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# E-Bayesian and Hierarchical Bayesian Estimation for Inverse Rayleigh Distribution Based on Left Censoring Scheme

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## Abstract

This study is concerned with estimating the scale parameter and the reversed hazard rate of the Inverse Rayleigh distribution based on left censoring, one of the most noticeable distributions in lifetime studies. Even though different estimation methods are employed, each method suffers from its problems such as complexity of calculations, high risk, *etc.* Results derived under squared error, entropy, and precautionary loss functions. E-Bayesian and H-Bayesian estimations are obtained based on different priors of the hyper parameters to investigate the influence on these estimations. We investigated the asymptotic behaviors of E-Bayesian estimates and relations among them. Finally, a comparison among the Bayes, H-Bayes, and E-Bayes estimates in different sample sizes made using real and the simulated data. Numerical study shows that the newly presented method is more efficient than previous methods and is also easy to operate.

*Key words:* Inverse Rayleigh distribution; Left censoring; Bayesian estimation; E-Bayesian estimation; H-Bayesian estimation.

**AMS Subject Classifications:** 62F15; 62N05

## 1. Introduction

Several authors used Inverse Rayleigh (IR) distribution to model applications in the area of reliability. Voda (1972) used this distribution to model the lifetimes of several experimental units. Several works related to inference using complete samples based on parameters of inverse Rayleigh (IR) distribution are available in the literature. El-Helbawy and Abd-El-Monem (2005) developed Bayes estimators for the parameters of the IR distribution using different loss functions. For more works related to inference using IR distribution, one can refer to Soliman *et al.* (2010), Dey (2012), Feroze and Aslam (2012) and Shawky and Badr (2012). In the context of reliability and survival analysis, censoring is unavoidable, and there

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are different censoring schemes available. One of the practical censoring schemes is the left censoring, and it occurs when we cannot identify the exact time the event occurred.

Considering the advantage of using the E-Bayesian estimation method recently, many papers are published in the literature using this approach. Han (2009) proposed the E-Bayesian estimate of the failure rate of exponential distribution using type-1 censoring. E-Bayesian estimates of Burr type XII distribution parameters using type-2 censoring had proposed by Jaheen and Okasha (2011). Okasha and Wang (2016) derived E-Bayesian estimators of the geometric distribution parameters when samples are available only in the form of records. Kızılaslan (2017) discusses the E-Bayesian estimation of the proportional hazard rate model. E-bayesian and hierarchical bayesian estimates of the power function distribution parameters had proposed by Abdul-Sathar and Athirakrishnan (2019). This paper aims to propose E-Bayesian and H-Bayesian estimates of the inverse Rayleigh distribution parameters when left-censored data are available. We additionally provide estimates of the reversed hazard rate using three different loss functions. The asymptotic performance of the proposed estimators for different priors is also studied.

The organization of the rest of the works is as follows. We discuss Bayesian estimation of the scale parameter and reversed hazard rate of the IR distribution using left-censored data in Section 2. In Section 3, we discuss the H-Bayesian estimation of the scale parameter and the reversed hazard rate. E-Bayesian estimators of the scale parameter and reversed hazard rate are discussed in Section 4. The properties exhibited by all these estimators discusses in Section 5. The estimator's performance using simulated and real data sets discuss respectively in Sections 6 and 7. Finally, concluding remarks about the proposed study are given in Section 8.

## 2. Bayesian estimation

In this section, we derive the Bayesian estimators of the parameter $\lambda$ of IR distribution using left-censored data under the squared error loss function (SELF), the entropy loss function (ELF), and the precautionary loss function (PLF). The pdf, cdf, and reversed hazard rate of the one-parameter IR distribution are respectively given by

$$f(x;\lambda) = \frac{2\lambda}{x^3} e^{\frac{-\lambda}{x^2}}, \quad x > 0, \quad \lambda > 0, \tag{1}$$

$$F(x;\lambda) = e^{\frac{-\lambda}{x^2}}, \quad x > 0, \quad \lambda > 0, \tag{2}$$

and

$$\bar{h(t)} = \frac{2\lambda}{t^3}, t > 0. \tag{3}$$

Let $\underline{X} = X_{(r+1)}, ..., X_{(n)}$ be the last $(n-r)$ order statistics using a random sample of size $n$ from IR distribution. Likelihood function in this context is given as

$$L(X_{(r+1)}, ..., X_{(n)}|\lambda) \propto \lambda^{n-r} e^{-\lambda \tau_{(ir)}}, \tag{4}$$

where $\tau_{(ir)} = rx^{-2}_{(r+1)} + \sum_{i=r+1}^{n} x^{-2}_{(i)}$. The prior for the parameter $\lambda$ assumes Gamma distribution with density function

$$\pi(\lambda|a,b) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}, \quad \lambda > 0, \quad a,b > 0,$$

where $a$ and $b$ are the hyper parameters. Here we only consider the case of $a = 1$, then the density function $\pi(\lambda|a,b)$ reduces to

$$\pi(\lambda|b) = be^{-b\lambda}, \quad b > 0. \tag{5}$$

Hence the posterior distribution using (4) and (5) simplifies to

$$f(r|\lambda) = \frac{(\tau_{(ir)} + b)^{n-r+1}}{\Gamma(n-r+1)}\lambda^{(n-r)}e^{-\lambda(\tau_{(ir)}+b)}, \quad \lambda > 0 \tag{6}$$

Now we derive the Bayes estimators of $\lambda$ and reversed hazard rate of left censored IR distribution under three different loss functions.
Using SELF, the Bayes estimators of $\lambda$ and reversed hazard rate simplify to

$$\hat{\lambda}_{B1} = E(\lambda|\underline{x}) = \frac{n-r+1}{\tau_{(ir)} + b}, \tag{7}$$

$$\hat{h(t)}_{B1} = E\left(\frac{2\lambda}{t^3}\middle|\underline{x}\right) = \frac{2(n-r+1)}{t^3(\tau_{(ir)} + b)}. \tag{8}$$

The Bayes estimators of $\lambda$ and reversed hazard rate using ELF simplifies to

$$\hat{\lambda}_{B2} = \left[E\left(\frac{1}{\lambda}\middle|\underline{x}\right)\right]^{-1} = \frac{n-r}{\tau_{(ir)} + b}. \tag{9}$$

$$\hat{h(t)}_{B2} = \left[E\left(\left(\frac{2\lambda}{t^3}\right)^{-1}\middle|\underline{x}\right)\right]^{-1} = \frac{2(n-r)}{t^3(\tau_{(ir)} + b)}. \tag{10}$$

The Bayes estimators of $\lambda$ and reversed hazard rate using PLF simplifies to

$$\hat{\lambda}_{B3} = \sqrt{E(\lambda^2|\underline{x})} = \sqrt{\frac{(n-r+1)(n-r+2)}{(\tau_{(ir)} + b)^2}}. \tag{11}$$

$$\hat{h(t)}_{B3} = \sqrt{E\left(\left(\frac{2\lambda}{t^3}\right)^2\middle|\underline{x}\right)} = \frac{2}{t^3}\sqrt{\frac{(n-r+1)(n-r+2)}{(\tau_{(ir)} + b)^2}}. \tag{12}$$

