^m - p_3^m)\, dp_2\, dp_3 = \alpha.$$

For $\alpha = 0.05$ the above equation can be solved using the MATLAB function `fsolve()` for m resulting in $m^* = 0.6848$ and $c^* = (1/3)^{1/m^*} = 0.2010$. Analogous to the $n = 2$ case, different rejection regions and hence different integral expressions must be evaluated for $m > m^*$ and $m < m^*$. Before we do that for $m = 0$ (geometric mean) we have $-2n \ln(\bar{P}_3(0)) \sim \chi^2_{2n}$ and hence $c_n(\alpha) = \exp(-(1/2n)\chi^2_{2n,\alpha})$. Therefore

$$c_3(0, 0.05) = \exp(-(1/6)\chi^2_{6,.05}) = 0.1226.$$

Omitting the analytical details we give in Table 3 the critical values $c_3(m, 0.05)$ for selected m values. These are used in the type I error rate and power simulations in Section 8.

5. Asymptotic null distribution of the unweighted Hölder mean

The exact null distribution of the Hölder mean is difficult to derive in general for $n > 2$. Hence we resort to asymptotics. The P_i^m are i.i.d. with a beta distribution with parameters $a = 1/m$ and $b = 1$. The mean and variance of this distribution are

$$E\left(P_i^m\right) = \frac{1}{m+1} \quad \text{and} \quad \mathrm{Var}\left(P_i^m\right) = \frac{m^2}{(m+1)(2m+1)}. \tag{12}$$

Note that $\mathrm{Var}\left(P_i^m\right)$ exists (is finite) for $m > -1/2$ and does not exist (is either infinite or negative) for $m \leq -1/2$. So the standard Lindeberg-Lévy central limit theorem (CLT) applies in the former case, but not in the latter in which case $\bar{P}_n(m)$ is not asymptotically normal. Hence we treat the two cases separately.

5.1. The case $m > -1/2$

The case $m = 0$ is covered in Part 5 of Theorem 2 since it does not require asymptotics.

By the CLT,

$$\frac{\left(\frac{1}{n}\sum_{=1}^{n} P_i^m - \frac{1}{m+1}\right)}{\sqrt{\frac{m^2}{n(m+1)(2m+1)}}} \longrightarrow N(0,1)$$

as $n \to \infty$. The lower α critical value for $\frac{1}{n}\sum_{i=1}^{n} P_i^m$ is then given by

$$\frac{1}{m+1} - z_\alpha \sqrt{\frac{m^2}{n(m+1)(2m+1)}}, \tag{13}$$

where z_α is the $100(1-\alpha)$ percentile of the $N(0,1)$ distribution. However, we require the asymptotic critical values of $\bar{P}_n(m) = \left(\frac{1}{n}\sum_{i=1}^{n} P_i^m\right)^{1/m}$. One method (Method 1) is to take the $(1/m)$th power of (13). Another method (Method 2) is to use the delta method to find the mean and variance of $\bar{P}_n(m)$ and apply the CLT approximation to it.

The delta method gives

$$E(\bar{P}_n(m)) = E\left(\frac{1}{n}\sum_{i=1}^{n} P_i^m\right)^{1/m} \approx \left(\frac{1}{m+1}\right)^{1/m} \tag{14}$$

and

$$\text{Var}(\bar{P}_n(m)) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} P_i^m\right)^{1/m} \approx \frac{(m+1)^{1-2/m}}{n(2m+1)}. \tag{15}$$

The derivation of these two formulae is given in the Appendix. The lower α critical value for $\bar{P}_n(m)$ using Method 2 is given by

$$\left(\frac{1}{m+1}\right)^{1/m} - z_\alpha \sqrt{\frac{(m+1)^{1-2/m}}{n(2m+1)}}. \tag{16}$$

The critical values obtained by both these methods are given in Table 4 for selected values of m and n. Which method gives more accurate results depends on m and n.

Table 4: Asymptotic lower $\alpha = 0.05$ critical values of $\bar{P}_n(m)$ for $m > -1/2$

m	Method	n				
		10	20	50	100	1000
−0.25	Method 1	0.1752	0.2065	9.2403	0.2600	0.2970
	Method 2	0.1148	0.1739	0.2263	0.2527	0.2963
0	Exact	0.2079	0.2481	0.2884	0.3104	0.3490
0.5	Method 1	0.2668	0.3142	0.3594	0.3834	0.4247
	Method 2	0.2442	0.3029	0.3549	0.3811	0.4244
1.0	Method 1	0.2877	0.3499	0.4050	0.4329	0.4788
	Method 2	0.2877	0.3499	0.4050	0.4329	0.4788
2.0	Method 1	0.2544	0.3787	0.4618	0.4984	0.5536
	Method 2	0.3447	0.4129	0.4733	0.5038	0.5541

The accuracy of these critical values is evaluated by simulating their associated type I errors in Section 8.3.

5.2. The case $-1 < m \leq -1/2$

Here $X_i = P_i^m$ follows a beta distribution with $a = -1/m$ and $b = 1$. Its p.d.f. and c.d.f. are given by

$$f_{X_i}(x) = \left(-\frac{1}{m}\right) x^{\frac{1}{m}-1} \quad \text{and} \quad F_{X_i}(x) = 1 - x^{\frac{1}{m}} \quad \text{for } x \geq 1. \tag{17}$$

The variance of this distribution is ∞, so the standard Lindeberg-Lévy CLT does not apply and $(1/n)\sum_{i=1}^{n} P_i^m$ is not asymptotically normal. So we apply the generalized CLT (Gnedenko and Kolmogorov, 1954; Ibragimov and Linnik, 1971; Petrov, 1975) stated below.

Theorem 4: Let $X_1, \ldots, X_n$ be i.i.d. random variables with the distribution function $F_X(x)$ satisfying the conditions

$$F_X(x) \sim k_1|x|^{-a^*} \quad \text{as} \quad x \longrightarrow -\infty$$

and

$$1 - F_X(x) \sim k_2|x|^{-a^*} \quad \text{as} \quad x \longrightarrow +\infty$$

with $a^* > 0$. Then there exist sequences $\{\mu_n\}$ and $\{\sigma_n\}$ where $\sigma_n > 0$ such that the distribution of the centered and normalized sum

$$Z_n = \frac{\sum_{i=1}^{n} X_i - \mu_n}{\sigma_n}$$

weakly converges to a stable distribution (denoted by $S(a, b)$) with parameters $a = \min\{a^*, 2\}$ and $b = (k_2 - k_1)/(k_2 + k_1)$ as $n \to +\infty$. The centering and normalizing values μ_n and σ_n depend on the parameters a and b. □

Let $c^*(\alpha)$ denote the upper α critical value of the stable distribution $S(1, 1)$. Then the critical value of the $S(a, b)$ distribution is

$$c(\alpha) = a + bc^*(\alpha).$$

A discussion of the stable distribution and methods of approximating the critical value $c^*(\alpha)$ is given in the Appendix.

The asymptotic critical value of $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} P_i^m$ is $\mu_n + c(\alpha)\sigma_n$. Then the critical value of $\bar{P}_n(m) = [(1/n)\sum_{i=1}^{n} P_i^m]^{1/m}$ can be approximated by making the corresponding transformation as

$$c_n(m, \alpha) = [(1/n)(\mu_n + c(\alpha)\sigma_n)]^{1/m}. \tag{18}$$

Table 6 gives the critical values computed using this method, which we refer to as Method 0. The results regarding the values of μ_n and σ_n used in the three cases discussed below are due to Mijnheer (1975), Samorodnitsky and Taqqu (1994) and Uchaikin and Zolotarev (1999). Some selected values of $c^*(\alpha)$ are given in Table 5.

