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Bulk Behaviour of Skew-Symmetric Patterned Random Matrices

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Abstract

Limiting Spectral Distributions (LSDs) of real symmetric patterned matrices have been well-studied. In this article, we consider skew-symmetric/anti-symmetric patterned random matrices and establish the LSDs of several common matrices.

For the skew-symmetric Wigner, skew-symmetric Toeplitz and the skew-symmetric Circulant, the LSDs (on the imaginary axis) are the same as those in the symmetric cases.

However, for the skew-symmetric Hankel and the skew-symmetric Reverse Circulant, we obtain new LSDs. We also show the existence of LSDs for the triangular versions of these matrices.

We then introduce a related modification of the symmetric matrices by changing the sign of the lower triangle part of the matrices. In this case, the modified Wigner, modified Hankel and the modified Reverse Circulant have the same LSDs as their usual symmetric counterparts while new LSDs are obtained for the modified Toeplitz and the modified Symmetric Circulant.

Key words: Patterned random matrices; Limiting spectral distribution; Skew-symmetric Circulant, Hankel, Reverse Circulant, Toeplitz, Wigner matrices; Semi-circular law; Link function; Catalan word; Symmetric word.

AMS Subject Classifications: 60B20, 60B10

I knew Professor Aloke Dey for more than two decades. Aloke-da was a caring elder brother as well as a close family friend. I took his advice on numerous occasions and he was a pillar of support. A quintessential gentleman, he always maintained the most cordial relation with everyone, irrespective of which side of an issue the person was. His demise has left a large void and I miss him sorely. Fond memories survive.

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1. Introduction

Suppose A_n is an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. The empirical spectral measure μ_n of A_n is the random measure

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}, \tag{1}$$

where δ_x is the Dirac delta measure at x. The corresponding random probability distribution (on $\mathbb{R}$ or $\mathbb{R}^2$, depending on whether the eigenvalues are real or complex) is known as the *Empirical Spectral Distribution* (ESD) and is denoted by F^{A_n}.

The sequence $\{F^{A_n}\}$ is said to converge (weakly) almost surely to a non-random distribution function F if, outside a null set, as $n \to \infty$, $F^{A_n}(\cdot) \to F(\cdot)$ at all continuity points of F. F is known as the *Limiting Spectral Distribution* (LSD).

There has been a lot of recent work on obtaining the LSDs of large dimensional patterned random matrices. These matrices may be defined as follows (Bose and Sen (2008)). Let $(a_i)_{i \geq 1}$ be a sequence of random variables, called an *input sequence*. Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}_+$ be the set of all positive integers. Let

$$L_n : \{1, 2, \dots n\}^2 \to \mathbb{Z} \text{ (or } \mathbb{Z}^2), \; n \geq 1, \tag{2}$$

be a sequence of functions. We shall write $L_n = L$ and call it the *link* function. By a slight abuse of notation, we shall write $\mathbb{Z}_+^2$ as the common domain of $\{L_n\}_{n \geq 1}$. Matrices of the form

$$A_n = n^{-1/2}((a_{L(i,j)}))_{1 \leq i,j \leq n} \tag{3}$$

are called *patterned matrices*. If $L(i,j) = L(j,i)$ for all i, j, then the matrix is symmetric. We shall denote the LSD of $\{n^{-1/2}A_n\}$, if it exists, by $\mathcal{L}_A$.

The real symmetric patterned matrices that have received particular attention in the literature are the Wigner, Toeplitz, Hankel, Reverse Circulant and the Symmetric Circulant matrices. Their link functions are given in Table 1.

Table 1: Some common symmetric patterned matrices and their link functions.

Matrix	Notation	Link function
Wigner	W_n	$L_W(i,j) = (\min\{i,j\}, \max\{i,j\})$
Toeplitz	T_n	$L_T(i,j) = \lvert i-j \rvert$
Hankel	H_n	$L_H(i,j) = i+j$
Symmetric Circulant	SC_n	$L_{SC}(i,j) = \frac{n}{2} - \lvert \frac{n}{2} - \lvert i-j \rvert \rvert$
Reverse Circulant	RC_n	$L_{RC}(i,j) = (i+j)(\text{mod } n)$

While the LSDs of the Wigner, Reverse Circulant and the Symmetric Circulant are known explicitly, very little is known about the LSDs of the Hankel and the Toeplitz (see,

Table 2: Skew-symmetric patterned matrices and their LSDs.

Matrix	Notation (M)	LSD of iM
Skew-symmetric Wigner	$\widetilde{W}_n$	Same as W_n
Skew-symmetric Toeplitz	$\widetilde{T}_n$	Same as T_n
Skew-symmetric Hankel	$\widetilde{H}_n$	New LSD
Skew-symmetric Circulant	$\widetilde{SC}_n$	Same as SC_n
Skew-symmetric Reverse Circulant	$\widetilde{RC}_n$	New LSD

e.g., Bose (2018)). Existence of LSD is also known for the upper triangular versions of these matrices, though the nature of these limits is not known.

In this article, we study the existence of the LSDs of skew-symmetric/anti-symmetric patterned matrices. Recall that a matrix S is called skew-symmetric if $S = -S^{\top}$. In the Physics literature, the term "anti-symmetric" is more common. Technically, if S is a skew-symmetric matrix, then iS is called an anti-symmetric matrix, where i is the imaginary unit. Note that iS is Hermitian. Anti-symmetric Gaussian matrices appeared in the classic work of Mehta (2004) who, among other things, gave an expression for the joint distribution of their eigenvalues. Singular values of skew-symmetric Gaussian Wigner matrices are useful in Statistics too, *e.g.*, in the paired comparisons model (see Kuriki (1993, 2010)). Recently, Dumitriu and Forrester (2010) obtained tridiagonal realizations of anti-symmetric Gaussian β-ensembles.

We first establish the existence of the LSDs of several real skew-symmetric patterned random matrices and identify the limits in some cases. For the skew-symmetric Wigner, skew-symmetric Toeplitz and the skew-symmetric Circulant, the LSDs (on the imaginary axis) are the same as those in the symmetric cases. However, for the skew-symmetric Hankel and the skew-symmetric Reverse Circulant, we obtain new LSDs (see Figure 1). See Table 2 for a summary. We also show the existence of the LSDs for the triangular versions of these matrices that were introduced in Basu *et al.* (2012). While the LSDs are known for the Hermitian versions of some of these matrices, we show that the limits for the skew-symmetric versions may be derived from the proofs for symmetric matrices using simple arguments.

We also introduce a related modification of the symmetric matrices by changing the sign of the lower triangle part below the main anti-diagonal. In this case, the modified Wigner, the modified Hankel and the modified Reverse Circulant have the same LSDs as their symmetric counterparts whereas new LSDs are obtained for the modified Toeplitz and the modified Symmetric Circulant (see Figure 2). See Table 3 for a summary.

2. Preliminaries

We shall use the method of moments to establish the existence of LSDs. For any matrix A, let $\beta_h(A)$ denote the h-th moment of the ESD of A. We quote the following lemma from Bose (2018) which is easy to prove.

Table 3: Modified patterned matrices and their LSDs.