## 3. Hierarchical Bayesian estimation

Lindley and Smith (1972) first introduced the idea of hierarchical prior distribution. For the parameter $\lambda$, the hierarchical prior density function is defined as

$$\pi(\lambda) = \int_0^c \pi(\lambda|b)\pi(b)db.$$

Hierarchical Bayesian (H-Bayesian) estimation of $\lambda$ is obtained based on three different distributions of the hyper parameter $b$. The influence of the different prior distributions on the H-Bayesian estimation of $\lambda$ is studied by using these distributions. The following distributions of $b$ may be used

$$\pi_1(b) = \frac{2(c-b)}{c^2}, \quad 0 < b < c, \tag{13}$$

$$\pi_2(b) = \frac{1}{c}, \quad 0 < b < c, \tag{14}$$

$$\pi_3(b) = \frac{2b}{c^2}, \quad 0 < b < c, \tag{15}$$

### 3.1. Hierarchical Bayesian estimation of $\lambda$

For $\pi_1(b)$, the hierarchical prior density function simplifies to

$$\pi_4(\lambda) = \frac{2}{c^2}\int_0^c b(c-b)e^{-b\lambda}db, \lambda > 0. \tag{16}$$

Using Bayesian theorem, the hierarchical posterior density for $\lambda$ can be defined as

$$\begin{aligned} H_1(\lambda|\underline{x}) &= \frac{\pi_4(\lambda)L(r|\lambda)}{\int_0^\infty \pi_4(\lambda)L(r|\lambda)d\lambda} \\ &= \frac{\int_0^c b(c-b)\lambda^{n-r}e^{-\lambda(\tau_{(ir)}+2b)}(\tau_{(ir)}+b)^{n-r+1}db}{\int_0^c b(c-b)\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{n-r+1}}(\tau_{(ir)}+b)^{n-r+1}db}. \end{aligned} \tag{17}$$

The H-Bayesian estimators of $\lambda$ under SELF is given as

$$\hat{\lambda}_{HS1} = \frac{\int_0^c b(c-b)\frac{\Gamma(n-r+2)}{(\tau_{(ir)}+2b)^{(n-r+2)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b(c-b)\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}, \tag{18}$$

Similarly, the H-Bayesian estimators of $\lambda$ under ELF and PLF are given respectively as

$$\hat{\lambda}_{HE1} = \frac{\int_0^c b(c-b)\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b(c-b)\frac{\Gamma(n-r)}{(\tau_{(ir)}+2b)^{(n-r)}}(\tau_{(ir)}+b)^{(n-r+1)}db}, \tag{19}$$

and

$$\hat{\lambda}_{HP1} = \sqrt{\frac{\int_0^c b(c-b)\frac{\Gamma(n-r+3)}{(\tau_{(ir)}+2b)^{(n-r+3)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b(c-b)\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}}, \tag{20}$$

For $\pi_2(b)$, the hierarchical prior density function simplifies to

$$\pi_5(\lambda) = \frac{1}{c}\int_0^c be^{-b\lambda}db, \lambda > 0. \tag{21}$$

Using Bayesian theorem, the hierarchical posterior density for $\lambda$ can be defined as

$$\begin{aligned} H_2(\lambda|\underline{x}) &= \frac{\pi_5(\lambda)L(r|\lambda)}{\int_0^\infty \pi_5(\lambda)L(r|\lambda)d\lambda} \\ &= \frac{\int_0^c b\lambda^{n-r}e^{-\lambda(\tau_{(ir)}+2b)}(\tau_{(ir)}+b)^{n-r+1}db}{\int_0^c b\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{n-r+1}}(\tau_{(ir)}+b)^{n-r+1}db}. \end{aligned} \tag{22}$$

The H-Bayesian estimator of $\lambda$ under SELF is given as

$$\hat{\lambda}_{HS2} = \frac{\int_0^c b\frac{\Gamma(n-r+2)}{(\tau_{(ir)}+2b)^{(n-r+2)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}, \tag{23}$$

Similarly, the H-Bayesian estimators of $\lambda$ under ELF and PLF are given respectively as

$$\hat{\lambda}_{HE2} = \frac{\int_0^c b\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b\frac{\Gamma(n-r)}{(\tau_{(ir)}+2b)^{(n-r)}}(\tau_{(ir)}+b)^{(n-r+1)}db}, \tag{24}$$

and

$$\hat{\lambda}_{HP2} = \sqrt{\frac{\int_0^c b\frac{\Gamma(n-r+3)}{(\tau_{(ir)}+2b)^{(n-r+3)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}}, \tag{25}$$

For $\pi_3(b)$, the hierarchical prior density function simplifies to

$$\pi_6(\lambda) = \frac{2}{c^2}\int_0^c b^2 e^{-b\lambda}db, \lambda > 0. \tag{26}$$

Using Bayesian theorem, the hierarchical posterior density for $\lambda$ can be defined as

$$\begin{aligned} H_3(\lambda|\underline{x}) &= \frac{\pi_6(\lambda)L(r|\lambda)}{\int_0^\infty \pi_6(\lambda)L(r|\lambda)d\lambda} \\ &= \frac{\int_0^c b^2\lambda^{n-r}e^{-\lambda(\tau_{(ir)}+2b)}(\tau_{(ir)}+b)^{n-r+1}db}{\int_0^c b^2\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{n-r+1}}(\tau_{(ir)}+b)^{n-r+1}db}. \end{aligned} \tag{27}$$

The H-Bayesian estimator of $\lambda$ under SELF is given as

$$\hat{\lambda}_{HS3} = E(\lambda|\underline{x}) = \frac{\int_0^c b^2\frac{\Gamma(n-r+2)}{(\tau_{(ir)}+2b)^{(n-r+2)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b^2\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}. \tag{28}$$

Similarly, the H-Bayesian estimators of $\lambda$ under ELF and PLF are given respectively as

$$\hat{\lambda}_{HE3} = \left[E\left(\lambda^{-1}|\underline{x}\right)\right]^{-1} = \frac{\int_0^c b^2\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b^2\frac{\Gamma(n-r)}{(\tau_{(ir)}+2b)^{(n-r)}}(\tau_{(ir)}+b)^{(n-r+1)}db}, \tag{29}$$

and

$$\hat{\lambda}_{HP3} = \sqrt{E(\lambda^2|\underline{x})} = \sqrt{\frac{\int_0^c b^2\frac{\Gamma(n-r+3)}{(\tau_{(ir)}+2b)^{(n-r+3)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b^2\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}}. \tag{30}$$

### 3.2. Hierarchical Bayesian estimation of reversed hazard rate

Based on SELF, ELF and PLF, the H-Bayesian estimators of the reversed hazard rate is computed for the three different distributions of the hyperparameter $b$ given by (13), (14) and (15). For $\pi_1(b)$, the H-Bayesian estimator of the reversed hazard rate is obtained from (17). Under SELF, the H-Bayesian estimator of reversed hazard rate is given as

$$\hat{h(t)}_{HS1} = \frac{\frac{2}{t^3}\int_0^c b(c-b)\frac{\Gamma(n-r+2)}{(\tau_{(ir)}+2b)^{(n-r+2)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b(c-b)\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}. \tag{31}$$

The H-Bayesian estimators of reversed hazard rate under ELF and PLF are given as

$$\hat{h(t)}_{HE1} = \frac{\frac{2}{t^3}\int_0^c b(c-b)\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b(c-b)\frac{\Gamma(n-r)}{(\tau_{(ir)}+2b)^{(n-r)}}(\tau_{(ir)}+b)^{(n-r+1)}db}. \tag{32}$$

and

$$\hat{h(t)}_{HP1} = \frac{2}{t^3}\sqrt{\frac{\int_0^c b(c-b)\frac{\Gamma(n-r+3)}{(\tau_{(ir)}+2b)^{(n-r+3)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b(c-b)\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}}. \tag{33}$$