We now apply Theorem 4 to different cases for values of $m \leq -1/2$.

Table 5: Selected values of $c^*(\alpha)$

α	0.01	0.025	0.05	0.10	0.20
$c^*(\alpha)$	65.9760	27.1899	14.0048	7.1287	3.3843

Case 1 ($m = -1/2$)

From (17) we obtain the p.d.f. and c.d.f. of $X_i = P_i^{-1/2}$ as

$$f_{X_i}(x) = 2x^{-3} \quad \text{and} \quad F_{X_i}(x) = 1 - x^{-2} \quad \text{for } x \geq 1.$$

Therefore $k_1 = 0, k_2 = 1$ and $a^* = 2$. So $a = 2$ and $b = 1$. For these values of a and b it has been shown that (see the previously mentioned references)

$$\mu_n = nE\left(X_i\right) = nE\left(P_i^{-1/2}\right) = 2n \quad \text{and} \quad \sigma_n = \sqrt{n \ln n}.$$

Furthermore, the stable law $S(2,1)$ is simply the $N(0, (\sqrt{2})^2)$ distribution, so

$$\frac{\sum_{i=1}^{n} P_i^{-1/2} - 2n}{\sqrt{2n \ln n}} \quad \longrightarrow N(0, 1^2).$$

Thus the lower α critical value of $\sum_{i=1}^{n} P_i^{-1/2}$ is $2n - z_\alpha\sqrt{2n \ln n}$ from which the lower α critical value of $\bar{P}_n(-1/2)$ can be approximated as

$$c_n\left(-1/2, \alpha\right) = \left[\frac{1}{n}\left\{2n - z_\alpha\sqrt{2n \ln n}\right\}\right]^{-2}.$$

Case 2 ($-1 < m < -1/2$)

From (17) we get $k_1 = 0, k_2 = 1$ and $a^* = 1/m$. Thus we have $a = -1/m$ ($1 < a < 2$) and $b = 1$. For these values of a and b it has been shown that (see the previously mentioned references)

$$\mu_n = nE(X_i) = nE\left(P_i^m\right) = \frac{na}{a-1} \quad \text{and} \quad \sigma_n = \left(\frac{n\pi}{2\Gamma(a)\sin(a\pi/2)}\right)^{1/a}.$$

Therefore

$$\frac{\sum_{i=1}^{n} P_i^m - \frac{na}{a-1}}{\left(\frac{n\pi}{2\Gamma(a)\sin(a\pi/2)}\right)^{1/a}} \quad \longrightarrow S(a, b).$$

The asymptotic lower α critical value of $S(a, b)$ is

$$c(\alpha) = -\frac{1}{m} + c^*(\alpha).$$

Hence the approximate lower α critical value of $\bar{P}_n(m)$ is

$$c_n(m, \alpha) = \left[\frac{1}{n}\left\{\frac{na}{a-1} - c(\alpha)\left(\frac{n\pi}{2\Gamma(a)\sin(a\pi/2)}\right)^{1/a}\right\}\right]^{1/m}.$$

Case 3 ($m \leq -1$) The case $m = -1$ corresponds to the harmonic mean and is discussed in detail in Gou and Tamhane (2024). So we consider only the case $m < -1$. From (17) we get $k_1 = 0, k_2 = 1$ and $a^* = 1/m$. Thus we have $a = -\frac{1}{m}$ and $b = 1$ where $0 < a < 1$. For these values of a and b it has been shown that (see the previously mentioned references)

$$\mu_n = 0 \quad \text{and} \quad \sigma_n = \left(\frac{n\pi}{2\Gamma(a)\sin(a\pi/2)} \right)^{1/a}.$$

Therefore

$$\frac{\sum_{i=1}^{n} P_i^m}{\left(\frac{n\pi}{2\Gamma(a)\sin(a\pi/2)}\right)^{1/a}} \longrightarrow S(a,b).$$

Since a and b are the same as in Case 2, $c(\alpha)$ is also the same. Hence the approximate lower α critical value of $\bar{P}_n(m)$ is

$$\left[\frac{1}{n} \left\{ -c(\alpha) \left(\frac{n\pi}{2\Gamma(a)\sin(a\pi/2)} \right)^{1/a} \right\} \right]^{1/m}.$$

Table 6: Asymptomatic lower $\alpha = 0.05$ critical values of $\bar{P}_n(m)$ for $m \leq -1/2$ using Method 0

m	n				
	10	20	50	100	1000
−0.5	0.1030	0.1189	0.1423	0.1601	0.2079
−1.0	0.0412	0.0400	0.0386	0.0376	0.0346
−2.0	0.0159	0.0112	0.0071	0.0050	0.0016
−3.0	0.0110	0.0069	0.0037	0.0024	0.0005

The accuracy of these critical values is evaluated by simulating their associated type I errors in Section 8.3.

6. A closed multiple testing procedure (MTP)

Our testing strategy will be to use the closure method (Marcus *et al.*, 1976) based on $\bar{P}_n(m)$ with a preselected m as the test statistic. The closure method begins by testing the overall null hypothesis $H_0 = \cap_{i=1}^{n} H_i$ at level α. If H_0 is rejected then it tests all subset null hypotheses of size $n-1$ each at level α. If any subset null hypothesis is not rejected then all its subsets are accepted by implication. This ensures coherence (Gabriel, 1969). On the other hand, if any subset null hypothesis of size $n' \leq n$ is rejected then all its subsets of size $n'-1$ that are not already accepted by implication are tested each at level α.

This procedure does not have a simple stepwise shortcut like the Holm and the Hochberg procedures have. However, these computations can be substantially reduced as follows. When testing all subsets of size $n' \leq n$, first test the subset with the largest P-values. If it is significant then all other subsets of size n' will also be significant and need not be tested. Otherwise test the subset with the smallest P-values. If it is nonsignificant then all other subsets of size n' will also be nonsignificant and need not be tested. This method is illustrated in the numerical example in Section 7. A simple `R` code can be used to compute the Hölder means.

Dobriban (2020) has given an alternative shortcut which he called fast closed testing (FACT) algorithm. It is particularly efficient when n is large. He showed that when the hypotheses are exchangeable, we don't need to test all $2^n - 1$ intersection hypotheses, but only $n(n+1)/2$ of them. For example, if $n = 5$ then instead of testing all $2^5 - 1 = 31$ subsets, we only need to test $5(5+1)/2 = 15$ of them, a saving of 50%. As n grows larger, obviously saving increases. Here we don't use this algorithm as it would require much explanation.

7. Numerical example

Consider a dose response study in which $n = 5$ doses are tested for efficacy, labeled from the highest to the lowest as 1 through 5. Suppose that the P-values for the comparisons with placebo (zero dose) are as follows:

$$P_1 = 0.01,\ P_2 = 0.02,\ P_3 = 0.03,\ P_4 = 0.04,\ P_5 = 0.30.$$

Denote the corresponding hypotheses by $H_1, \ldots, H_5$. Because of space constraints we will only briefly illustrate three MTPs: harmonic mean MTP (denoted by HMP), geometric mean MTP (denoted by GMP) and arithmetic mean MTP (denoted by AMP). We will use $\alpha = 0.05$.

Harmonic Mean Procedure (HMP): The critical values for HMP are

$$c_1 = 0.0500,\ c_2 = 0.0460,\ c_3 = 0.0443,\ c_4 = 0.0433,\ c_5 = 0.0425.$$

Step 1: Test the whole set $\{1, 2, 3, 4, 5\}$. The harmonic mean for this set is $0.0236 < c_5 = 0.0425$, so we reject it.