Matrix	Notation	LSD
Modified Wigner	$\widehat{W}_n$	Same as W_n
Modified Toeplitz	$\widehat{T}_n$	New LSD
Modified Hankel	$\widehat{H}_n$	Same as H_n
Modified Symmetric Circulant	$\widehat{SC}_n$	New LSD
Modified Reverse Circulant	$\widehat{RC}_n$	Same as RC_n

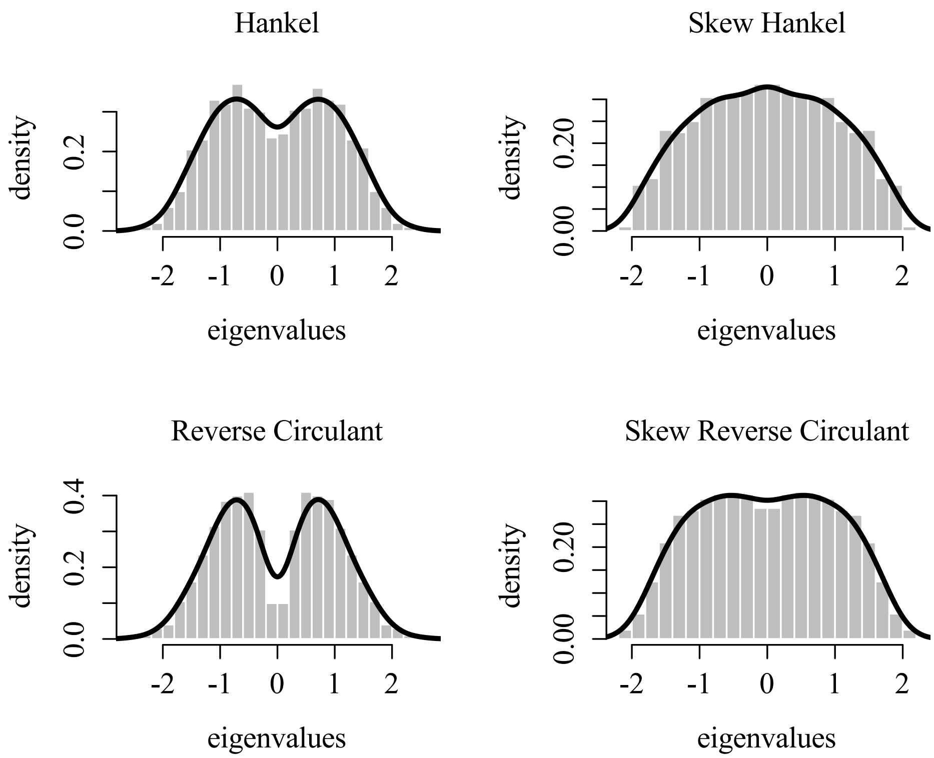

Figure 1: Histograms and kernel density estimates of the spectra of $n^{-1/2}H_n$, $n^{-1/2}i\widetilde{H}_n$, $n^{-1/2}RC_n$ and $n^{-1/2}i\widetilde{RC}_n$ with $n = 1000$ and $\mathcal{N}(0,1)$ entries.

Lemma 1: Let $\{A_n\}$ be a sequence of random matrices with all real eigenvalues. Suppose there exists a sequence $\{\beta_h\}$ such that

(i) for every $h \geq 1$, $\mathbb{E}(\beta_h(A_n)) \to \beta_h$,

(ii) $\sum_{n=1}^{\infty} \mathbb{E}[\beta_h(A_n) - \mathbb{E}(\beta_h(A_n))]^4 < \infty$ for every $h \geq 1$ and

(iii) the sequence $\{\beta_h\}$ satisfies Carleman's condition, $\sum \beta_{2h}^{-1/2h} = \infty$.

Then the LSD of F^{A_n} exists and equals F with moments $\{\beta_h\}$.

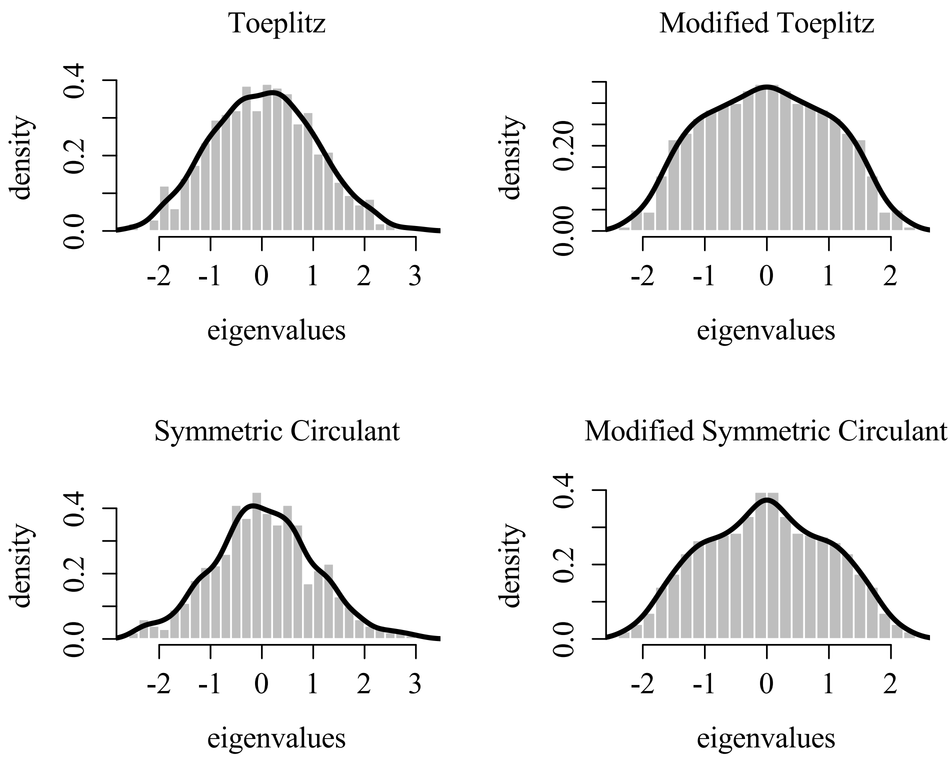

Figure 2: Histograms and kernel density estimates of the spectra of $n^{-1/2}T_n$, $n^{-1/2}\widehat{T}_n$, $n^{-1/2}SC_n$ and $n^{-1/2}\widehat{SC}_n$ with $n = 1000$ and $\mathcal{N}(0,1)$ entries.

To prove the existence of any LSD, we shall make use of the general notation and theory developed in Bose and Sen (2008) for patterned matrices (see also Bose (2018)). First observe that all the link functions in Table 1 satisfy the so called Property B: the total number of times any particular variable appears in any row is uniformly bounded. Moreover, the product of the total number of different variables in the matrix and the maximum number of times any variable appears in the matrix is $O(n^2)$. These two facts imply that the general theory applies to the link functions in Table 1.

We shall consider the following sets of assumptions on the input sequence.

(A1). $(a_i)_{i\geq 1}$ are independent and uniformly bounded with mean 0, and variance 1.

(A2). $(a_i)_{i\geq 1}$ are i.i.d. with mean 0 and variance 1.