For $\pi_2(b)$, the H-Bayesian estimator of the reversed hazard rate is obtained from (22). Under SELF, the H-Bayesian estimator of reversed hazard rate is given as

$$\hat{h(t)}_{HS2} = \frac{\frac{2}{t^3}\int_0^c b\frac{\Gamma(n-r+2)}{(\tau_{(ir)}+2b)^{(n-r+2)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}. \tag{34}$$

The H-Bayesian estimators of reversed hazard rate under ELF and PLF are given as

$$\hat{h(t)}_{HE2} = \frac{\frac{2}{t^3}\int_0^c b\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b\frac{\Gamma(n-r)}{(\tau_{(ir)}+2b)^{(n-r)}}(\tau_{(ir)}+b)^{(n-r+1)}db}. \tag{35}$$

and

$$\hat{h(t)}_{HP2} = \frac{2}{t^3}\sqrt{\frac{\int_0^c b\frac{\Gamma(n-r+3)}{(\tau_{(ir)}+2b)^{(n-r+3)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}}. \tag{36}$$

For $\pi_3(b)$, the H-Bayesian estimator of the reversed hazard rate is obtained from (27). Under SELF, the H-Bayesian estimator of reversed hazard rate is given as

$$\hat{h(t)}_{HS3} = \frac{\frac{2}{t^3}\int_0^c b^2\frac{\Gamma(n-r+2)}{(\tau_{(ir)}+2b)^{(n-r+2)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b^2\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}. \tag{37}$$

The H-Bayesian estimators of reversed hazard rate under ELF and PLF are given as

$$\hat{h(t)}_{HE3} = \frac{\frac{2}{t^3}\int_0^c b^2\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b^2\frac{\Gamma(n-r)}{(\tau_{(ir)}+2b)^{(n-r)}}(\tau_{(ir)}+b)^{(n-r+1)}db}. \tag{38}$$

and

$$\hat{h(t)}_{HP3} = \frac{2}{t^3}\sqrt{\frac{\int_0^c b^2 \frac{\Gamma(n-r+3)}{(\tau_{(ir)}+2b)^{(n-r+3)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b^2 \frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}}. \tag{39}$$

## 4. E-Bayesian estimation

According to Han (1997) the E-Bayesian estimate of $\lambda$ is defined as

$$\hat{\lambda}_E = \int_b \hat{\lambda}_B(b)\pi(b)db. \tag{40}$$

where $\hat{\lambda}_B(b)$ is the Bayesian estimator of $\lambda$ with prior density $\pi(b)$. From (40), we can see that E-Bayesian estimation is the expectation of Bayesian estimator of the parameters for the hyper parameter. E-Bayesian estimation based on three different prior distributions of the hyper parameter (13), (14) and (15) are used to investigate the influence of different prior distributions on the E-Bayesian estimation of $\lambda$ and reversed hazard rate.

### 4.1. E-Bayesian estimation for $\lambda$

Based on SELF, ELF and PLF, the E-Bayesian estimators of $\lambda$ is computed for the three different distributions of the hyperparameter $b$ given by (13), (14) and (15). For $\pi_1(b)$, the E-Bayesian estimate of $\lambda$ under SELF is obtained from (7) and (13) as

$$\hat{\lambda}_{ES1} = \int_0^c \hat{\lambda}_{B1}(b)\pi_1(b)db = \frac{2(n-r+1)}{c^2}\left\{(\tau_{(ir)}+c)\ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right) - c\right\}. \tag{41}$$

Similarly, the E-Bayesian estimates of $\lambda$ under ELF and PLF are computed from (9), (11) and (13) and are given respectively, by

$$\hat{\lambda}_{EE1} = \frac{2(n-r)}{c^2}\left\{(\tau_{(ir)}+c)\ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right) - c\right\}, \tag{42}$$

and

$$\hat{\lambda}_{EP1} = 2\sqrt{\frac{(n-r+1)(n-r+2)}{c}}\left\{(\tau_{(ir)}+c)\ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right) - c\right\}. \tag{43}$$

For $\pi_2(b)$, the E-Bayesian estimate of $\lambda$ under SELF is obtained from (7) and (14) as

$$\hat{\lambda}_{ES2} = \frac{n-r+1}{c}\ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right), \tag{44}$$

Similarly, the E-Bayesian estimates of $\lambda$ under ELF and PLF are computed from (9), (11) and (14) and are given respectively, by

$$\hat{\lambda}_{EE2} = \frac{n-r}{c}\ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right), \tag{45}$$

and

$$\hat{\lambda}_{EP2} = \sqrt{\frac{(n-r+1)(n-r+2)}{c}} \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right), \tag{46}$$

For $\pi_3(b)$, the E-Bayesian estimate of $\lambda$ under SELF is obtained from (7) and (15) as

$$\hat{\lambda}_{ES3} = \frac{2(n-r+1)}{c^2}\left\{c - \tau_{(ir)} \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right)\right\}, \tag{47}$$

Similarly, the E-Bayesian estimates of $\lambda$ under ELF and PLF are computed from (9), (11) and (14) and are given respectively, by

$$\hat{\lambda}_{EE3} = \frac{2(n-r)}{c^2}\left\{c - \tau_{(ir)} \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right)\right\}, \tag{48}$$

and

$$\hat{\lambda}_{EP3} = 2\sqrt{\frac{(n-r+1)(n-r+2)}{c}}\left\{c - \tau_{(ir)} \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right)\right\}. \tag{49}$$

### 4.2. E-Bayesian estimation for reversed hazard rate

Based on SELF, ELF and PLF, the E-Bayesian estimators of reversed hazard rate is computed for the three different distributions of the hyperparameter $b$ given by (13), (14) and (15). For $\pi_1(b)$, the E-Bayesian estimate of reversed hazard rate under SELF is obtained from (8) and (13) as

$$\hat{h(t)}_{ES1} = \frac{4(n-r+1)}{c^2t^3}\left\{(\tau_{(ir)}+c) \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right) - c\right\}. \tag{50}$$

Similarly, the E-Bayesian estimates of reversed hazrd rate under ELF and PLF are computed from (10), (12) and (13) and are given respectively, by

$$\hat{h(t)}_{EE1} = \frac{4(n-r)}{c^2t^3}\left\{(\tau_{(ir)}+c) \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right) - c\right\}, \tag{51}$$

and

$$\hat{h(t)}_{EP1} = \frac{4}{t^3}\sqrt{\frac{(n-r+1)(n-r+2)}{c}}\left\{(\tau_{(ir)}+c) \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right) - c\right\}. \tag{52}$$

For $\pi_2(b)$, the E-Bayesian estimate of reversed hazard rate under SELF is obtained from (8) and (14) as

$$\hat{h(t)}_{ES2} = \frac{2(n-r+1)}{ct^3} \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right). \tag{53}$$

Similarly, the E-Bayesian estimates of reversed hazard rate under ELF and PLF are computed from (10), (12) and (14) and are given respectively, by

$$\hat{h(t)}_{EE2} = \frac{2(n-r)}{ct^3} \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right), \tag{54}$$

and

$$\hat{h(t)}_{EP2} = \frac{2}{t^3}\sqrt{\frac{(n-r+1)(n-r+2)}{c}} \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right). \tag{55}$$

For $\pi_3(b)$, the E-Bayesian estimate of reversed hazard rate under SELF is obtained from (8) and (15) as

$$\hat{h(t)}_{ES3} = \frac{4(n-r+1)}{c^2t^3}\left\{c - \tau_{(ir)} \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right)\right\}. \tag{56}$$

Similarly, the E-Bayesian estimates of reversed hazard rate under ELF and PLF are computed from (10), (12) and (15) and are given respectively, by

$$\hat{h(t)}_{EE3} = \frac{4(n-r)}{c^2t^3}\left\{c - \tau_{(ir)} \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right)\right\}, \tag{57}$$

and

$$\hat{h(t)}_{EP3} = \frac{4}{t^3}\sqrt{\frac{(n-r+1)(n-r+2)}{c}}\left\{c - \tau_{(ir)} \ln\left(\frac{\tau_{(ir)}+c}{\tau_{(ir)}}\right)\right\}. \tag{58}$$

## 5. Properties

In this section, we discussed the important properties of E-Bayesian estimators including the relation of this estimators with the hierarchical Bayesian estimators. In the following theorem, we gives the relationship of E-Bayes estimators of $\lambda$ under different loss functions.