Step 2: Test the subset $\{2, 3, 4, 5\}$ of size 4 with the largest P-values. The harmonic mean for this subset is $0.0358 < c_4 = 0.0433$, so we reject it.

Step 3: Test the subset $\{3, 4, 5\}$ of size 3 with the largest P-values. The harmonic mean for this subset is $0.0486 > c_3 = 0.0443$. Therefore we accept intersection hypotheses associated with all subsets of $\{3, 4, 5\}$. The next largest harmonic mean is associated with the subset $\{2, 4, 5\}$ and is $0.0383 < c_3 = 0.0443$, which is thus rejected and hence all other subsets of size 3 are rejected.

Step 4: Test only those subsets of size 2 that include 1 or 2 or both. The subset $\{2, 5\}$ has the largest harmonic mean $0.0375 < c_2 = 0.0460$ and hence the subsets $\{1, 5\}$ is also rejected.

Step 5: Test only $\{1\}$ and $\{2\}$. Since P_1 and P_2 are $< c_1 = 0.05$, both H_1 and H_2 are rejected.

Thus HMP rejects two hypotheses, H_1 and H_2.

Having explained how HMP operates, we will present the application of GMP and AMP rather briefly, since they operate similarly.

Geometric Mean Procedure (GMP): The critical values for GMP are

$$c_1 = 0.0500,\ c_2 = 0.0933,\ c_3 = 0.1226,\ c_4 = 0.1439,\ c_5 = 0.1603.$$

At the first step we get $\bar{P}_5(0, \{1, 2, 3, 4, 5\}) = 0.0373 < c_5 = 0.1603$, so reject $\{1, 2, 3, 4, 5\}$. Next $\bar{P}_4(0, \{2, 3, 4, 5\}) = 0.0518 < c_4 = 0.1439$, so reject $\{2, 3, 4, 5\}$. Next $\bar{P}_3(0, \{3, 4, 5\}) = 0.0711 < c_3 = 0.1226$, so reject $\{3, 4, 5\}$. Next $\bar{P}_2(0, \{4, 5\}) = 0.1095 > c_2 = 0.0933$ and $\bar{P}_2(0, \{3, 5\}) = 0.0949 > c_2 = 0.0933$, so these subsets are accepted while the subset $\{1, 2\}$ is rejected since $\bar{P}_2(0, \{1, 2\}) = 0.0141 < c_2 = 0.0933$. Finally since P_1 and P_2 are $< c_1 = 0.05$, both H_1 and H_2 are rejected.

Arithmetic Mean Procedure (AMP): The critical values for AMP are

$$c_1 = 0.0500,\ c_2 = 0.1581,\ c_3 = 0.2231,\ c_4 = 0.2617,\ c_5 = 0.2869.$$

At the first step we get $\bar{P}_5(1, \{1, 2, 3, 4, 5\}) = 0.0800 < c_5 = 0.2869$, so reject $\{1, 2, 3, 4, 5\}$. Next $\bar{P}_4(1, \{2, 3, 4, 5\}) = 0.0975 < c_4 = 0.2617$, so reject $\{2, 3, 4, 5\}$. Next $\bar{P}_3(1, \{3, 4, 5\}) = 0.1233 < c_3 = 0.2231$, so reject $\{3, 4, 5\}$. Next $\bar{P}_2(1, \{2, 5\}) = 0.1600, \bar{P}_2(1, \{3, 5\}) = 0.1605, \bar{P}_2(1, \{4, 5\}) = 0.1700$ are all $> c_2 = 0.1581$ and so are not rejected while all other pairs of hypotheses are rejected including $\{1, 5\}$ for which $\bar{P}_2(1, \{1, 5\}) = 0.1550$. So only H_1 remains to be tested and since $P_1 = 0.01 < c_1 = 0.05$, it is rejected. Thus AMP only rejects H_1.

8. Type I error and power simulations

8.1. Power simulations for $n = 2$

To save space, we report only the power of the test of $H_0 = H_1 \cap H_2$ for $n = 2$. Note that if the closed test procedure is consonant (Gabriel, 1969)), *i.e.*, if it rejects H_0 then it also rejects at least one of H_1 or H_2. Therefore the power of the test of H_0 is also the power of the corresponding closed MTP. It is easy to show that the closed MTP given above is not consonant for $n = 2$ if $c_2(m, \alpha) > \alpha$. In that case it is possible to have $P_1, P_2 > \alpha$ but $\bar{P}_2(m) < c_2(m, \alpha)$. So H_0 is rejected but neither H_1 nor H_2. For example, consider $m = 1$ (arithmetic mean). Let $P_1 = P_2 = 0.15$. Then $\bar{P}_2(1) = 0.15 < c_2(1, 0.05) = 0.1581$, but $P_1 = P_2 = 0.15 > c_1(1, 0.05) = 0.05$. From Table 2 we see that MTPs are consonant if $m \leq -1$ for $\alpha = 0.05$.

The power comparison setup is as follows. Let $X_1 \sim N(\mu_1, 1)$ and $X_2 \sim N(\mu_2, 1)$ with $\mathrm{Corr}(X_1, X_2) = \rho \geq 0$. Further let $P_1 = 1 - \Phi(X_1)$ and $P_2 = 1 - \Phi(X_2)$. Under the alternative hypothesis ($\mu_1 \neq 0$ or $\mu_2 \neq 0$) the power can be expressed as a bivariate normal integral for all m. So it can be evaluated using numerical integration and does not need to be simulated. The integral expressions for power are omitted for brevity. The power is evaluated for $m = -\infty, -2, -1, -1/2, 0, 1/2, 1, 2, +\infty$ and for six configurations of (μ_1, μ_2) either $\mu_1 = 0$ and $\mu_2 = 1, 2, 3$ or $\mu_1 = \mu_2 = 1, 2, 3$. The power results for $\rho = 0$ are given in Table 7.

We also conducted power comparisons for $\rho = -0.5$ and $\rho = +0.5$, but we don't show them in Table 7. Furthermore, we also evaluated the Pr(Type I Error) under the overall null hypothesis $\mu_1 = \mu_2 = 0$ for $\rho = 0, -0.5$ and $+0.5$. This probability is 0.05 under $\rho = 0$ by design and is confirmed by simulation and hence is not shown in Table 7. We see that for $\rho = -0.5$ the Pr(Type I Error) is slightly liberal for $m = -2$ and $-\infty$ while it is conservative for other values pf m. On the other hand, for $\rho = +0.5$ the Pr(Type I Error) is slightly conservative for $m = -2$ and $-\infty$ while it is quite liberal for other values of m.

Table 7: Power for rejecting $H_0 = H_1 \cap H_2$ for selected values of m, (μ_1, μ_2) and $\alpha = 0.05$

m	P(Type I Error)		Power					
	(μ_1, μ_2)							
	$(0,0)$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(1,1)$	$(2,2)$	$(3,3)$
	$\rho = -0.5$	$\rho = +0.5$	$\rho = 0$					
$-\infty$	0.0506	0.0459	0.1909	0.5303	0.8559	0.3110	0.7678	0.9781
-2	0.0508	0.0478	0.1915	0.5311	0.8562	0.3159	0.7783	0.9809
-1	0.0478	0.0507	0.1928	0.5320	0.8561	0.3283	0.7982	0.9849
-0.5	0.0424	0.0572	0.1945	0.5309	0.8538	0.3487	0.8245	0.9890
0	0.0260	0.0736	0.1925	0.5086	0.8307	0.3886	0.8646	0.9937
0.5	0.0094	0.0957	0.1601	0.3182	0.4502	0.4024	0.8729	0.9939
1	0.0102	0.1005	0.1464	0.2481	0.3000	0.3838	0.8425	0.9867
2	0.0111	0.1021	0.1389	0.2175	0.2471	0.3679	0.8167	0.9800
∞	0.0124	0.1024	0.1996	0.2208	0.8559	0.3538	0.7966	0.9751

Figure 2 shows the plots of power with left panel showing the plots when H_1 is true and H_2 is false ($\mu_1 = 0, \mu_2 = 1$, 2 or 3) and right panel showing the plots when both H_1 and H_2 are equally false ($\mu_1 = \mu_2 = 1$, 2 or 3).