(A3). $(a_i)_{i\geq 1}$ are independent with mean 0, variance 1, and uniformly bounded moments of all orders.

Note that Assumption (A1) implies Assumption (A3). Traditionally, LSD results are stated under Assumption (A1) while Assumption (A3) is appropriate for studying the joint convergence of more than one sequence of matrices. It turns out that, for the matrices under our consideration, if LSDs exist under Assumption (A1), then the same LSDs continue to hold under Assumptions (A2) or (A3). Thus in our proofs, without loss of any generality, Assumption (A1) is assumed to hold. Below we give a brief outline of the reasoning. The reader may consult Bose (2018) for detailed justifications in the similar context of symmetric patterned matrices.

(i) When the entries satisfy Assumption (A1), the main idea is to show that the expected moments of the ESD of A_n converge and these limit moments determine a unique distribution. Moreover, these limit moments depend only on the pattern and not on the specific distribution of the entries. We thus call this limit *universal*.

(ii) If the entries of the matrix under consideration satisfy Assumption (A2), then one considers the same matrix but where the entries are truncated suitably and standardized to have mean 0 and variance 1. This matrix satisfies Assumption (A1) and hence has the same (universal) limit. Then one shows that the original matrix and the modified matrix are close in a suitable metric as $n \to \infty$. This leads us to conclude that the same universal limit persists under Assumption (A2).

(iii) Finally, suppose that the entries satisfy Assumption (A3). Then we compute the moments of the ESD again. Using the "uniformly bounded moments" assumption and Property B of the link function, it can be shown that the third or higher order moments of the variables do not influence the LSD (somewhat like the central limit theorem, for example), and we have the same limit as obtained under Assumption (A1).

The *Moment-Trace Formula* plays a key role in this approach. A function

$$\pi : \{0, 1, \cdots, h\} \to \{1, 2, \cdots, n\}$$

with $\pi(0) = \pi(h)$ is called a *circuit* of length h. The dependence of a circuit on h and n is suppressed. Then, for any $n \times n$ square matrix $A = ((a_{L(i,j)}))$, we have

$$\beta_h(A) = \frac{1}{n}\operatorname{tr}(A^h) = \frac{1}{n}\sum_{\pi \text{ circuit of length } h} a_\pi,$$

where

$$a_\pi := a_{L(\pi(0),\pi(1))}a_{L(\pi(1),\pi(2))}\cdots a_{L(\pi(h-1),\pi(h))}.$$

If $L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))$, with $i < j$, we shall use the notation (i, j) to denote such a match of the L-values. From the general theory, it follows that circuits where there are only pair-matches are relevant when computing limits of moments.

Two circuits π_1 and π_2 are equivalent if and only if their L-values respectively match at the same locations, *i.e.* if, for all i, j,

$$L(\pi_1(i-1), \pi_1(i)) = L(\pi_1(j-1), \pi_1(j)) \Leftrightarrow L(\pi_2(i-1), \pi_2(i)) = L(\pi_2(j-1), \pi_2(j)).$$

Any equivalence class can be indexed by a partition of $\{1, 2, \cdots, h\}$. We label these partitions by *words* w of length h of letters where the first occurrence of each letter is in alphabetical order. For example, if $h = 4$, then the partition $\{\{1, 3\}, \{2, 4\}\}$ is represented by the word *abab*. This identifies all circuits π for which $L(\pi(0), \pi(1)) = L(\pi(2), \pi(3))$ and $L(\pi(1), \pi(2)) = L(\pi(3), \pi(1))$. Let $w[i]$ denote the i-th entry of w. The equivalence class corresponding to w is

$$\Pi(w) := \{\pi \mid w[i] = w[j] \Leftrightarrow L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))\}.$$

By varying w, we obtain all the equivalence classes. It is important to note that, for any fixed h, even as $n \to \infty$, the number of words (equivalence classes) remains finite but the number of circuits in any given $\Pi(w)$ may grow indefinitely. Henceforth we shall denote the set of all words of length h by $\mathcal{A}_h$.

Notions of matches carry over to words. A word is *pair-matched* if every letter appears exactly twice in that word. The set of all pair-matched words of length $2k$ is denoted by $\mathcal{W}_{2k}$. For technical reasons, it is often easier to deal with a class larger than $\Pi(w)$:

$$\Pi^*(w) := \{\pi \mid w[i] = w[j] \Rightarrow L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))\}.$$

Any i (or $\pi(i)$ by abuse of notation) is a *vertex*. It is *generating* if either $i = 0$ or $w[i]$ is the first occurrence of a letter. Otherwise, it is called non-generating. For example, if $w = abbcab$, then $\pi(0), \pi(1), \pi(2), \pi(4)$ are generating and $\pi(3), \pi(5), \pi(6)$ are non-generating. The set of generating vertices (indices) is denoted by S. By Property B, a circuit is completely determined, up to finitely many choices, by its generating vertices.

From the general theory for symmetric random matrices it follows that the LSD exists if, for each $w \in \mathcal{W}_{2k}$, the following limit exists:

$$p(w) = \lim_n n^{-(k+1)} \# \Pi^*(w).$$

3. A Unified Framework for Real Skew-symmetric Matrices

If A is an $n \times n$ skew-symmetric matrix, then all its eigenvalues $\{\lambda_j\}$ are purely imaginary (and has one zero eigenvalue when n is odd), and every eigenvalue occurs in conjugate pairs. As discussed in the introduction, the Hermitian matrix iA will then have real spectrum. Consider the ESD of iA on $\mathbb{R}$:

$$F^{iA}(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{\{i\lambda_j \leq x\}}.$$

Note that F^{iA} is a symmetric (about zero) distribution. Therefore, in order to apply the moment method, it suffices to deal with only the even moments. Note that

$$\begin{aligned}\beta_{2k}(iA) &= \int x^{2k}\, dF^{iA}(x) \\ &= \frac{1}{n} \sum_{j=1}^{n} (i\lambda_j)^{2k} = (-1)^k \frac{1}{n} \sum_{j=1}^{n} \lambda_j^{2k} = (-1)^k \frac{1}{n} \operatorname{tr}(A^{2k}).\end{aligned}$$

Let $\{A_n\}$ be a sequence of $n \times n$ patterned random matrices with the symmetric link function L. Let

$$s_{ij} = (1 - \delta_{ij})(-1)^{\mathbf{1}_{\{i>j\}}},$$

where δ_{ij} is the Kronecker-delta. Let $S_n = ((s_{ij}))$ be the $n \times n$ matrix

$$S_n = \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & 0 \end{pmatrix}_{n\times n}.$$

Then we can construct $\widetilde{A}_n$, the skew-symmetric version of A_n by

$$\widetilde{A}_n = S_n \odot A_n,$$

where $\odot$ denotes the Schur-Hadamard/entrywise product.