**Theorem 1:** The relationship of E-Bayes estimators of $\lambda$ using respectively the SELF, ELF and PLF are given as

i) $\hat{\lambda}_{EEi} < \hat{\lambda}_{ESi} < \hat{\lambda}_{EPi}, i = 1, 2, 3$

ii) $\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{ESi} = \lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{EEi} = \lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{EPi} = 0.$

**Proof:**

i) The relationship $\hat{\lambda}_{EE1} < \hat{\lambda}_{ES1} < \hat{\lambda}_{EP1}$ is a particular case of $\hat{\lambda}_{EEi} < \hat{\lambda}_{ESi} < \hat{\lambda}_{EPi}$ and it is same as

$$n - r < n - r + 1 < \sqrt{(n-r+1)(n-r+2)}. \tag{59}$$

We use the concept of mathematical induction for proving the relation. For $n$=1, we have $1 - r < (2 - r) < \sqrt{(2-r)(3-r)}$. Hence the result is true for $n$=1. Squaring the above equation, we get

$$(n-r)^2 < (n-r+1)^2 < (n-r+1)(n-r+2). \tag{60}$$

Now assume that the result hold for $n$=$k$. That is

$$(k-r)^2 < (k-r+1)^2 < (k-r+1)(k-r+2). \tag{61}$$

Now, we prove the result for *n=k+1*, so we have

$$\begin{aligned}((k+1)+r+1)((k+1)+r+2) &= (k-r+2)(k-r+3)\\ &= (k-r+1)(k-r+2)\\ &\quad +2(k-r+2). \end{aligned} \tag{62}$$

Using (33), we get

$$\begin{aligned}(k-r)^2+2(k-r+2) &< (k-r+1)^2+2(k-r+2)\\ &< (k-r+1)(k-r+2)+2(k-r+2). \end{aligned} \tag{63}$$

we have

$$(k-r)^2+2(k-r+2) = ((k+1)-r)^2+3 > ((k+1)-r)^2. \tag{64}$$

Also, we have

$$(k-r+1)^2+2(k-r+2) = ((k+1)-r+1)^2+1 > ((k+1)-r+1)^2. \tag{65}$$

Using (33) to (36) ,we have

$$((k+1)-r)^2 < ((k+1)-r+1)^2 < ((k+1)-r+1)((k+1)-r+2). \tag{66}$$

Hence the result.

ii) From the derivation of $\hat{\lambda}_{ES1}$ , we have

$$\hat{\lambda}_{ES1} = \frac{2(n-r+1)}{c^2}\int_0^c \frac{c-b}{\tau_{(ir)}+b}db.$$

Using the generalized mean value theorem, we can find atleast one number $b_1 \in (0,c)$ such that

$$\hat{\lambda}_{ES1} = \frac{2(n-r+1)}{c^2}\frac{1}{\tau_{(ir)}+b_1}\int_0^c (c-b)db.$$

Taking the limit as $\tau_{(ir)} \to \infty$

$$\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{ES1} = 0. \tag{67}$$

Using the generalized mean value theorem, we can find atleast one number $b_2 \in (0,c)$ such that

$$\hat{\lambda}_{EE1} = \frac{2(n-r)}{c^2}\frac{1}{\tau_{(ir)}+b_2}\int_0^c (c-b)db.$$

Taking the limit as $\tau_{(ir)} \to \infty$

$$\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{EE1} = 0. \tag{68}$$

Using the generalized mean value theorem, we can find atleast one number $b_3 \in (0,c)$ such that, we have

$$\hat{\lambda}_{EP1} = \frac{2\sqrt{(n-r+2)(n-r+1)}}{c^2(\tau_{(ir)}+b_3)}\int_0^c (c-b)db.$$

Taking the limit as $\tau_{(ir)} \to \infty$

$$\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{EP1} = 0. \tag{69}$$

Using (38) to (40), we have the proof. From the above theorem, we can see that, E-Bayesian estimators for $\lambda$ are different for different loss functions. It can also be noted that the estimators are asymptotically equal or close to each other when $\tau_{(ir)}$ is sufficiently large. The rest of the proof is same as the above. In the following theorem we provide the relationship of E-Bayes estimators of reversed hazard rate for different loss functions. The proof is similar to the above theorem and hence omitted. □

**Theorem 2:** The relationship of E-Bayes estimators of reversed hazard rate using respectively the SELF, ELF and PLF are given as

i) $\hat{h(t)}_{EE1} < \hat{h(t)}_{ES1} < \hat{h(t)}_{EP1}$

ii) $\lim_{\tau_{(ir)}\to\infty} \hat{h(t)}_{ES1} = \lim_{\tau_{(ir)}\to\infty} \hat{h(t)}_{EE1} = \lim_{\tau_{(ir)}\to\infty} \hat{h(t)}_{EP1} = 0.$

In the following theorem, we gives the relationship between E-Bayes and hierarchical Bayes estimators of $\lambda$ under the same loss function.

**Theorem 3:** The relation between E-Bayes and hierarchical Bayes estimators of $\lambda$ for SELF, ELF and PLF are respectively given as

i) $\lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{\lambda}_{ESi} = \lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{\lambda}_{HSi} = 0, i = 1, 2, 3.$

ii) $\lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{\lambda}_{EEi} = \lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{\lambda}_{HEi} = 0, i = 1, 2, 3.$

iii) $\lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{\lambda}_{EPi} = \lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{\lambda}_{HPi} = 0, i = 1, 2, 3.$

**Proof:**

i) Under SELF, from the above theorem, using (22), we get

$$\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{ES1} = 0. \tag{70}$$

Using the result $\Gamma(n+r+2) = (n+r+1)\Gamma(n+r+1)$ and by using the generalized mean value theorem, we can find atleast one number $b_4 \in (0, c)$

$$\int_0^c b(c-b)(\tau_{(ir)}+b)^{(n-r+1)} \frac{(n-r+1)\Gamma(n-r+1)}{(\tau_{(ir)}+2b)(\tau_{(ir)}+2b)^{(n-r+1)}} db = \frac{n-r+1}{(\tau_{(ir)}+2b_4)} \int_0^c b(c-b)(\tau_{(ir)}+b)^{(n-r+1)} \frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}} db.$$

$$\begin{aligned} \therefore \hat{\lambda}_{HS1} &= \frac{\int_0^c b(c-b)(\tau_{(ir)}+b)^{(n-r+1)} \frac{\Gamma(n-r+2)}{(\tau_{(ir)}+2b)^{(n-r+2)}} db}{\int_0^c b(c-b)(\tau_{(ir)}+b)^{(n-r+1)} \frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}} db} \\ &= \frac{n-r+1}{(\tau_{(ir)}+2b_4)} \end{aligned} \tag{71}$$