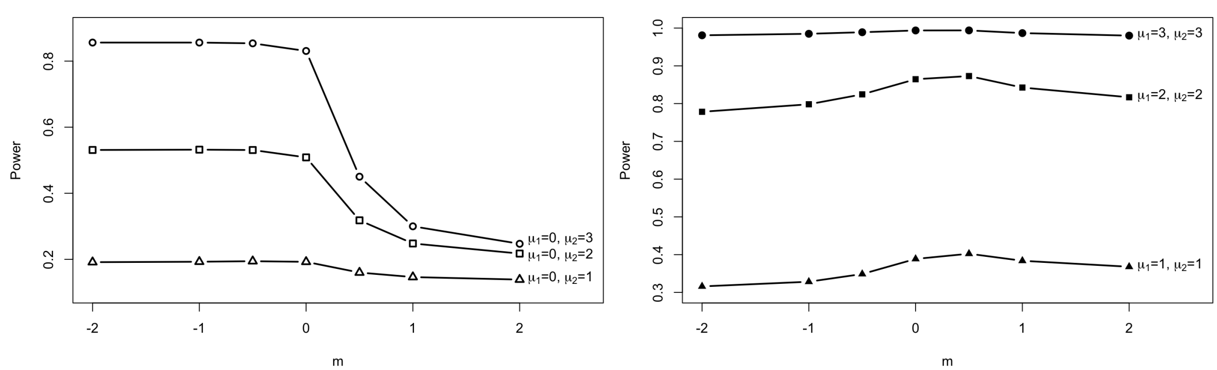

Figure 2: Plots of power for rejecting $H_0 = H_1 \cap H_2$ using different m (left panel: $\mu_1 = 0, \mu_2 = 1, 2, 3$, right panel: $\mu_1 = \mu_2 = 1, 2, 3$)

The Pr(Type I Error) is fairly well controlled when $\rho = -0.5$, but not when $\rho = +0.5$. The power results show that the maximum (or nearly maximum) power is achieved close to $m = -1$ (harmonic mean) when one hypothesis is true and the other is false. When both hypotheses are equally false, $m = 0.5$ yields the maximum power. The plots are fairly flat in the vicinity of the maximum, so any value of m close to the true optimum would work nearly equally well.

8.2. Type I error and power simulations for $n = 3$

Using the critical values obtained in Section 4 we evaluated the type I error of rejecting $H_0 = H_1 \cap H_2 \cap H_3$ (which is also the FWER of any consonant closed MTP) under independence and positive and negative dependence. The P-values are generated in the same manner as for $n = 2$ by simulating equicorrelated trivariate normal random variables with zero means and common correlation $\rho = 0$ for independence, $\rho = 0.5$ for positive dependence and $\rho = -0.25$ for negative dependence and transforming them to P-values. The number of replications were 10^6. The simulation results for type I error are presented in Table 8. Notice that the type I error rate is controlled accurately under independence for all m and conservatively for $\rho = -0.25$ when $m \geq -1.0$; however, it is not controlled for $\rho = 0.5$ when $m \geq -1.0$.

Table 8: Simulated type I error for rejecting $H_0 = H_1 \cap H_2 \cap H_3$ under independence ($\rho = 0$), positive dependence ($\rho = 0.5$) and negative dependence ($\rho = -0.25$)

m	$\rho = 0$	$\rho = 0.5$	$\rho = -0.25$
$-\infty$	0.0501	0.0435	0.0506
−2.0	0.0501	0.0454	0.0500
−1.0	0.0500	0.0512	0.0477
−0.5	0.0499	0.0636	0.0416
0.0	0.0495	0.0941	0.0231
0.5	0.0497	0.1270	0.0102
1.0	0.0496	0.1401	0.0106
2.0	0.0497	0.1447	0.0126
$+\infty$	0.0496	0.1438	0.0152

Next we consider power for rejecting $H_0 = H_1 \cap H_2 \cap H_3$. We considered three different configurations: $(\mu_1, \mu_2, \mu_3) = (0, 0, \delta), (0, \delta, \delta)$ and (δ, δ, δ) where $\delta = 2$. The simulated powers for different m are summarized in Table 9.

Table 9: Simulated powers for rejecting $H_0 = H_1 \cap H_2 \cap H_3$ under independence for different m ($\rho = -0.25$)

m	(μ_1, μ_2, μ_3)		
	(0,0,2)	(0,2,2)	(2,2,2)
$-\infty$	0.4709	0.7043	0.8354
−2.0	0.4720	0.7146	0.8507
−1.0	0.4734	0.7362	0.8798
−0.5	0.4717	0.7656	0.9147
0.0	0.4324	0.8008	0.9537
0.5	0.2529	0.7307	0.9626
1.0	0.1829	0.5161	0.9414
2.0	0.1496	0.3994	0.9023
$+\infty$.	0.1291	0.3342	0.8633

First we note that as in the case of $n = 2$, maximum power is achieved at $m = -1$ (harmonic mean) when only one H_i is false, with optimum m increasing as more hypotheses

Table 10: Simulated type I error for rejecting $H_0 = \cap_{i=1}^n H_i$ for $n \geq 10$ using asymptotic approximations to critical values Using Method 1 and Method 2 When $\alpha = 0.05$

m	Method*	n				
		10	20	50	100	1000
-2	Method 0	0.0498	0.0500	0.0502	0.0501	0.0499
-0.5	Method 0	0.0543	0.0499	0.0455	0.0430	0.0362
-0.25	Method 1	0.0784	0.0803	0.0819	0.0818	0.0797
	Method 2	0.0208	0.0365	0.0515	0.0591	0.0719
0.25	Method 1	0.0414	0.0389	0.0370	0.0354	0.0334
	Method 2	0.0168	0.0219	0.0260	0.0280	0.0311
0.5	Method 1	0.0261	0.0248	0.0239	0.0233	0.0222
	Method 2	0.0132	0.0161	0.0184	0.0194	0.0210
1	Method 1	0.0092	0.0095	0.0098	0.0098	0.0100
	Method 2	0.0092	0.0095	0.0098	0.0098	0.0100
2	Method 1	0.0003	0.0009	0.0016	0.0016	0.0021
	Method 2	0.0059	0.0043	0.0034	0.0030	0.0025

* Method 0 uses the generalized central limit theorem (GCLT); see Section 5.2. Methods 1 and 2 use the central limit theorem (CLT); see Section 5.1.

become false: optimum $m = -0.5$ when two hypotheses are false and optimum $m = 0$ (geometric mean) when all three hypotheses are false. The power first increases with m and then decreases rapidly as m approaches $+\infty$.

8.3. Type I error simulations for $n \geq 10$

To check the accuracy of the asymptotic approximations to the critical values computed using Method 1 and Method 2 in Tables 4 and 6 we performed simulations of type I error for rejecting the overall null hypothesis $H_0 = \cap_{i=1}^n H_i$. The results are reported in Table 10. These results show that the asymptotic approximations are not very accurate and better approximations need to be found. Method 2 gives generally conservative approximations (estimated type I error rate is $< \alpha = 0.05$) except for $m = -0.25$ and $n \geq 50$, while Method 1 gives anti-conservative approximations for all values of n when $m = -0.25$; otherwise it is conservative. Generally, Method 2 is more conservative than Method 1.