We shall assume without loss of generality that (A1) holds. The moment-trace formula for $i\widetilde{A}_n$ may be written as

$$\beta_{2k}(n^{-1/2}i\widetilde{A}_n) = (-1)^k \frac{1}{n^{1+k}} \sum_{\pi \text{ circuit of length } 2k} s_\pi a_\pi.$$

Therefore

$$\mathbb{E}\beta_{2k}(n^{-1/2}i\widetilde{A}_n) = (-1)^k \frac{1}{n^{1+k}} \sum_{\pi \text{ circuit of length } 2k} s_\pi \mathbb{E} a_\pi.$$

Using the concept of words, we may rewrite the above equality as

$$\mathbb{E}\beta_{2k}(n^{-1/2}i\widetilde{A}_n) = (-1)^k \frac{1}{n^{1+k}} \sum_{w\in\mathcal{A}_{2k}} \sum_{\pi\in\Pi(w)} s_\pi \mathbb{E} a_\pi.$$

Suppose L satisfies Property B. Let $C^L_{h,3+}$ denote the set of L-matched h-circuits on $\{1, \cdots, n\}$ with at least one edge of order ≥ 3. Then Lemma 1(a) of Bose and Sen (2008) says that there is a constant C depending on L and h such that

$$\# C^L_{h,3+} \leq C n^{\lfloor (h+1)/2 \rfloor}.$$

Combining this with the observation that $|s_\pi| \leq 1$ it is easy to see that

$$\lim_n \frac{1}{n^{1+k}} \sum_{\pi\in C^L_{2k,3+}} s_\pi \mathbb{E} a_\pi = 0.$$

Therefore

$$\lim_n \mathbb{E}\beta_{2k}(n^{-1/2}i\widetilde{A}_n) = (-1)^k \lim_n \frac{1}{n^{1+k}} \sum_{w\in\mathcal{W}_{2k}} \sum_{\pi\in\Pi(w)} s_\pi \mathbb{E} a_\pi.$$

Since, by our assumptions, $\mathbb{E} a_\pi = 1$ for any pair-matched circuit π, the above expression reduces to

$$\lim_n \mathbb{E}\beta_{2k}(n^{-1/2}i\widetilde{A}_n) = (-1)^k \sum_{w\in\mathcal{W}_{2k}} \lim_n \frac{1}{n^{1+k}} \sum_{\pi\in\Pi(w)} s_\pi, \tag{4}$$

provided the limits on the right-hand side exist. In fact, since $\Pi^*(w) \setminus \Pi(w) \subseteq C^L_{2k,3+}$, one has

$$\lim_n \frac{1}{n^{1+k}} \sum_{\pi \in \Pi(w)} s_\pi = \lim_n \frac{1}{n^{1+k}} \sum_{\pi \in \Pi^*(w)} s_\pi,$$

and thus one can write

$$\lim_n \mathbb{E}\beta_{2k}(n^{-1/2} i\widetilde{A}_n) = (-1)^k \sum_{w \in \mathcal{W}_{2k}} \lim_n \frac{1}{n^{1+k}} \sum_{\pi \in \Pi^*(w)} s_\pi, \tag{5}$$

provided the limits exist for each w. If we define

$$p_{\widetilde{A}}(w) := (-1)^k \lim_n \frac{1}{n^{1+k}} \sum_{\pi \in \Pi(w)} s_\pi,$$

then (5) becomes

$$\lim_n \mathbb{E}\beta_{2k}(n^{-1/2} i\widetilde{A}_n) = \sum_{w \in \mathcal{W}_{2k}} p_{\widetilde{A}}(w). \tag{6}$$

In this context, we recall the analogous expression for symmetric matrices A_n from Bose and Sen (2008):

$$\lim_n \mathbb{E}\beta_{2k}(n^{-1/2} A_n) = \sum_{w \in \mathcal{W}_{2k}} p_A(w),$$

where

$$p_A(w) := \lim_n \frac{1}{n^{1+k}} \#\Pi(w) = \lim_n \frac{1}{n^{1+k}} \#\Pi^*(w)$$

is assumed to exist for each $w \in \mathcal{W}_{2k}$.

It is not difficult to show that if the limits exist in (5), then Condition (iii) of Lemma 1 follows (see Theorem 3 of Bose and Sen (2008) for the argument in the symmetric case; in the skew-symmetric case too, one can use their argument verbatim because $|s_\pi| \leq 1$). In fact, the limiting moments are *sub-Gaussian*, *i.e.* the even moments are dominated by the even moments of some Gaussian distribution. The verification of Condition (ii) is also easy since

$$\prod_{j=1}^{4} \mathbb{E}(s_{\pi_j} a_{\pi_j} - \mathbb{E}s_{\pi_j} a_{\pi_j}) = s_{\pi_1} s_{\pi_2} s_{\pi_3} s_{\pi_4} \prod_{j=1}^{4} \mathbb{E}(a_{\pi_j} - \mathbb{E}a_{\pi_j})$$

and the arguments given in the proof of Lemma 2 of Bose and Sen (2008) apply with minor modifications.

In the next section, we shall consider several skew-symmetric patterned matrices and show that Condition (i) of Lemma 1 holds by arguing that the limits on the right-hand side of (4) holds in each case.

4. Some Specific Matrices

First note that

$$s_\pi = (-1)^{\sum_{j=1}^{2k} \mathbf{1}_{\{\pi(j-1) > \pi(j)\}}} \prod_{j=1}^{2k} (1 - \delta_{\pi(j-1),\pi(j)}).$$

It is convenient to use some graph theoretic terminology to deal with the above expression. Consider the complete directed graph DK_n on $V = \{1, \cdots n\}$. Note that π defines a directed circuit of length $2k$ on this graph. Call the numerical value of each vertex its *level*. Associate with each π a marking-vector $(\epsilon_1, \cdots, \epsilon_{2k})$, where

$$\epsilon_j = (-1)^{\mathbf{1}_{\{\pi(j-1)>\pi(j)\}}}(1 - \delta_{\pi(j-1),\pi(j)}).$$

Note that if a traveler moves along the circuit π, starting from $\pi(0)$, and marks each move $\pi(j-1) \rightsquigarrow \pi(j)$ by ϵ_j, then moving to a higher (respectively lower) level corresponds to a mark of 1 (respectively -1) and remaining at the same level corresponds to marking with 0. Then

$$s_\pi = \prod_{j=1}^{2k} \epsilon_j.$$

Note that a circuit π contains a *loop* if and only if $s_\pi = 0$.

We first tackle the skew-symmetric Wigner matrix $n^{-1/2}\widetilde{W}_n$. To do so recall the concept of *Catalan* words from Bose (2018). A Catalan word of length 2 is just a double letter *aa*. In general, a Catalan word of length $2k$, $k > 1$, is a word $w \in \mathcal{W}_{2k}$ containing a double letter such that if one deletes the double letter the reduced word becomes a Catalan word of length $2k-2$. For example, *abba*, *aabbcc*, *abccbdda* are Catalan words whereas *abab*, *abccab*, *abcddcab* are not. The set of all Catalan words of length $2k$ will be denoted by $\mathcal{C}_{2k}$. It is known that

$$\#\mathcal{C}_{2k} = \frac{1}{k+1}\binom{2k}{k},$$

the ubiquitous Catalan number from Combinatorics. It is known that $\#\mathcal{C}_{2k}$ also equals the $2k$-th moment of the semi-circular law, the LSD of the Wigner matrix.

Theorem 1: If the input sequence satisfies (A1) or (A2) or (A3), then the LSD of $n^{-1/2}i\widetilde{W}_n$ is the semi-circular law.