Taking limit as $\tau_{(ir)} \to \infty$

$$\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{HS1} = 0. \tag{72}$$

Hence using (41) and (43), we have

$$\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{ES1} = \lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{HS1} = 0. \tag{73}$$

ii) Under ELF, from the above theorem, using (22), we get

$$\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{EE1} = 0. \tag{74}$$

Using the result $\Gamma(n-r+1) = (n-r)\Gamma(n-r)$ and by sing the generalized mean value theorem, we can find atleast one number $b_5 \in (0, c)$ such that

$$\int_0^c b(c-b)(\tau_{(ir)}+b)^{(n-r+1)} \frac{(n-r)\Gamma(n-r)}{(\tau_{(ir)}+2b)(\tau_{(ir)}+2b)^{(n-r)}} db = \frac{(n-r)}{(\tau_{(ir)}+2b_5)} \int_0^c b(c-b)(\tau_{(ir)}+b)^{(n-r+1)} \frac{\Gamma(n-r)}{(\tau_{(ir)}+2b)^{(n-r)}} db. \tag{75}$$

Using (18) we have

$$\begin{aligned} \hat{\lambda}_{HE1} &= \frac{\int_0^c b(c-b)(\tau_{(ir)}+b)^{(n-r+1)} \frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}} db}{\int_0^c b(c-b)(\tau_{(ir)}+b)^{(n-r+1)} \frac{\Gamma(n-r)}{(\tau_{(ir)}+2b)^{(n-r)}} db} \\ &= \frac{(n-r)}{(\tau_{(ir)}+2b_5)}. \end{aligned} \tag{76}$$

Taking limit as $\tau_{(ir)} \to \infty$

$$\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{HE1} = 0. \tag{77}$$

Hence using (45) and (47), we have

$$\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{EE1} = \lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{HE1} = 0. \tag{78}$$

iii) Under PLF, from the above theorem, using (22), we get

$$\lim_{\tau_{(ir)}\to\infty} \hat{\lambda}_{EP1} = 0. \tag{79}$$

Using the result $\Gamma(n+a+2) = (n+a+1)\Gamma(n+a+1)$ and by using the generalized mean value theorem, we can find atleast one number $b_6 \in (0, c)$ such that

$$\int_0^c b(c-b) \frac{\Gamma(n-r+3)}{(\tau_{(ir)}+2b)^{(n-r+3)}} (\tau_{(ir)}+b)^{(n-r+1)} db = \int_0^c b(c-b) \frac{\Gamma(n-r+3)}{(\tau_{(ir)}+2b)^{(n-r+3)}} (\tau_{(ir)}+b)^{(n-r+1)} db. \tag{80}$$

Using (20) we have

$$\hat{\lambda}_{HP1} = \sqrt{\frac{\int_0^c b(c-b)\frac{\Gamma(n-r+3)}{(\tau_{(ir)}+2b)^{(n-r+3)}}(\tau_{(ir)}+b)^{(n-r+1)}db}{\int_0^c b(c-b)\frac{\Gamma(n-r+1)}{(\tau_{(ir)}+2b)^{(n-r+1)}}(\tau_{(ir)}+b)^{(n-r+1)}db}}$$
$$= \frac{\sqrt{(n-r+2)(n-r+1)}}{(\tau_{(ir)}+2b_6)}. \tag{81}$$

Taking limit as $\tau_{(ir)} \rightarrow \infty$

$$\lim_{\tau_{(ir)}\rightarrow\infty} \hat{\lambda}_{HP1} = 0. \tag{82}$$

Hence using (49) and (51), we have

$$\lim_{\tau_{(ir)}\rightarrow\infty} \hat{\lambda}_{EP1} = \lim_{\tau_{(ir)}\rightarrow\infty} \hat{\lambda}_{HP1} = 0. \tag{83}$$

□

The rest of the proof can be proved in the similar way and omitted. In the following theorem, we gives the relationship between E-Bayes and hierarchical Bayes estimators of reversed hazard rate under the same loss function. The proof is similar to the above theorem and hence omitted.

**Theorem 4:** The relation between E-Bayes and hierarchical Bayes estimators of reversed hazard rate for SELF, ELF and PLF are respectively given as

i) $\lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{h(t)}_{ESi} = \lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{h(t)}_{HSi} = 0, i = 1, 2, 3.$

ii) $\lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{h(t)}_{EEi} = \lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{h(t)}_{HEi} = 0, i = 1, 2, 3.$

iii) $\lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{h(t)}_{EPi} = \lim_{\tau_{(ir)}\longrightarrow ß\infty} \hat{h(t)}_{HPi} = 0, i = 1, 2, 3.$

## 6. Monte Carlo Simulation

In this section, we inspect the performance of the proposed estimators using a simulation study. We use the following steps for performing the study.

Step 1: Generate samples of sizes $n$=500,1000 and 1500 from the inverse Rayleigh distribution with pdf (1) for $\lambda = 13$.

Step 2: Fix the value of $c = 1$.

Step 3: For computing the Bayesian estimators, use (7), (8), (9), (10), (11) and (12), for E-Bayesian estimators, use (40), (41), (42), (43), (44), (45), (46), (47) and (48) and for calculating hierarchical Bayesian estimators, use (70), (75) and (80).

Step 4: Repeat steps 1-3, 10000 times and compute the MSE.