9. Concluding remarks and practical recommendations

In this paper we have exhaustively studied the null distribution of the Hölder mean with the exact distribution for $n = 2$ and the asymptotic distribution for large n. We have also obtained the exact critical values for $n = 3$. The exact null distribution in closed form is also available for all $n > 2$ in special cases, *e.g.*, minimum, maximum and geometric mean and can be obtained using the convolution method in other cases, in particular, the harmonic mean and arithmetic mean. The asymptotic approximations to critical values are generally too conservative and better approximations need to be found.

These null distributions and their critical values are employed in a closed MTP. The power of the test of $H_0 = \cap_{i=1}^n$ for $n = 2$ and 3 for different values of m is evaluated for six different configurations of (μ_1, μ_2) and three different configurations of (μ_1, μ_2, μ_3) and optimum chpoces of m are found. The power comparisons show that for $n = 2$ if only one null hypothesis is false ($\mu_1 = 0, \mu_2 > 0$) then the test based on the harmonic mean ($m = -1$) gives the maximum power and if both null hypotheses are equally false ($\mu_1 = \mu_2 > 0$) then the test based on the Hölder mean with $m = -0.5$ gives the maximum power. Similarly, for $n = 3$, if only one null hypothesis is false. Similarly, maximum power is achieved at $m = -1$ (harmonic mean) when only one H_i is false, with optimum m increasing as more hypotheses become false: optimum $m = -0.5$ when two hypotheses are false and optimum $m = 0$ (geometric mean) when all three hypotheses are false. Since the power plots in the vicinity of the maximum power are fairly flat, our practical recommendation is to use $m = -0.5$ or$m = -1$.

In this paper the power comparisons are limited to the test of H_0 for $n = 2$ and 3. Power comparisons are not made for $n \geq 10$ because the asymptotic critical values are too conservative.

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ANNEXURE

Appendix: Proofs and Derivations

Proof of Theorem 1 As seen from Figure 3, for $m < m^*$, the rejection boundary is convex while for $m > m^*$, the rejection boundary is concave. The corresponding rejection regions are below the rejection boundaries. Therefore the regions of integration are different for evaluating the integral below:

$$F_2(x;m) = \Pr\left\{\frac{P_1^m + P_2^m}{2} \le x^m\right\} = \int (2x^m - y^m)dy. \tag{19}$$

The two regions are

$$R_1 = \{0 \le P_1 \le (2x^m - P_2^m)^{1/m}, 0 \le P_2 \le 1\} \quad (m \le m^*).$$

and

$$R_2 = \{0 \le P_1 \le 2^{1/m}x, 0 \le P_2 \le (2x^m - P_1^m)^{1/m} \quad (m > m^*)$$

<u>The Case $(m \le m^*)$</u>: By integrating (19) over the region R_1, we get

$$\begin{aligned}
&F_2(x;m)\\
&= \int_0^{(2x^m-1)^{1/m}} (2x^m - y^m)^{1/m} dy\\
&= \int_0^{(2x^m-1)^{1/m}} dy + \int_{(2x^m-1)^{1/m}}^1 (2x^m - y^m)^{1/m} dy\\
&= (2x^m-1)^{1/m} + (2^{1/m}x)^2 \int_{(1-1/2x^m)^{1/m}}^{1/2^{1/m}x} (1-u^m)^{1/m} du \ \text{ (by putting } \ u = y/(2^{1/m}x)).
\end{aligned}$$

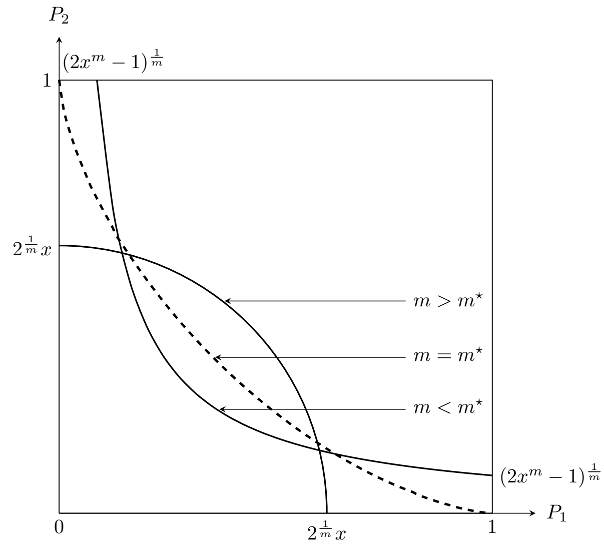

Figure 3: Rejection regions R_1 for $m \le m^*$ and R_2 for $m > m^*$ which are below the respective boundaries

Hence

$$F_2(x;m) = (2x^m-1)^{1/m} + 4^{1/m}x^2\left[\int_0^{1/2^{1/m}x}(1-u^m)^{1/m}du - \int_0^{1-1/2^{1/m}x}(1-u^m)^{1/m}du\right].$$

Now put $u^m = v$. Hence $du = (1/m)v^{1/m-1}dv$. Thus we get

$$\begin{aligned} F_2(x;m) &= (2x^m-1)^{\frac{1}{m}} + \frac{4^{\frac{1}{m}}x^2}{m}\left[\int_0^{1/2x^m} v^{1/m-1}(1-v)^{1/m}dv - \int_0^{1-1/2x^m} v^{1/m-1}(1-v)^{1/m}dv\right] \\ &= (2x^m-1)^{1/m} + \frac{4^{1/m}x^2}{m}\left[B_{1/2x^m}\left(\frac{1}{m},\frac{1}{m}+1\right) - B_{1-1/2x^m}\left(\frac{1}{m},\frac{1}{m}+1\right)\right]. \end{aligned}$$

<u>The Case $m > m^*$</u>: By integrating (19) over the region R_2, we get

$$\begin{aligned} F_2(x;m) &= \int_0^{2^{1/m}x}(2x^m-y^m)^{1/m}dy \\ &= 4^{1/m}x^2\int_0^1(1-u^m)^{1/m}du \quad \text{(by putting } u = \tfrac{y}{2^{1/m}x}\text{)}. \end{aligned}$$

Now put $u^m = v$. Hence $du = (1/m)v^{1/m-1}dv$. Thus we get

$$\begin{aligned} F_2(x;m) &= \frac{4^{1/m}x^2}{m}\int_0^1 v^{1/m-1}(1-v)^{1/m}dv \\ &= \frac{4^{1/m}x^2}{m}B_1\left(\frac{1}{m},\frac{1}{m}+1\right) \\ &= \frac{4^{1/m}x^2}{m}\frac{\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{1}{m}+1\right)}{\Gamma\left(\frac{2}{m}+1\right)}. \end{aligned}$$

assuming $2^{\frac{1}{m}}x \leq 1$ that is equivalent to $x \leq 2^{-1/m}$.