Proof: It is well known (see, *e.g.*, Bose (2018)) that, for the symmetric Wigner matrix, only Catalan words contribute in the limit. In fact, one has

$$p_W(w) = \lim_n \frac{1}{n^{1+k}} \#\Pi^*(w) = \begin{cases} 0 & \text{if } w \notin \mathcal{C}_{2k}, \\ 1 & \text{if } w \in \mathcal{C}_{2k}. \end{cases}$$

From this and the fact that $|s_\pi| \leq 1$ it follows that

$$|p_{\widetilde{W}}(w)| \begin{cases} = 0 & \text{if } w \in \mathcal{W}_{2k} \setminus \mathcal{C}_{2k}, \\ \leq 1 & \text{if } w \in \mathcal{C}_{2k}. \end{cases}$$

We shall prove that if w is a Catalan word, then $p_{\widetilde{W}}(w)$ exists and equals 1. Then (5) would imply that

$$\lim_n \mathbb{E}\beta_{2k}(n^{-1/2}i\widetilde{W}_n) = \#\mathcal{C}_{2k},$$

establishing the semi-circular limit for the ESD of $\{n^{-1/2}\widetilde{W}_n\}$.

We first observe that if we replace the diagonal entries by 0, then the LSD does not change. It follows from this observation that *circuits with loops do not have any contribution to* $p_{\widetilde{W}}(w)$. It now suffices for our purpose to prove that if $w \in \mathcal{C}_{2k}$ and $\pi \in \Pi^*(w)$, then

$$s_\pi = \begin{cases} (-1)^k & \text{if } \pi \text{ is loopless,} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

To prove this, suppose that a double letter appears at the i-th and the $(i+1)$-th positions. Consider a loopless $\pi \in \Pi^*(w)$. Since, $w[i] = w[i+1]$, we must have

$$L_W(\pi(i-1), \pi(i)) = L_W(\pi(i), \pi(i+1)).$$

Since π is loopless, it follows that we must have $\pi(i-1) = \pi(i+1) \neq \pi(i)$. There are two possibilities: either $\pi(i-1) < \pi(i)$ or $\pi(i-1) > \pi(i)$. In the first case, $\epsilon_i = 1$ and $\epsilon_{i+1} = -1$, while, in the second case, $\epsilon_i = -1$ and $\epsilon_{i+1} = -1$. In either case, we have

$$\epsilon_i \epsilon_{i+1} = -1.$$

Now delete the double letter and think of π as a circuit of length $2k-2$ by identifying the vertices $(i-1)$ and $(i+1)$ as identical and deleting the vertex i. The resulting word w' is still Catalan and the resulting circuit π' is loopless and lies in $\Pi^*(w')$. Apply the above procedure again. Clearly, we will need k iterations of this procedure to empty the word w and each such iteration contributes one -1, which proves (7) and hence the theorem. □

Remark 1: Basu *et al.* (2012) considered upper/lower triangular versions of the Wigner, W_n^Δ. Its LSD $\mathcal{L}_{W^\Delta}$ is different from the semi-circular law, but its free convolution with itself is the semi-circular law. It follows from the proof of Theorem 1 and their moment calculations that the LSD of $i\widetilde{W}^\Delta$ is again $\mathcal{L}_{W^\Delta}$.

The existence of the LSD of the symmetric Toeplitz matrix T_n was first established by Hammond and Miller (2005) and Bryc *et al.* (2006). The properties of the limit law $\mathcal{L}_T$ are not well understood. We now consider the skew-symmetric Toeplitz $\widetilde{T}_n$.

Theorem 2: If the input sequence satisfies (A1) or (A2) or (A3), then the LSD of $n^{-1/2} i\widetilde{T}_n$ is $\mathcal{L}_T$, the LSD of the symmetric Toeplitz.

Proof: Let $w \in \mathcal{W}_{2k}$ and $s(i) := \pi(i) - \pi(i-1)$. Define

$$\Pi^{**}(w) := \{\pi \mid w[i] = w[j] \Rightarrow s(i) + s(j) = 0\}.$$

Then Bose and Sen (2008) show that

$$p_T(w) = \lim_n \frac{1}{n^{1+k}} \#\Pi^*(w) = \lim_n \frac{1}{n^{1+k}} \#\Pi^{**}(w). \tag{8}$$

As in the Wigner case, circuits with loops do not contribute and to establish our goal it suffices to prove that if $w \in \mathcal{W}_{2k}$ and $\pi \in \Pi^{**}(w)$, then

$$s_\pi = \begin{cases} (-1)^k & \text{if } \pi \text{ is loopless,} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

The proof of this is much easier than the Wigner case as all the difficulty is relegated to the proof of (8). Consider a loopless circuit $\pi \in \Pi^{**}(w)$. Note that $w[i] = w[j]$ implies that $s(i) + s(j) = 0$ and since π is loopless, we have

$$s(i)s(j) = -s(j)^2 < 0.$$

This immediately implies that

$$\epsilon_i \epsilon_j = (-1)^{\mathbf{1}_{\{s(i)<0\}} + \mathbf{1}_{\{s(j)<0\}}} = -1.$$

Since w is pair-matched, there are exactly k matches from each of which comes one -1. This establishes (9) and completes the proof. □

Remark 2: Basu *et al.* (2012) considered upper/lower triangular versions of the Toeplitz, T_n^{Δ}. They proved the existence of the LSD but it could not be identified. It follows from the proof of Theorem 2 and their moment calculations that the LSD of $i\widetilde{T}^{\Delta}$ is again $\mathcal{L}_{T^{\Delta}}$, exactly paralleling the Wigner case.

The Symmetric Circulant matrix SC_n and the Palindromic Toeplitz matrix PT_n have the standard Gaussian distribution $\mathcal{N}(0,1)$ as their LSD (see Bose (2018)). We now consider the skew-symmetric versions $\widetilde{SC}_n$ and $\widetilde{PT}_n$.

Theorem 3: If the input sequence satisfies (A1) or (A2) or (A3), then the LSDs of $n^{-1/2} i\widetilde{SC}_n$ and $n^{-1/2} i\widetilde{PT}_n$ are the same as the LSDs of their symmetric counterparts, *i.e.* the standard Gaussian distribution.

Proof: We first tackle $\widetilde{SC}_n$. From Bose and Sen (2008), it is known that, for any $w \in \mathcal{W}_{2k}$, if one defines

$$\Pi'(w) := \{\pi \mid w[i] = w[j] \Rightarrow s(i) + s(j) = 0, \pm n\},$$

then one actually has

$$p_{SC}(w) = \lim_n \frac{1}{n^{1+k}} \#\Pi^*(w) = \lim_n \frac{1}{n^{1+k}} \#\Pi'(w) = 1.$$

Once again, circuits with loops have no role to play and to prove the desired result it suffices to prove that if $w \in \mathcal{W}_{2k}$ and $\pi \in \Pi'(w)$, then

$$s_\pi = \begin{cases} (-1)^k & \text{if } \pi \text{ is loopless,} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Due to the similarity with the Toeplitz link function, the proof of the above is similar to that in the Toeplitz case. Let π be a loopless circuit from $\Pi'(w)$. Suppose that $w[i] = w[j]$. Then we have $s(i) + s(j) = 0, \pm n$. We treat each of these three cases separately:

1. $s(i) + s(j) = 0$. This is the same as the Toeplitz case and we conclude that $\epsilon_i \epsilon_j = -1$.