**Table 1: MSE for Bayesian, E-Bayesian and H-Bayesian estimates of $\lambda$ for simulated data**

| | $n=500$ | | | $n=1000$ | | | $n=1500$ | | | CP | ACI |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $r=50$ | $r=100$ | $r=150$ | $r=100$ | $r=200$ | $r=300$ | $r=200$ | $r=300$ | $r=400$ | | |
| $\hat{\lambda}_{B1}$ | 0.6168 | 0.4591 | 0.4501 | 0.2479 | 0.2384 | 0.1945 | 0.1653 | 0.1473 | 0.1414 | 92.3 % | (11.5621, 13.7678) |
| $\hat{\lambda}_{B2}$ | 0.5334 | 0.4832 | 0.4403 | 0.2521 | 0.2431 | 0.1991 | 0.1701 | 0.1500 | 0.1439 | 94.3 % | (11.4505, 13.8129) |
| $\hat{\lambda}_{B3}$ | 0.4918 | 0.4543 | 0.4260 | 0.2459 | 0.2363 | 0.1924 | 0.1631 | 0.1460 | 0.1403 | 99.1 % | (11.0848, 14.2832) |
| $\hat{\lambda}_{ES1}$ | 0.3851 | 0.3791 | 0.3727 | 0.2376 | 0.2113 | 0.1752 | 0.1378 | 0.1338 | 0.1312 | 90.6 % | (11.8765, 14.0324) |
| $\hat{\lambda}_{ES2}$ | 0.3970 | 0.3844 | 0.3758 | 0.2365 | 0.2082 | 0.1771 | 0.1428 | 0.1358 | 0.1321 | 95.8 % | (11.5952, 14.1703) |
| $\hat{\lambda}_{ES3}$ | 0.4212 | 0.4036 | 0.3822 | 0.2378 | 0.2084 | 0.1810 | 0.1491 | 0.1387 | 0.1342 | 95.3 % | (11.5601, 14.0589) |
| $\hat{\lambda}_{EE1}$ | 0.3912 | 0.3788 | 0.3782 | 0.2374 | 0.2100 | 0.1762 | 0.1402 | 0.1348 | 0.1317 | 98.2 % | (11.4222, 14.4218) |
| $\hat{\lambda}_{EE2}$ | 0.4088 | 0.3940 | 0.3794 | 0.2374 | 0.2085 | 0.1791 | 0.1458 | 0.1372 | 0.1331 | 97.4 % | (11.4457, 14.2521) |
| $\hat{\lambda}_{EE3}$ | 0.4387 | 0.4167 | 0.3903 | 0.2399 | 0.2102 | 0.1839 | 0.1527 | 0.1406 | 0.1357 | 98.7 % | (11.2301, 14.3229) |
| $\hat{\lambda}_{EP1}$ | 0.3830 | 0.3804 | 0.3702 | 0.2379 | 0.2121 | 0.1748 | 0.1367 | 0.1333 | 0.1310 | 93.1 % | (11.8072, 14.1385) |
| $\hat{\lambda}_{EP2}$ | 0.3920 | 0.3802 | 0.3747 | 0.2362 | 0.2083 | 0.1763 | 0.1414 | 0.1351 | 0.1317 | 96.9 % | (11.5367, 14.2634) |
| $\hat{\lambda}_{EP3}$ | 0.4135 | 0.3977 | 0.3789 | 0.2370 | 0.2078 | 0.1798 | 0.1474 | 0.1379 | 0.1335 | 94.5 % | (11.6169, 14.0357) |
| $\hat{\lambda}_{HS1}$ | 0.6005 | 0.4286 | 0.4392 | 0.3148 | 0.2663 | 0.1816 | 0.1879 | 0.1819 | 0.1651 | 90.1 % | (11.7812, 13.9065) |
| $\hat{\lambda}_{HS2}$ | 0.6861 | 0.4473 | 0.4210 | 0.3345 | 0.2789 | 0.1912 | 0.1983 | 0.1904 | 0.1734 | 96.3 % | (11.4715, 14.1431) |
| $\hat{\lambda}_{HS3}$ | 0.6861 | 0.4473 | 0.4310 | 0.3345 | 0.2789 | 0.1912 | 0.1992 | 0.1940 | 0.1743 | 96.0 % | (11.4920, 14.1226) |
| $\hat{\lambda}_{HE1}$ | 0.5910 | 0.4334 | 0.4138 | 0.2987 | 0.2568 | 0.2044 | 0.1770 | 0.1752 | 0.1704 | 96.1 % | (11.6000, 14.3119) |
| $\hat{\lambda}_{HE2}$ | 0.4987 | 0.4332 | 0.4138 | 0.2855 | 0.2503 | 0.1945 | 0.1756 | 0.1703 | 0.1686 | 98.9 % | (11.2829, 14.6215) |
| $\hat{\lambda}_{HE3}$ | 0.5535 | 0.4331 | 0.4139 | 0.2929 | 0.2537 | 0.2002 | 0.1743 | 0.1725 | 0.1658 | 90.9 % | (11.8395, 14.0574) |
| $\hat{\lambda}_{HP1}$ | 0.6668 | 0.4322 | 0.4173 | 0.3299 | 0.2759 | 0.1890 | 0.1968 | 0.1914 | 0.1723 | 90.7 % | (11.7994, 13.9757) |
| $\hat{\lambda}_{HP2}$ | 0.6200 | 0.4330 | 0.4187 | 0.2794 | 0.2505 | 0.1885 | 0.1760 | 0.1741 | 0.1739 | 94.9 % | (11.6174, 14.1429) |
| $\hat{\lambda}_{HP3}$ | 0.5110 | 0.4339 | 0.4202 | 0.2863 | 0.2502 | 0.1955 | 0.1710 | 0.1673 | 0.1604 | 94.1 % | (11.6524, 14.0932) |

Step 5: For creating the credible intervals, we first order $\lambda_1, \lambda_2, ..., \lambda_N$ as $\lambda_{(1)} < \lambda_{(2)} < ... < \lambda_{(N)}$ and $h_1, h_2, ..., h_N$ as $h_{(1)} < h_{(2)} < ... < h_{(N)}$. The $100(1-\gamma)$ symmetric credible intervals of $\lambda$ and reversed hazard rate are obtained respectively as $(\lambda_{(N\gamma/2)}, \lambda_{(N(1-\gamma/2))})$ and $(h_{(N\gamma/2)}, h_{(N(1-\gamma/2))})$.

The MSE, average credible intervals (ACI) and coverage probabilities (CP) of the estimators computed using the simulated data are reported in Tables 1 and 2.

From Tables 1 and 2, we have the following conclusions.

- For a fixed value of $n$ and $r$ the MSE is less for E-Bayesian estimators as compared to Bayesian and Hierarchical Bayesian estimators.
- The performance of the proposed estimators are better than Bayesian and Hierarchical Bayesian estimators in terms of MSE.

## 7. Real data set

To study the performance of the estimators derived in this article, for real life situations, we considered the real data set reported by Ma and Gui (2020) representing 23 deep-groove ball bearing failure times. We fit inverse Rayleigh distribution to the data and the corresponding p-value and test statistic value for the Kolmogorov-Smirnov test are 0.6942 and 0.1415 respectively. Using MLE we estimated $\hat{\lambda} = 0.2244$. Using the bootstrapping concept, we computed the MSE, average credible interval (ACI) and coverage probability (CP) of the estimators and are given in Tables 3 and 4.