Proof of Theorem 2:

Part 1 ($m = -\infty$):

Assume that $P_1 = P_{\min}$ is unique. Then

$$\begin{aligned}\left(\sum_{i=1}^{n} w_i P_i^m\right)^{1/m} &= P_1 \left(w_1 + \sum_{i=2}^{n} w_i \left(\frac{P_i}{P_1}\right)^m\right)^{1/m} \\ &\to P_1 \text{ as } m \to -\infty \text{ since } w_1^{1/m} \to 1 \text{ and } \left(\frac{P_i}{P_1}\right)^m \to 0 \ \ \forall i > 2.\end{aligned}$$

The c.d.f. of $P_1 = P_{\min}$ is

$$F_n(x; -\infty, w) = \Pr\{P_{\min} \leq x\} = 1 - \prod_{i=1}^{n} \Pr\{P_i > x\} = 1 - (1-x)^n.$$

Equating this to α and solving for x, we get

$$c_n(-\infty, \alpha) = 1 - (1-\alpha)^{1/n}$$

Part 2 ($m = -2$):

We have

$$\begin{aligned}F_2(x; -2, w) &= \Pr\left\{\left(\frac{w_1}{P_1^2} + \frac{w_2}{P_2^2}\right)^{-1/2} \leq x\right\} \\ &= \Pr\left\{P_2 \leq \sqrt{\frac{w_2 x^2 P_1^2}{P_1^2 - w_1 x^2}}\right\}.\end{aligned}$$

Now note that if $P_1 \leq \sqrt{\frac{w_1 x^2}{1 - w_2 x^2}}$ then $P_2 \leq 1$. Hence the above probability equals

$$\begin{aligned}F_2(x; -2, w) &= \Pr\left\{P_1 \leq \sqrt{\frac{w_1 x^2}{1 - w_2 x^2}}, P_2 \leq 1\right\} + \Pr\left\{P_1 > \sqrt{\frac{w_1 x^2}{1 - w_2 x^2}}, P_2 \leq \sqrt{\frac{w_2 x^2 P_1^2}{P_1^2 - w_1 x^2}}\right\} \\ &= \sqrt{\frac{w_1 x^2}{1 - w_2 x^2}} + \int_{\sqrt{\frac{w_1 x^2}{1 - w_2 x^2}}}^{1} \sqrt{\frac{w_2 x^2 y^2}{y^2 - w_1 x^2}} dy \\ &= \sqrt{\frac{w_1 x^2}{1 - w_2 x^2}} + \left.\frac{\sqrt{w_2 x^2 y^2 (y^2 - w_1 x^2)}}{y}\right|_{\sqrt{\frac{w_1 x^2}{1 - w_2 x^2}}}^{1} \\ &= \sqrt{\frac{w_1 x^2}{1 - w_2 x^2}} + x\sqrt{w_2(1 - w_1 x^2)} - \sqrt{\frac{w_1 x^2}{1 - w_2 x^2}} + x\sqrt{w_1(1 - w_2 x^2)} \\ &= x\left(\sqrt{w_1(1 - w_2 x^2)} + \sqrt{w_2(1 - w_1 x^2)}\right).\end{aligned}$$

Part 3 ($m = -1$):

This case corresponds to *weighted harmonic mean.* Its c.d.f. is derived in the following.

$$
\begin{aligned}
F_2(x;-1,w) &= \Pr\left(\left(\frac{w_1}{P_1}+\frac{w_2}{P_2}\right)^{-1} \le x\right)\\
&= \Pr\left\{P_2 \le \frac{w_2 x P_1}{P_1 - w_1 x}\right\}\\
&= \Pr\left\{P_1 \le \frac{w_1 x}{1-w_2 x}, P_2 \le 1\right\} + \Pr\left\{P_1 > \frac{w_1 x}{1-w_2 x}, P_2 \le \frac{w_2 x P_1}{P_1 - w_1 x}\right\}\\
&= \frac{w_1 x}{1-w_2 x} + \int_{\frac{w_1 x}{1-w_2 x}}^{1} \frac{w_2 x y}{y - w_1 x} dy\\
&= \frac{w_1 x}{1-w_2 x} + \left\{w_2 x y + w_1 w_2 x^2 \ln\left|\frac{y}{w_2 x} - \frac{w_1}{w_2}\right|\right\}\Big|_{\frac{w_1 x}{1-w_2 x}}^{1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{w_1 x}{1-w_2 x} + \frac{w_2 x(1-x)}{1-w_2 x} + w_1 w_2 x^2 \ln\left[\frac{(1-w_1 x)(1-w_2 x)}{w_1 w_2 x^2}\right]\\
&= x + w_1 w_2 x^2 \ln\left[\frac{(1-w_1 x)(1-w_2 x)}{w_1 w_2 x^2}\right]\\
&= x + w_1 w_2 x^2 \ln\left[1 + \frac{1-x}{w_1 w_2 x^2}\right].
\end{aligned}
$$

In Step 5 above we have used the standard formula from Rektorys (1969): For $a \neq b \neq 0$,

$$\int \frac{y dy}{ay+b} = \frac{y}{a} - \frac{b}{a^2}\ln|ay+b|.$$

Part 4 ($m = -0.5$):

$$
\begin{aligned}
& F_2(x;-0.5)\\
=\ & \frac{1}{\left(\frac{2}{\sqrt{x}}-1\right)^2} + \int_{\frac{1}{\left(\frac{2}{\sqrt{x}}-1\right)^2}}^{1} \frac{1}{\left(\frac{2}{\sqrt{x}}-\frac{1}{\sqrt{y}}\right)^2} dy\\
=\ & \frac{1}{\left(\frac{2}{\sqrt{x}}-1\right)^2} + \frac{x}{8}\cdot\left(x^{3/2}/(\sqrt{x}-2\sqrt{y}) + 4\sqrt{xy} + 3x\ln(2\sqrt{y}-\sqrt{x}) + 2y\right)\Big|_{\frac{1}{\left(\frac{2}{\sqrt{x}}-1\right)^2}}^{1}\\
=\ & \frac{x}{(2-\sqrt{x})^2} + \frac{x}{8}\left(-\frac{x^{3/2}}{2-\sqrt{x}} + 4\sqrt{x} + 3x\ln(2-\sqrt{x}) + 2\right.\\
& \left. -x + 2\sqrt{x} - \frac{4x}{2-\sqrt{x}} - 3x\ln\left(\frac{x}{2-\sqrt{x}}\right) - \frac{2x}{(2-\sqrt{x})^2}\right)\\
=\ & \frac{x}{(2-\sqrt{x})^2} + \frac{x}{8}\left(6\sqrt{x} - x - \frac{x(4+\sqrt{x})}{2-\sqrt{x}} + 3x\ln\left(\frac{(2-\sqrt{x})^2}{x}\right) + 2 - \frac{2x}{(2-\sqrt{x})^2}\right).
\end{aligned}
$$

The lower α critical value is obtained by solving the equation obtained by setting the above expression equal to α.

Part 5 ($m = 0$):

We have

$$F_2(x; 0, w) = \Pr\left(P_1^{w_1} P_2^{w_2} \leq x\right).$$

Now note that if $P_1 \leq x^{1/w_1}$ then $P_2 \leq 1$. Therefore

$$\begin{aligned}
F_2(x; 0, w) &= \Pr\{P_1 \leq x^{1/w_1}, P_2 \leq 1\} + \Pr\left\{P_1 > x^{1/w_1}, P_2 \leq \frac{x^{1/w_2}}{P_1^{w_1/w_2}}.\right\} \\
&= x^{1/w_1} + \int_{x^{1/w_1}}^{1} \frac{x^{1/w_2}}{y^{w_1/w_2}} dy \\
&= x^{1/w_1} + x^{1/w_2} \frac{w_1}{w_2 - w_1}\left[y^{\frac{w_2 - w_1}{w_2}}\right]_{x^{1/w_1}}^{1} \\
&= x^{1/w_1} + x^{1/w_2} \frac{w_1}{w_2 - w_1}\left[1 - x^{\frac{w_2 - w_1}{w_1 w_2}}\right] \\
&= \begin{cases} x^2(1 - 2\ln x), & w_1 = w_2 = 1/2 \\ \frac{1}{1 - \frac{w_2}{w_1}} x^{\frac{1}{w_1}} + \frac{1}{1 - \frac{w_1}{w_2}} x^{\frac{1}{w_2}} & w_1 \neq w_2. \end{cases}
\end{aligned}$$