2. $s(i)+s(j)=n$. Note that $s(i)=n-s(j)$ and since π is loopless,
$$|s(j)|=|\pi(j)-\pi(j-1)|\leq n-1.$$
Therefore $s(i)=n-s(j)>0$. By symmetry, $s(j)>0$. Therefore, in this case, $\epsilon_i\epsilon_j=1$.

3. $s(i)+s(j)=-n$. Note that $s(i)=-(n+s(j))$, and therefore $s(i)$, and by symmetry $s(j)$, are both negative ceding $\epsilon_i\epsilon_j=1$.

Therefore, combining the above cases,
$$s_\pi=(-1)^{k-e_\pi},$$
where e_π is the number of matches (i,j) where $s(i)+s(j)=\pm n$. It suffices to show that e_π is even. But note that
$$\sum_{i=1}^{2k}s(i)=\pi(2k)-\pi(0)=0,$$
which cannot occur unless e_π is even. This establishes (10) and completes the proof for $\widetilde{SC}_n$.

To prove the same for $\widetilde{PT}_n$ we take the approach of Bose and Sen (2008). We need the following version of the well known interlacing inequality. We omit its proof.

Suppose A is a real skew-symmetric matrix with eigenvalues $i\lambda_j$ with $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$. Let B be the $(n-1)\times(n-1)$ principal submatrix of A with eigenvalues $i\mu_k$ with $\mu_1\geq\mu_2\geq\cdots\geq\mu_{n-1}$. Then one has
$$\lambda_1\geq\mu_1\geq\lambda_2\geq\mu_2\geq\cdots\geq\mu_{n-1}\geq\lambda_n,$$
i.e. the imaginary parts of the eigenvalues of B are interlaced between the imaginary parts of the eigenvalues of A.

As a consequence
$$||F^A-F^B||_\infty\leq\frac{1}{n}. \tag{11}$$
Now note that the $n\times n$ principal submatrix of $\widetilde{SC}_{n+1}$ is $\widetilde{PT}_n$. Therefore, from (11), we can conclude that $n^{-1/2}i\widetilde{PT}_n$ also has the standard Gaussian law as its LSD. □

Remark 3: Basu *et al.* (2012) considered the upper/lower triangular versions of the symmetric Circulant, SC_n^Δ. They proved the existence of the LSD but it could not be identified. It follows from the proof of Theorem 3 and their moment calculations that the LSD of $i\widetilde{SC}^\Delta$ is again $\mathcal{L}_{SC^\Delta}$.

The skew-symmetric matrices considered so far have the same LSD (on the imaginary axis) as their corresponding symmetric versions. However, simulations suggest that the LSDs of $n^{-1/2}i\widetilde{H}_n$ and $n^{-1/2}i\widetilde{RC}_n$ exist and are different from those of $n^{-1/2}H_n$ and $n^{-1/2}RC_n$ respectively. See Figure 1. We now establish this rigorously.

In this context, *symmetric* words play the key role. A word $w \in \mathcal{W}_{2k}$ is called symmetric if each letter in w occurs once each in an odd and an even position. For example, the word *aabb* is symmetric and the word *abab* is not. We shall denote the set of symmetric words of length $2k$ by $\mathcal{S}_{2k}$. All Catalan words are symmetric. An example of a non-Catalan symmetric word is *abcabc*. It is easy to prove that

$$\#\mathcal{S}_{2k} = k!.$$

Theorem 4: If the input sequence satisfies (A1) or (A2) or (A3), then the LSDs of $n^{-1/2} i\widetilde{H}_n$ and $n^{-1/2} i\widetilde{RC}_n$ exist and are different from the LSDs of $n^{-1/2} H_n$ and $n^{-1/2} RC_n$ respectively.

Proof: We first consider the skew-symmetric Hankel. First suppose $w \in \mathcal{C}_{2k}$. It is known that then $p_H(w) = 1$. By an argument similar to that given in the proof of Theorem 1 one can show that $p_{\widetilde{H}}(w) = 1$.

Now suppose that w is not symmetric. It is known that then $p_H(w) = 0$. Since, $|s_\pi| \leq 1$, it follows that $p_{\widetilde{H}}(w)$ also vanishes.

More generally, for any pair-matched word w, the limit $p_{\widetilde{H}}(w)$ can be shown to exist using the same Riemann approximation technique that is used in the Hankel case (see, for example, Bose and Sen (2008)). We omit the details.

We now show that this LSD is not the same as in the symmetric Hankel case. Since $|s_\pi| \leq 1$, it is clear that the limit is *sub-Hankel*, *i.e.* $\lim_n \beta_{2k}(n^{-1/2} i\widetilde{H}_n) \leq \lim_n \beta_{2k}(n^{-1/2} H_n)$ for all $k \geq 1$. It is thus enough to show that $\lim_n \beta_{2k}(n^{-1/2} i\widetilde{H}_n) < \lim_n \beta_{2k}(n^{-1/2} H_n)$ for some $k \geq 1$. Since Catalan words contribute 1 to both of these and non-symmetric words do not contribute at all, we need to look at non-Catalan symmetric words. The first such word is $w = abcabc$. We shall show that $p_{\widetilde{H}}(abcabc) < \frac{1}{2} = p_H(abcabc)$.

So let us consider the word $w = abcabc$ and its four generating vertices, *viz.*, $\pi(0)$, $\pi(1)$, $\pi(2)$, $\pi(3)$. Writing $\nu_i = \pi(i)/n$ and expressing the $\frac{1}{n^4}\#\Pi^*(w)$ as a Riemann sum, we know from Bose and Sen (2008) that, for the Hankel matrix,

$$p_H(w) = \int_{I^4} \mathbf{1}_{\{0<\nu_0+\nu_1-\nu_3<1,\, 0<\nu_2+\nu_3-\nu_0<1\}} d\nu_3 d\nu_2 d\nu_1 d\nu_0,$$

where I^4 is the unit 4-cube. Let P be the subset of I^4 where the integrand above is positive. For the skew-symmetric case, however, there are many $\pi \in \Pi^*(w)$ such that $s_\pi = -1$, which means that there are lots of cancellations. More formally, for any $\pi \in \Pi^*(w)$, we have

$$\nu_4 = \nu_0 + \nu_1 - \nu_3,$$
$$\nu_5 = \nu_2 + \nu_3 - \nu_0.$$

If we define

$$g(\nu) = s_\pi = (-1)^{\sum_{j=1}^{2k} \mathbf{1}_{\{\nu_{j-1}<\nu_j\}}},$$

then by resorting to the Riemann approximation technique it is easy to see that

$$p_{\widetilde{H}}(w) = (-1)^3 \int_{I^4} g(\nu) \mathbf{1}_{\{0<\nu_0+\nu_1-\nu_3<1,\, 0<\nu_2+\nu_3-\nu_0<1\}} d\nu_3 d\nu_2 d\nu_1 d\nu_0.$$