**Table 2: MSE for Bayesian, E-Bayesian H-Bayesian estimates of $\hat{h(t)}$ for simulated data**

| | $n = 500$ | | | $n = 1000$ | | | $n = 1500$ | | | CP | ACI |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $r = 50$ | $r = 100$ | $r = 150$ | $r = 100$ | $r = 200$ | $r = 300$ | $r = 200$ | $r = 300$ | $r = 400$ | | |
| $\hat{h}_{B1}$ | 0.0152 | 0.0038 | 0.0011 | 0.0059 | 0.0018 | 0.0006 | 0.0022 | 0.0008 | 0.0006 | 92.1 % | (0.6645, 2.0725) |
| $\hat{h}_{B2}$ | 0.0157 | 0.0039 | 0.0012 | 0.0060 | 0.0018 | 0.0006 | 0.0023 | 0.0008 | 0.0006 | 97.5 % | (0.6519, 2.1202) |
| $\hat{h}_{B3}$ | 0.0150 | 0.0037 | 0.0010 | 0.0058 | 0.0018 | 0.0006 | 0.0022 | 0.0008 | 0.0005 | 96.8 % | (0.6577, 2.1129) |
| $\hat{h}_{ES1}$ | 0.0125 | 0.0031 | 0.0005 | 0.0053 | 0.0016 | 0.0004 | 0.0019 | 0.0007 | 0.0004 | 99.0 % | (0.6636, 2.2009) |
| $\hat{h}_{ES2}$ | 0.0130 | 0.0032 | 0.0006 | 0.0054 | 0.0016 | 0.0004 | 0.0020 | 0.0007 | 0.0004 | 91.4 % | (0.6783, 2.0970) |
| $\hat{h}_{ES3}$ | 0.0135 | 0.0033 | 0.0007 | 0.0055 | 0.0017 | 0.0005 | 0.0021 | 0.0007 | 0.0005 | 94.1 % | (0.6701, 2.1060) |
| $\hat{h}_{EE1}$ | 0.0127 | 0.0031 | 0.0005 | 0.0053 | 0.0016 | 0.0004 | 0.0020 | 0.0007 | 0.0004 | 95.8 % | (0.6737, 2.1367) |
| $\hat{h}_{EE2}$ | 0.0132 | 0.0032 | 0.0006 | 0.0054 | 0.0016 | 0.0005 | 0.0020 | 0.0007 | 0.0005 | 97.2 % | (0.6657, 2.1447) |
| $\hat{h}_{EE3}$ | 0.0139 | 0.0034 | 0.0008 | 0.0056 | 0.0017 | 0.0005 | 0.0021 | 0.0007 | 0.0005 | 92.4 % | (0.6707, 2.0890) |
| $\hat{h}_{EP1}$ | 0.0125 | 0.0031 | 0.0005 | 0.0053 | 0.0016 | 0.0004 | 0.0019 | 0.0007 | 0.0004 | 98.4 % | (0.6683, 2.1850) |
| $\hat{h}_{EP2}$ | 0.0128 | 0.0032 | 0.0006 | 0.0054 | 0.0016 | 0.0004 | 0.0020 | 0.0007 | 0.0004 | 97.4 % | (0.6680, 2.1550) |
| $\hat{h}_{EP3}$ | 0.0134 | 0.0033 | 0.0007 | 0.0055 | 0.0017 | 0.0005 | 0.0020 | 0.0007 | 0.0005 | 99.3 % | (0.6532, 2.1964) |
| $\hat{h}_{HS1}$ | 0.0110 | 0.0035 | 0.0020 | 0.0072 | 0.0021 | 0.0009 | 0.0032 | 0.0011 | 0.0007 | 95.3 % | (1.6906, 2.0910) |
| $\hat{h}_{HS2}$ | 0.0113 | 0.0040 | 0.0024 | 0.0076 | 0.0023 | 0.0008 | 0.0034 | 0.0012 | 0.0008 | 95.9 % | (1.6799, 2.0910) |
| $\hat{h}_{HS3}$ | 0.0106 | 0.0043 | 0.0028 | 0.0080 | 0.0024 | 0.0007 | 0.0037 | 0.0013 | 0.0009 | 94.1 % | (1.7096, 2.0915) |
| $\hat{h}_{HE1}$ | 0.0104 | 0.0035 | 0.0016 | 0.0069 | 0.0020 | 0.0008 | 0.0031 | 0.0010 | 0.0008 | 95.6 % | (1.7032, 2.1112) |
| $\hat{h}_{HE2}$ | 0.0104 | 0.0037 | 0.0013 | 0.0066 | 0.0019 | 0.0007 | 0.0028 | 0.0009 | 0.0007 | 98.0 % | (1.6711, 2.1423) |
| $\hat{h}_{HE3}$ | 0.0104 | 0.0042 | 0.0015 | 0.0068 | 0.0020 | 0.0009 | 0.0030 | 0.0010 | 0.0008 | 92.7 % | (1.7246, 2.0876) |
| $\hat{h}_{HP1}$ | 0.0107 | 0.0041 | 0.0023 | 0.0075 | 0.0023 | 0.0008 | 0.0034 | 0.0012 | 0.0008 | 99.7 % | (1.5973, 2.1971) |
| $\hat{h}_{HP2}$ | 0.0107 | 0.0037 | 0.0009 | 0.0064 | 0.0019 | 0.0006 | 0.0026 | 0.0010 | 0.0007 | 97.7 % | (1.6665, 2.1258) |
| $\hat{h}_{HP3}$ | 0.0108 | 0.0037 | 0.0013 | 0.0067 | 0.0019 | 0.0008 | 0.0029 | 0.0009 | 0.0008 | 98.4 % | (1.6518, 2.1383) |

It can also be noted that the estimators are satisfying the inequalities mentioned in Theorems 1 and 2. From the Tables, we can conclude that E-Bayesian estimators perform better than Bayesian and H-Bayesian estimators in terms of MSE.

## 8. Conclusion

The Bayesian, E-Bayesian and H-Bayesian techniques are used for estimating the parameter and reversed hazard rate of the inverse Rayleigh distribution based on left censoring. A real data and the Monte Carlo simulation are used for computing the estimates and the comparisons of these estimation methods are also carried out.Using E-Bayesian method we can see that the complex integrals involved in the calculation of hierarchical estimation methods are reduced to some extent. One of the important finding of the study is the close dependency of the proposed method with existing method and are established in Theorems 3 and 4. Another finding of the present study is the superiority of the proposed estimators with existing estimators. We also study the effect of various loss functions theoretically and are presented in Theorems 1 and 2. Important concluding remarks from our study are listed below:

1. Results showed that the MSE of the estimates decreases as the sample size increases.
2. The MSE of the E-Bayesian estimates of $\lambda$ is less than the MSE of the Bayesian and H-Bayesian estimates, so E-Bayesian estimators perform better than the other two existing estimation methods.
3. The MSE of Bayesian, H-Bayesian and E-Bayesian estimates decrease when r increases.

**Table 3: Comparison of MSE of the proposed estimators of $\lambda$ with Bayesian estimates for real data**

| | $n = 23$ | | | | CP | ACI |
|---|---|---|---|---|---|---|
| | $r = 2$ | $r = 4$ | $r = 8$ | $r = 12$ | | |
| $\hat{\lambda}_{B1}$ | 0.0106 | 0.0083 | 0.0054 | 0.0052 | 97.4 % | (0.0839, 0.4024) |
| $\hat{\lambda}_{B2}$ | 0.0115 | 0.0082 | 0.0048 | 0.0047 | 98.8 % | (0.0606, 0.4036) |
| $\hat{\lambda}_{B3}$ | 0.0103 | 0.0084 | 0.0059 | 0.0056 | 95.9 % | (0.0992, 0.3980) |
| $\hat{\lambda}_{ES1}$ | 0.0106 | 0.0085 | 0.0057 | 0.0054 | 93.0 % | (0.1134, 0.3768) |
| $\hat{\lambda}_{ES2}$ | 0.0106 | 0.0084 | 0.0056 | 0.0054 | 96.0 % | (0.0959, 0.3933) |
| $\hat{\lambda}_{ES3}$ | 0.0106 | 0.0084 | 0.0055 | 0.0053 | 98.8 % | (0.0630, 0.4252) |
| $\hat{\lambda}_{EE1}$ | 0.0115 | 0.0084 | 0.0049 | 0.0049 | 95.1 % | (0.0974, 0.3705) |
| $\hat{\lambda}_{EE2}$ | 0.0115 | 0.0083 | 0.0049 | 0.0048 | 97.2 % | (0.0816, 0.3853) |
| $\hat{\lambda}_{EE3}$ | 0.0115 | 0.0083 | 0.0049 | 0.0048 | 96.9 % | (0.0845, 0.3815) |
| $\hat{\lambda}_{EP1}$ | 0.0104 | 0.0087 | 0.0062 | 0.0058 | 97.8 % | (0.0804, 0.4208) |
| $\hat{\lambda}_{EP2}$ | 0.0103 | 0.0086 | 0.0061 | 0.0057 | 98.9 % | (0.0619, 0.4383) |
| $\hat{\lambda}_{EP3}$ | 0.0103 | 0.0086 | 0.0060 | 0.0057 | 97.5 % | (0.0844, 0.4148) |
| $\hat{\lambda}_{HS1}$ | 0.0106 | 0.0082 | 0.0053 | 0.0052 | 91.8 % | (0.1188, 0.3663) |
| $\hat{\lambda}_{HS2}$ | 0.0106 | 0.0081 | 0.0053 | 0.0051 | 92.4 % | (0.1163, 0.3677) |
| $\hat{\lambda}_{HS3}$ | 0.0106 | 0.0081 | 0.0053 | 0.0051 | 98.3 % | (0.0729, 0.4110) |
| $\hat{\lambda}_{HE1}$ | 0.0115 | 0.0085 | 0.0050 | 0.0050 | 91.6 % | (0.1140, 0.3558) |
| $\hat{\lambda}_{HE2}$ | 0.0115 | 0.0083 | 0.0049 | 0.0048 | 99.8 % | (0.0202, 0.4460) |
| $\hat{\lambda}_{HE3}$ | 0.0115 | 0.0083 | 0.0048 | 0.0047 | 93.7% | (0.1051, 0.3602) |
| $\hat{\lambda}_{HP1}$ | 0.0102 | 0.0083 | 0.0057 | 0.0054 | 91.3 % | (0.1234, 0.3714) |
| $\hat{\lambda}_{HP2}$ | 0.0104 | 0.0087 | 0.0061 | 0.0058 | 98.7 % | (0.0661, 0.4347) |
| $\hat{\lambda}_{HP3}$ | 0.0103 | 0.0085 | 0.0060 | 0.0056 | 96.7 % | (0.0925, 0.4059) |

4. The MSE of E-Bayesian estimates under ELF is less than the MSE of E-Bayesian estimates under SELF and PLF, so E-Bayesian estimators under ELF perform better than the E-Bayesian estimator SELF and PLF.