For an alternative proof of the case $w_1 = w_2 = 1/2$, note that for any $n \geq 2$,

$$\begin{aligned}
F_2(x; 0, w) &= \Pr\left\{\left(\prod_{i=1}^{n} P_i\right)^{1/n} \leq x\right\} \\
&= \Pr\left\{-\frac{2}{n}\sum_{i=1}^{n} \ln P_i > -2\ln x\right\} \\
&= \Pr\left\{-2\sum_{i=1}^{n} \ln P_i > -2n\ln x\right\} \\
&= \Pr\left\{\chi^2_{2n} > -2n\ln x\right\}.
\end{aligned}$$

Now consider $n = 2$. Then by putting $u = t/2$ in the integral below we get

$$F_n(x; 0, w) = \int_{-4\ln x}^{\infty} \frac{1}{2^2\Gamma(2)} t e^{-t/2} dt = \int_{-2\ln x}^{\infty} u e^{-u} du = x^2(1 - 2\ln x).$$

Part 6 ($m = 0.5$):

We have

$$\begin{aligned}
F_2(x, 0.5) &= \int_0^{2^2 x} (2\sqrt{x} - \sqrt{y})^2 dy \\
&= -\frac{8}{3}\sqrt{xy^3} + 4xy + y^2/2\Big|_0^{4x} \\
&= \frac{8x^2}{3}.
\end{aligned}$$

Equating $8x^2/3 = \alpha$ we get the lower α critical value as $c_2(0.5, \alpha) = \sqrt{3\alpha/8}$. These expressions for the c.d.f. and the α critical value are valid for all α less than or equal to

$$\int_0^1 (1 - x^{1/2})^2 dx = \frac{x^2}{2} - \frac{4x^{3/2}}{3} + x \Big|_0^1 = \frac{1}{6}.$$

Part 7 ($m = 1$):

Assuming $w_1 \leq w_2$, the c.d.f. $F_2(x; 1, w)$ is given by the areas of the regions in the (P_1, P_2) space as follows.

1. ($0 \leq x \leq w_1$): In this case the region of interest is the triangle shown in Figure 4 (a). Its area equals
$$F_2(x; 1, w) = \frac{1}{2}\left(\frac{x}{w_1} \times \frac{x}{w_2}\right) = \frac{x^2}{2w_1 w_2}.$$
2. ($w_1 < x \leq w_2$): In this case the region of interest is the quadrilateral shown in Figure 4 (b). Its area equals
$$F_2(x; 1, w) = \frac{1}{2}\left(\frac{x}{w_2} + \frac{x - w_1}{w_2}\right) = \frac{2x - w_1}{2w_2}.$$
3. ($w_2 < x \leq 1$): In this case the region of interest is the trapezoid shown in Figure 4 (c). Its area equals
$$\begin{aligned} F_2(x; 1, w) &= 1 - \frac{1}{2}\left(1 - \frac{x - w_1}{w_2}\right)\left(1 - \frac{x - w_2}{w_1}\right) \\ &= 1 - \frac{1}{2}\left(\frac{w_2 - x + w_1}{w_2}\right)\left(\frac{w_1 - x + w_2}{w_1}\right) \\ &= 1 - \frac{(1 - x)^2}{2w_1 w_2}. \end{aligned}$$

Part 8 ($m = 2$):

Assume $w_1 \leq w_2$. We consider three cases.

Case 1 ($x \leq \sqrt{w_1}$) Then

$$\begin{aligned} F_2(x; 2, w) &= \Pr\{\sqrt{w_1 P_1^2 + w_2 P_2^2} \leq x\} \\ &= \Pr\{w_1 P_1^2 + w_2 P_2^2 \leq x^2\} \\ &= \Pr\left\{\frac{P_1^2}{(x^2/w_1)} + \frac{P_2^2}{(x^2/w_2)} \leq 1\right\} \\ &= \frac{\pi x^2}{4\sqrt{w_1 w_2}}. \end{aligned}$$

using the formula for the area of an ellipse.

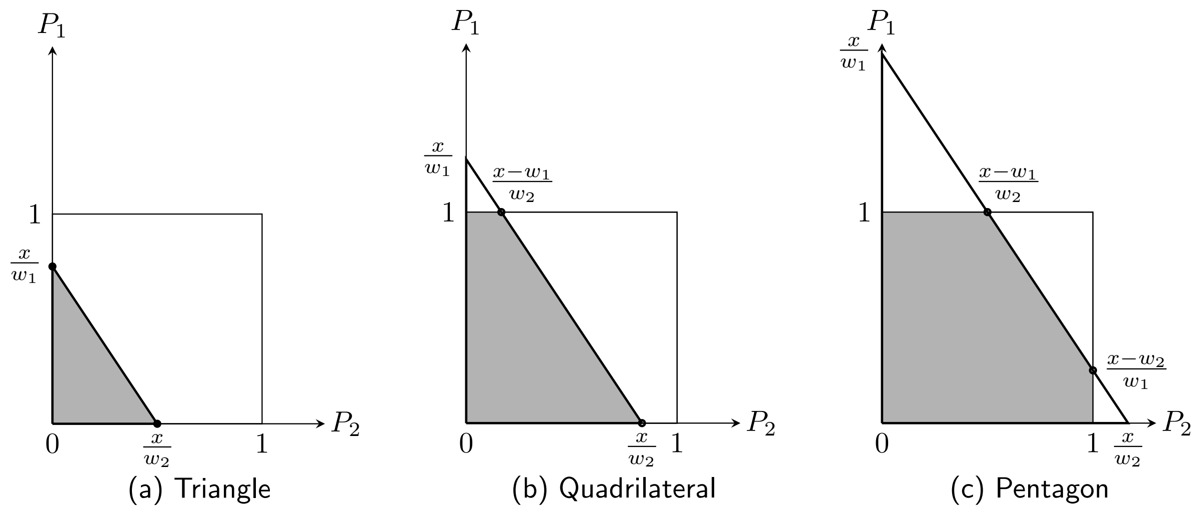

Figure 4: Rejection regions for weighted arithmetic mean

Case 2 ($\sqrt{w_1} < x \leq \sqrt{w_2}$)

$$\begin{aligned}
F_2(x;2,w) &= \Pr\{\sqrt{w_1P_1^2 + w_2P_2^2} \leq x\} \\
&= \int_0^1 \sqrt{\frac{x^2 - w_1y^2}{w_2}}dy \\
&= \left.\frac{\left(y\sqrt{w_1(x^2 - w_1y^2)} + x^2\tan^{-1}\left(\frac{\sqrt{w_1}y}{\sqrt{x^2-w_1y^2}}\right)\right)}{2\sqrt{w_1w_2}}\right|_0^1 \\
&= \frac{\sqrt{w_1(x^2 - w_1)} + x^2\tan^{-1}\left(\frac{\sqrt{w_1}y}{\sqrt{x^2-w_1y^2}}\right)}{2\sqrt{w_1w_2}}.
\end{aligned}$$