We shall show that on a subset of P of positive Lebesgue measure, $g(\nu) = 1$. Consider the set $U = P \cap \{(\nu_0, \nu_1, \nu_2, \nu_3) \mid 0 < \nu_0 < \nu_1 < \nu_2 < \nu_3 < 1\} \subseteq I^4$. We claim that on U, one has $g(\nu) = 1$. To see this, note that we automatically have $\nu_j - \nu_{j-1} > 0$ for $j = 1, 2, 3$. Moreover,

$$\begin{aligned}\nu_4 - \nu_3 &= \nu_1 + \nu_0 - 2\nu_3 < 0,\\ \nu_5 - \nu_4 &= (\nu_2 - \nu_1) + 2(\nu_3 - \nu_0) > 0,\\ \nu_6 - \nu_5 &= 2\nu_0 - \nu_2 - \nu_3 < 0.\end{aligned}$$

Therefore, on U, we have $g(\nu) = (-1)^{1+1+1+(-1)+1+(-1)} = 1$. It now suffices to show that

$$\int_U \mathbf{1}_{\{0<\nu_0+\nu_1-\nu_3<1,\, 0<\nu_2+\nu_3-\nu_0<1\}} d\nu_3 d\nu_2 d\nu_1 d\nu_0 > 0.$$

With some easy manipulations with the constraints it is easy to show that

$$\begin{aligned}\int_U \mathbf{1}_{\{0<\nu_0+\nu_1-\nu_3<1,\, 0<\nu_2+\nu_3-\nu_0<1\}} d\nu &\geq \int_{\frac{1}{3}}^{\frac{1}{2}} \int_{\nu_0}^{\frac{1}{2}} \int_{1-\nu_1}^{\frac{1+\nu_0}{2}} \int_{\nu_2}^{1+\nu_0-\nu_2} d\nu_3 d\nu_2 d\nu_1 d\nu_0\\ &= \frac{19}{62208} > 0.\end{aligned}$$

This completes the proof for the skew-symmetric Hankel.

Now consider the skew-symmetric Reverse Circulant. By following the arguments in the Hankel case, it is easy to see that each word limit exists, thereby proving the existence of the LSD. Moreover, it is known that, for the Reverse Circulant, $p_{RC}(w) = 1$ if w is symmetric and 0 otherwise. In the present case, $p_{\widetilde{RC}}(w) \leq 1$ for all symmetric words and the non-symmetric words continue to contribute zero. It is also easy to show that if $w \in \mathcal{C}_{2k}$, then $p_{\widetilde{RC}}(w) = p_{RC}(w) = 1$. Thus, as before, it remains to seek out a symmetric non-Catalan word w such that $p(w) < 1$. Once again, we may look at $w = abcabc$ and prove this. Due to the similarity with the Hankel case, we skip the details. □

5. A Related Class of Symmetric Matrices

We have seen that skew-symmetry does not change the LSDs of the Wigner, Toeplitz and the Symmetric Circulant, whereas it changes the LSDs of the Hankel and the Reverse Circulant. We now investigate this issue a little more.

Let M_n be the $n \times n$ symmetric matrix whose upper and lower triangle entries are respectively $+1$ and -1, the anti-diagonal consisting of 0's. Then $M_n = ((m_{ij}))$ where

$$m_{ij} = \begin{cases} 1 & \text{if } i + j < n + 1,\\ 0 & \text{if } i + j = n + 1,\\ -1 & \text{if } i + j > n + 1.\end{cases}$$

We show that an LSD exists for the Schur-Hadamard product of M_n with each of the above five matrices. For a patterned matrix A_n, we denote by $\widehat{A}_n$ its modified version $M_n \odot A_n$.

Note that, for the Wigner and the Hankel cases, the Schur-Hadamard product is also of the same type (with a modified input sequence where the signs have changed for some elements of the sequence)–the fact that the anti-diagonal is zero does not affect the LSDs. Hence their LSDs remain unchanged due to the universality of LSDs with respect to the input variables as long as they satisfy Assumptions (A1) or (A2) or (A3). As we shall see, the LSD remains unchanged for the modified Reverse Circulant matrix too.

Note that $n^{-1/2}\widehat{T}_n$ and $n^{-1/2}\widehat{SC}_n$ are not Toeplitz and Symmetric Circulant matrices. We show that LSDs exist for both and are different from $\mathcal{L}_T$ and $\mathcal{N}(0,1)$ respectively. See Figure 2 for simulations.

Similar to the skew-symmetric case, define

$$\epsilon_i = (1 - \mathbf{1}_{\{\pi(i-1)+\pi(i)=n+1\}})(-1)^{\mathbf{1}_{\{\pi(i-1)+\pi(i)>n+1\}}},$$

and

$$m_\pi = \prod_{i=1}^{n} \epsilon_i.$$

Then we have the following analogue of (6):

$$\lim \mathbb{E}\beta_{2k}(n^{-1/2}\widehat{A}_n) = \sum_{w \in \mathcal{W}_{2k}} p_{\widehat{A}}(w), \tag{12}$$

where

$$p_{\widehat{A}}(w) := \lim_n \frac{1}{n^{1+k}} \sum_{\pi \in \Pi(w)} m_\pi = \lim_n \frac{1}{n^{1+k}} \sum_{\pi \in \Pi^*(w)} m_\pi$$

is assumed to exist for each $w \in \mathcal{W}_{2k}$. First we consider the LSD of $n^{-1/2}\widehat{RC}_n$.

Theorem 5: If the input sequence satisfies (A1) or (A2) or (A3), then the LSD of $n^{-1/2}\widehat{RC}_n$ is the same as the LSD of $n^{-1/2}RC_n$, *i.e.* $\mathcal{L}_{RC}$.

Proof: To prove this theorem, note that, by (12), it is enough to prove that $m_\pi = 1$ for each $\pi \in \Pi^*(w)$, where $w \in \mathcal{W}_{2k}$. Define

$$t(i) = \pi(i-1) + \pi(i) \text{ and } u(i) = t(i) - (n+1).$$

Call a circuit π *good* if $m_\pi \neq 0$. It is enough to consider only such circuits.

If $w[i] = w[j]$, then we have

$$t(i) \equiv t(j) \pmod{n},$$

which implies that $u(i) \equiv u(j) \pmod{n}$. Now note that

$$-(n-1) = 2 - (n+1) \le u(i) \le n + n - (n+1) = n - 1,$$

and hence

$$|u(i) - u(j)| \le 2(n-1).$$

So, we must have

$$u(i) - u(j) = 0, \pm n.$$

Observe that

1. If $u(i) - u(j) = 0$, then $\epsilon_i = \epsilon_j$, which yields $\epsilon_i \epsilon_j = 1$.
2. If $u(i) - u(j) = n$, then $u(i) = n + u(j) > 0$, and $u(j) = u(i) - n < 0$, as $|u(l)| \leq n - 1$ for any l. So, in this case, $\epsilon_i \epsilon_j = -1$.
3. If $u(i) - u(j) = -n$, then, again, $\epsilon_i \epsilon_j = -1$ by interchanging the roles of i and j in the previous argument.