5. We can conclude that the E-Bayesian estimators perform better than Bayesian and H-Bayesian estimators in terms of MSE.

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**Table 4: Comparison of MSE of the proposed estimators of $h\bar{(t)}$ with Bayesian estimates for real data**

| | $n = 23$ | | | | CP | ACI |
|---|---|---|---|---|---|---|
| | $r = 2$ | $r = 4$ | $r = 8$ | $r = 12$ | | |
| $\hat{h}_{B1}$ | 60.6478 | 46.0936 | 42.6285 | 41.8922 | 91.9 % | (-10.9944, 19.0189) |
| $\hat{h}_{B2}$ | 55.1956 | 41.6762 | 37.7979 | 36.1012 | 98.4 % | (-15.9456, 23.6054) |
| $\hat{h}_{B3}$ | 63.4686 | 48.3812 | 45.1682 | 44.9786 | 95.9 % | (-13.8673, 22.0723) |
| $\hat{h}_{ES1}$ | 61.5741 | 46.7373 | 43.315 | 42.7344 | 99.0 % | (-18.2402, 26.3238) |
| $\hat{h}_{ES2}$ | 61.3408 | 46.5753 | 43.1418 | 42.5214 | 98.8 % | (-17.6651, 25.7339) |
| $\hat{h}_{ES3}$ | 61.1081 | 46.4137 | 42.9692 | 42.3093 | 99.3 % | (-19.234, 27.2879) |
| $\hat{h}_{EE1}$ | 56.0256 | 42.2482 | 38.3836 | 36.7883 | 91.1 % | (-10.185, 17.9011) |
| $\hat{h}_{EE2}$ | 55.8165 | 42.1042 | 38.2359 | 36.6144 | 97.1 % | (-14.1523, 21.8543) |
| $\hat{h}_{EE3}$ | 55.6081 | 41.9606 | 38.0886 | 36.4414 | 95.0 % | (-12.2928, 19.9807) |
| $\hat{h}_{EP1}$ | 64.4444 | 49.062 | 45.9071 | 45.9024 | 92.8 % | (-11.7803, 20.0455) |
| $\hat{h}_{EP2}$ | 64.1986 | 48.8906 | 45.7207 | 45.6688 | 99.4 % | (-20.1433, 28.3934) |
| $\hat{h}_{EP3}$ | 63.9536 | 48.7197 | 45.5350 | 45.4362 | 90.9 % | (-10.7881, 19.0231) |
| $\hat{h}_{HS1}$ | 60.3752 | 45.9039 | 42.4273 | 41.6472 | 99.9 % | (-24.2466, 32.2537) |
| $\hat{h}_{HS2}$ | 60.105 | 45.7157 | 42.228 | 41.4053 | 96.1 % | (-13.6962, 21.6858) |
| $\hat{h}_{HS3}$ | 59.8369 | 45.5289 | 42.0308 | 41.1667 | 97.0 % | (-14.5804, 22.5527) |
| $\hat{h}_{HE1}$ | 56.448 | 42.539 | 38.6829 | 37.1418 | 98.4 % | (-16.0768, 23.8213) |
| $\hat{h}_{HE2}$ | 55.6528 | 41.9915 | 38.1201 | 36.478 | 97.7 % | (-14.8781, 22.569) |
| $\hat{h}_{HE3}$ | 55.4419 | 41.8461 | 37.9714 | 36.3037 | 99.7 % | (-20.5672, 28.2438) |
| $\hat{h}_{HP1}$ | 62.8967 | 47.9816 | 44.7371 | 44.4445 | 92.7 % | (-11.6261, 19.7953) |
| $\hat{h}_{HP2}$ | 64.3447 | 48.9925 | 45.8313 | 45.8067 | 98.0 % | (-16.4345, 24.6936) |
| $\hat{h}_{HP3}$ | 63.7582 | 48.5834 | 45.3870 | 45.251 | 95.6 % | (-13.6303, 21.8532) |

## References


Abdul-Sathar, E. I. and Athirakrishnan, R. B. (2019). E-Bayesian and hierarchical Bayesian estimation for the shape parameter and reversed hazard rate of power function distribution under different loss functions. *Journal of the Indian Society for Probability and Statistics*, **20**, 227–253.

Dey, S. (2012). Bayesian estimation of the parameter and reliability function of an inverse Rayleigh distribution. *Malaysian Journal of Mathematical Sciences*, **6**, 113–124.

El-Helbawy, A. and Abd-El-Monem (2005). Bayesian estimation and prediction for the inverse rayleigh lifetime distribution. In *Proceeding of the 40th annual conference of Statistics, Computer sciences and Operation Research*, pages 45–59, ISSR, Cairo University.

Feroze, N. and Aslam, M. (2012). On posterior analysis of inverse Rayleigh distribution under singly and doubly type II censored data. *International Journal of Probability and Statistics*, **1**, 145–152.

Han, M. (1997). The structure of hierarchical prior distribution and its applications. *Chinese Operations Research and Management Science*, **6**, 31–40.

Han, M. (2009). E-Bayesian estimation and hierarchical Bayesian estimation of failure rate. *Applied Mathematical Modelling*, **33**, 1915–1922.

Jaheen, Z. F. and Okasha, H. M. (2011). E-Bayesian estimation for the Burr type XII model based on type-2 censoring. *Applied Mathematical Modelling*, **35**, 4730–4737.

Kızılaslan, F. (2017). The E-Bayesian and hierarchical Bayesian estimations for the proportional reversed hazard rate model based on record values. *Journal of Statistical Computation and Simulation*, **87**, 2253–2273.

Lindley, D. V. and Smith, A. F. (1972). Bayes estimates for the linear model. *Journal of the Royal Statistical Society. Series B (Methodological)*, **34**, 1–41.

Ma, Y. and Gui, W. (2020). Entropy-based and non-entropy-based goodness of fit test for the inverse Rayleigh distribution with progressively type-II censored data. *Probability in the Engineering and Informational Sciences*, **35**, 1–19.

Okasha, H. M. and Wang, J. (2016). E-Bayesian estimation for the geometric model based on record statistics. *Applied Mathematical Modelling*, **40**, 658–670.

Shawky, A. and Badr, M. (2012). Estimations and prediction from the inverse Rayleigh model based on lower record statistics. *Life Science Journal*, **9**, 985–990.

Soliman, A., Amin, E. A., and Abd-El Aziz, A. A. (2010). Estimation and prediction from inverse Rayleigh distribution based on lower record values. *Applied Mathematical Sciences*, **4**, 3057–3066.

Voda, V. G. (1972). On the inverse Rayleigh distributed random varaible. *Reports of Statistical Application Research*, **19**, 13–21.