Case 3 ($x > \sqrt{w_2}$)

$$\begin{aligned}
F_2(x;2,w) &= \Pr\left(\sqrt{w_1P_1^2 + w_2P_2^2} \leq x\right) \\
&= \sqrt{\frac{x^2 - w_2}{w_1}} + \int_{\sqrt{\frac{x^2-w_2}{w_1}}}^1 \sqrt{\frac{x^2 - w_1y^2}{w_2}}dy \\
&= \sqrt{\frac{x^2 - w_2}{w_1}} + \left.\frac{\left(\sqrt{w_1}y\sqrt{x^2 - w_1y^2} + x^2\tan^{-1}\left(\frac{\sqrt{w_1}y}{\sqrt{x^2-w_1y^2}}\right)\right)}{2\sqrt{w_1w_2}}\right|_{\sqrt{\frac{p^2-w_2}{w_1}}}^1 \\
&= \sqrt{\frac{x^2 - w_2}{w_1}} + \frac{\sqrt{w_1(x^2 - w_1)} + x^2\tan^{-1}\left(\frac{\sqrt{w_1}}{\sqrt{x^2-w_1}}\right)}{2\sqrt{w_1w_2}}
\end{aligned}$$

$$
-\frac{\sqrt{w_1 w_2}\sqrt{\frac{x^2-w_2}{w_1}} + x^2 \tan^{-1}\left(\sqrt{\frac{x^2-w_2}{w_2}}\right)}{2\sqrt{w_1 w_2}}
$$

$$
= \frac{1}{2}\left(\sqrt{\frac{x^2-w_2}{w_1}} + \sqrt{\frac{x^2-w_1}{w_2}}\right) + \frac{x^2}{2\sqrt{w_1 w_2}}\left(\tan^{-1}\left(\sqrt{\frac{w_1}{x^2-w_2}}\right) - \tan^{-1}\left(\sqrt{\frac{x^2-w_2}{w_2}}\right)\right)
$$

Part 9 ($m = \infty$)

Assume that $P_n = P_{\max}$ is unique. Then

$$
\begin{aligned}
\left(\sum_{i=1}^n w_i P_i^m\right)^{1/m} &= P_n \left(\sum_{i=1}^n w_i \left(\frac{P_i}{P_n}\right)^m\right)^{1/m} \\
&= P_n \left(w_n + \sum_{i=1}^{n-1} w_i \left(\frac{P_i}{P_n}\right)^m\right)^{1/m} \\
&\to P_n \quad \text{as } m \to \infty \text{ since } w_n^{1/m} \to 1 \text{ and } \left(\frac{P_i}{P_n}\right)^m \to 0
\end{aligned}
$$

The c.d.f. of $P_n = P_{\max}$ is

$$
F_n(x; \infty, w) = \Pr\{P_{\max} \le x\} = \prod_{i=1}^n \Pr\{P_i \le x\} = x^n.
$$

Equating this to α and solving for x, we get $c_n(\infty, \alpha) = \alpha)^{1/n}$.

Proof of Theorem 3:
Consider two values of m, $m' < m''$.Then we have

$$
\Pr\{\bar{P}_n(m') \le c_n(m', \alpha)\} = \Pr\{\bar{P}_n(m'') \le c_n(m'', \alpha)\} = \alpha.
$$

From the power mean inequality we have $\bar{P}_n(m') \le \bar{P}_n(m'')$. Therefore

$$
\begin{aligned}
\Pr\{\bar{P}_n(m') \le c_n(m'', \alpha)\} &= \Pr\{\bar{P}_n(m') \le \bar{P}_n(m'') \le c_n(m'', \alpha)\} \\
&\quad + \Pr\{\bar{P}_n(m') \le c_n(m'', \alpha) \le \bar{P}_n(m'')\} \\
&\ge \Pr\{\bar{P}_n(m') \le \bar{P}_n(m'') \le c_n(m'', \alpha)\} \\
&= \Pr\{\bar{P}_n(m'') \le c_n(m'', \alpha)\} \\
&= \alpha.
\end{aligned}
$$

Since $\Pr\{\bar{P}_n(m') \le c_n(m'', \alpha)\} \ge \alpha$ and $\Pr\{\bar{P}_n(m') \le c_n(m', \alpha)\} = \alpha$, it follows that $c_n(m', \alpha) \le c_n(m'', \alpha)$.

Derivation of the Mean and Variance of $\frac{1}{n}\left(\sum_{i=1}^n P_i^m\right)^{1/m}$ Using the Delta Method
Denote $\frac{1}{n}\left(\sum_{i=1}^n P_i^m\right) = X$. From (12) it follows that

$$
E(X) = \frac{1}{m+1} \quad \text{and} \quad \text{Var}(X) = \frac{m^2}{n(m+1)(2m+1)}.
$$

Now let $g(X) = X^{1/m}$. By the delta method we have $E[g(X)] \approx \left(\frac{1}{m+1}\right)^{1/m}$ and

$$\begin{aligned}\mathrm{Var}[g(X)] &\approx \mathrm{Var}(X)[g'(m)]^2 \\ &= \frac{m^2}{n(m+1)(2m+1)}\left[-\frac{(m+1)^{-(1+1/m)}}{m}\right]^2 \\ &= \frac{m^2}{n(m+1)(2m+1)}\frac{(m+1)^{2(1-1/m)}}{m^2} = \frac{(m+1)^{1-2/m}}{n(2m+1)}.\end{aligned}$$

Distribution of $S(1,1)$

There is no explicit closed formula for the distribution of $S(1,1)$. However, we can calculate it numerically from its characteristic function given by

$$\varphi(t \mid a, b, \mu, \sigma) = \exp\{it\mu - |t\sigma|^a(1 - ib \cdot \mathrm{sgn}(t) \cdot \Psi)\} \tag{20}$$

where $i = \sqrt{-1}$, $a \in (0,2]$ is a stability parameter, $b \in [-1,1]$ is a skewness parameter, $\mu \in (-\infty, \infty)$ is a shift parameter, $\sigma > 0$ is a scale parameter and

$$\Psi = \begin{cases} \tan\left(\frac{\pi a}{2}\right) & (a \neq 1), \\ -\frac{2}{\pi}\ln|t| & (a = 1). \end{cases}$$

The p.d.f. of $S(1,1)$ can be found by applying the inverse Fourier transform to its characteristic function:

$$f^*(x) = \frac{1}{2n}\int_{-\infty}^{\infty}\varphi(t)e^{-ixt}dt. \tag{21}$$

The c.d.f. can be found from $F^*(x) = \int_{-\infty}^{x} f^*(t)dt$. These operations can be done numerically using the `MATLAB` function `makedist()`. Figure 5 shows the plots of the p.d.f. of $S(0.5,1), S(1,1)$ and $S(1.5,1)$computed using the above numerical method.

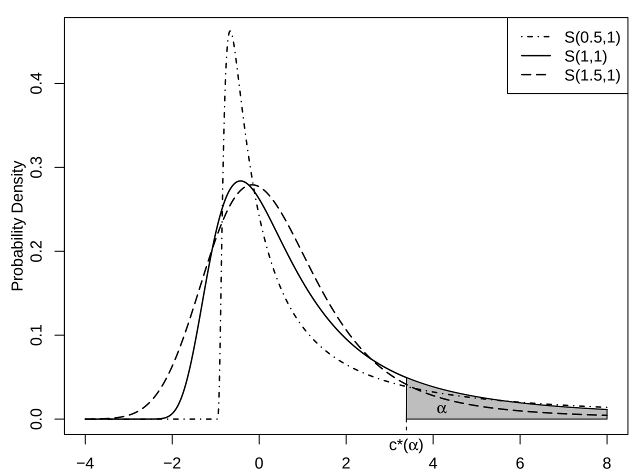

Figure 5: The p.d.f.s of the stable distributions $S(0.5,1)$, $S(1,1)$ and $S(1.5,1)$.