As a consequence

$$m_\pi = (-1)^{e_\pi},$$

where e_π is the number of matches (i, j) in π for which $u(i) - u(j) = t(i) - t(j) = \pm n$. Let further e_π^+ be the number of matches (i, j) in π for which $t(i) - t(j) = n$ and $e_\pi^- = e_\pi - e_\pi^+$. First notice that

$$\sum_{i=1}^{2k} t(i) = 2 \sum_{i=1}^{2k} \pi(i).$$

The same sum can be written as

$$\sum_{(i,j)\text{ match}} (t(i) + t(j)).$$

Notice then that

$$\begin{aligned} \sum_{(i,j)\text{ match}} (t(i) + t(j)) &= \sum_{(i,j)\text{ match}} (t(i) - t(j)) + 2 \sum_{(i,j)\text{ match}} t(j) \\ &= (e_\pi^+ - e_\pi^-) n + 2 \sum_{(i,j)\text{ match}} t(j) \\ &= n e_\pi - 2 n e_\pi^- + 2 \sum_{(i,j)\text{ match}} t(i). \end{aligned}$$

It follows from the above considerations that ne_π is always even. Now suppose that n is odd. It then follows that e_π is even and therefore $m_\pi = 1$. The case with n even seems to be more complicated. It is not clear why e_π has to be even. We shall use a little trick to bypass the need to pinpoint the parity of e_π in this case. Define, for $w \in \mathcal{W}_{2k}$,

$$\begin{aligned} q_n(w) &:= \frac{1}{n^{1+k}} \sum_{\pi \in \Pi^*(w)} m_\pi, \\ p_n(w) &:= \frac{1}{n^{1+k}} \# \Pi^*(w). \end{aligned}$$

Then it is known from Bose and Sen (2008) that

$$p_n(w) = p_{RC}(w) + o(1),$$

which implies, since $|q_n(w)| \leq |p_n(w)|$, that

$$|q_n(w)| = O(1). \tag{13}$$

We have already proved that (as we have proved that $m_\pi = 1$ for n odd)

$$q_{2n+1}(w) = p_{RC}(w) + o(1). \tag{14}$$

In the following lemma, we shall write $\Pi_n^*(w)$ instead of $\Pi^*(w)$ to explicitly denote the dependence on n.

Lemma 2: We have

$$\#\Pi_{n+1}^*(w) - \#\Pi_n^*(w) = o(n^{1+k}).$$

Proof: We have

$$p_n(w) = \frac{1}{n^{1+k}} \#\Pi_n^*(w) = p(w) + o(1),$$

which can be rewritten as

$$\#\Pi_n^*(w) = p(w)n^{1+k} + o(n^{1+k}).$$

As a consequence

$$\#\Pi_{n+1}^*(w) - \#\Pi_n^*(w) = p(w)((n+1)^{1+k} - n^{1+k}) + o(n^{1+k}),$$

from which the lemma follows since the first term is $O(n^k)$. □ We need another lemma.

Lemma 3: We have

$$q_{n+1}(w) - q_n(w) = o(1).$$

Proof: We have, using the triangle inequality,

$$\begin{aligned}
&|q_{n+1}(w) - q_n(w)| \\
&\quad = \left| \frac{1}{(n+1)^{1+k}} \sum_{\pi \in \Pi_{n+1}^*(w)} m_\pi - \frac{1}{n^{1+k}} \sum_{\pi \in \Pi_n^*(w)} m_\pi \right| \\
&\quad = \left| \frac{1}{(n+1)^{1+k}} \sum_{\pi \in \Pi_n^*(w)} m_\pi + \frac{1}{(n+1)^{1+k}} \sum_{\pi \in \Pi_{n+1}^*(w) \setminus \Pi_n^*(w)} m_\pi - \frac{1}{n^{1+k}} \sum_{\pi \in \Pi_n^*(w)} m_\pi \right| \\
&\quad \le \left| \frac{1}{(n+1)^{1+k}} \sum_{\pi \in \Pi_n^*(w)} m_\pi - \frac{1}{n^{1+k}} \sum_{\pi \in \Pi_n^*(w)} m_\pi \right| + \left| \frac{1}{(n+1)^{1+k}} \sum_{\pi \in \Pi_{n+1}^*(w) \setminus \Pi_n^*(w)} m_\pi \right| \\
&\quad =: \text{(I)} + \text{(II)}.
\end{aligned}$$

Using (13), we get

$$\text{(I)} \le \left| \left(\frac{n}{n+1} \right)^{1+k} - 1 \right| \times |q_n(w)| = o(1) \times O(1) = o(1).$$

On the other hand, by Lemma 2, we have

$$\text{(II)} \le \frac{1}{n^{1+k}} \#(\Pi_{n+1}^*(w) \setminus \Pi_n^*(w)) = o(1).$$

Together, the above two estimates imply the lemma. □ Coming back to the original problem, because of Lemma 3 and (14), we can write

$$\begin{aligned} q_{2n+2}(w) &= q_{2n+1}(w) + o(1) \\ &= p_{RC}(w) + o(1). \end{aligned}$$

This establishes, irrespective of the parity of n, that

$$q_n(w) = p_{RC}(w) + o(1),$$

which completes the proof of the theorem. □ Finally, we give the result on the LSDs of $n^{-1/2}\widehat{T}_n$ and $n^{-1/2}\widehat{SC}_n$.

Theorem 6: If the input sequence satisfies (A1) or (A2) or (A3), then the LSDs of $n^{-1/2}\widehat{T}_n$ and $n^{-1/2}\widehat{SC}_n$ exist and are different from the LSDs of $n^{-1/2}T_n$ and $n^{-1/2}SC_n$ respectively.

Proof: We shall outline the proof only for $n^{-1/2}\widehat{T}_n$. The proof for $n^{-1/2}\widehat{SC}_n$ is similar and is omitted.

Once again, the existence of the LSD, say $\mathcal{L}_{\widehat{T}}$, may be proven using the Riemann approximation technique. We show that $\mathcal{L}_{\widehat{T}}$ does not equal $\mathcal{L}_T$. As in the proof of Theorem 1 we can show that, for each Catalan word w, $p_{\widehat{T}}(w) = 1 = p_T(w)$. Thus we need to look at a non-Catalan pair-matched word. The first such word is $w = abab$. We shall show that $p_{\widehat{T}}(abab) \neq p_T(abab) = 2/3$, which would conclude proof. Using the Riemann approximation argument, it is easy to show that

$$p_{\widehat{T}}(w) = \int_{I^3} (-1)^{\sum_{i=1}^{4} \mathbf{1}_{\{\nu_i + \nu_{i-1} > 1\}}} \mathbf{1}_{\{0 \leq \nu_0 - \nu_1 + \nu_2 \leq 1\}} d\nu_2 d\nu_1 d\nu_0,$$

where $\nu_3 = \nu_0 - \nu_1 + \nu_2$ and $\nu_4 = \nu_0$. Now, similar to the skew-symmetric Hankel case, one can show that on a subset of positive Lebesgue measure the integrand above is negative. In fact, a calculation in `Mathematica` reveals that $p_{\widehat{T}}(abab) = 2/9$. This proves the theorem completely. □

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