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The Transmuted Inverted Nadarajah-Haghighi Distribution: Different Estimation Methods and Applications

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Abstract

In this paper, a new inverted model called the transmuted inverted Nadarajah-Haghighi distribution is introduced. Different estimation methods of the unknown parameters of the new distribution are utilized. These methods are maximum likelihood (MLE), least squares and weighted least squares, maximum product spacing estimation, AD and RAD estimation, CVM estimation and Bayesian estimation. Also, the potentiality of the new model is discussed via a real data set.

Key words: Inverted distributions; Nadarajah-Haghighi distribution; Maximum likelihood estimation (MLE); Least squares and weighted least squares estimation; Maximum product spacing estimation; AD and RAD estimation; CVM estimation; Bayesian estimation.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

Two-Parameter Nadarajah-Haghighi (NH) distribution was introduced by Nadarajah and Haghighi (2011) as an extension of exponential distribution and also as an alternative to the gamma, Weibull and exponentiated exponential distributions. They provided three motivations for introducing their distribution, for more details see their paper. Let Z to have Nadarajah-Haghighi distribution, $Z \sim NH(\alpha, \beta)$, then the cdf of Z takes the form

$$F(z) = 1 - e^{1-(1+\beta z)^{\alpha}}, \quad z > 0, \alpha, \beta > 0, \tag{1}$$

where β is the scale parameter and α is the shape parameter. When $\alpha = 1$, the exponential distribution is obtained. Nadarajah and Haghighi (2011) showed that its density can take decreasing and unimodal shapes and the hazard rate can take increasing, constant and decreasing shapes. In order to provide some flexibility, alternative generalizations of the Nadarajah and Haghighi distribution have been proposed. For example, Lemonte *et al.* (2015) introduced the Marshall-Olkin Nadarajah Haghighi distribution via the Marshall-Olkin generator (Marshall and Olkin, 1997). Its cdf is given by

$$F(z) = \frac{1 - e^{1-(1+\beta z)^{\alpha}}}{(1 - (1-\theta)e^{1-(1+\beta z)^{\alpha}})}, \quad z > 0,$$

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where the parameter $\alpha > 0$ and $\theta > 0$ control the shapes of the distribution, and the parameter $\beta > 0$ is the scale parameter. If $\beta = 1$, the NH distribution is obtained. They noted that this distribution is quite flexible and can be used effectively in modeling survival data, reliability problems, fatigue life studies and hydrological data. Also, it can have constant, decreasing, increasing, upside-down bathtub (unimodal), bathtub-shaped and decreasing-increasing-decreasing hazard rate functions.

Yousof and Karkmaz (2017) introduced the Topp-Leone Nadarajah-Haghighi model using the Topp-Leone generated family of distributions (Sangsanit and Bodhisuwan, 2016). If a random variable Z follows the Topp-Leone Nadarajah-Haghighi distribution, then its cdf of Z takes the form

$$F(z) = (1 - e^{2(1-(1+\beta z)^{\alpha})})^{\theta}, \quad z > 0, \alpha, \beta, \theta > 0,$$

They provided some plots of the pdf and hazard rate function for the distribution and showed that its hazard function allows different shapes.

Ogunde *et al.* (2017) introduced transmuted Nadarajah-Haghighi distribution as another generalization of Nadarajah-Haghighi distribution. Its cdf takes the form

$$F(z) = (1 - e^{1-(1+\beta z)^{\alpha}})(1 + \lambda e^{1-(1+\beta z)^{\alpha}}), \quad z > 0, \alpha, \beta > 0 \text{ and } |\lambda| \leq 1.$$

They showed that its hazard function allows different shapes such as decreasing and bathtub shapes.

On the other hand inverted distributions of random variables with positive support provide a valuable alternative for the regular distributions when the assumptions for the use of these distributions are not valid. Also, they may be used in Bayesian analysis of prior and posterior distribution of some parameters such as the scale parameter. Sheikh and Ahmed (1987) discussed characteristic features of the hazard functions based on this inverted class of distributions and explored their possible uses. Hazard functions and mean residual life of inverted normal inverted Gamma and inverted Weibull are compared with the normal, Gamma and Weibull hazards. For a general discussion of inverted distributions, see Folks (1983), Lehmann and Shaffer (1988) and Habibullah and Ahmed (2006).

Some authors discussed the inverse transformation method of baseline variables to obtain inverted distributions due to its usefulness to explore additional properties of the phonomenons which non inverted distributions cannot. Some of these distributions are: inverse exponential distribution (Keller and Kamath in 1982), inverse Rayleigh distribution (Voda in 1972), inverse Lindley distribution (Sharma *et al.*, 2015), inverted Nadarajah-Haghighi (Tahir *et al.*, 2018), inverse xgamma (Yadav *et al.*, 2019) etc.

Furthermore, some authors used the quadratic rank transmutation map (QRTM) approach to generate a generalization of an inverted distribution such as: Mahmoud and Mandouh (2013); Elbatal (2013); Khan (2019) ect. According to this approach, a random variable Z is said to have a transmuted distribution if its cumulative distribution function (cdf) satisfies the following relationship:

$$G(x) = (1 + \lambda)F(x) - \lambda F(x)^2, |\lambda| \leq 1, \tag{2}$$

where F(x) is the cdf of the baseline model and the corresponding probability density function takes the form:

$$g(x) = f(x)[(1+\lambda) - 2\lambda F(x)], |\lambda| \leq 1$$

(see Shaw and Buckley (2009)). The same approach has been used to introduce the transmuted form of inverted Nadarajah-Haghighi distribution.

2. The Transmuted Inverted N-H Distribution

Let the random variable X=1/Z, where Z follows the NH distribution whose cdf is given in (1), then cdf of the inverted N-H distribution takes the form

$$F(x) = e^{1-(1+\beta x^{-1})^{\alpha}}, \quad x > 0, \alpha, \beta > 0.$$

Using (2) and taking the inverted N-H distribution as the base distribution, one can generate the cdf of the transmuted inverted N-H (TINH) distribution as follows

$$G(x) = e^{1-(1+\beta x^{-1})^{\alpha}}(1+\lambda-\lambda e^{1-(1+\beta x^{-1})^{\alpha}}), \quad x > 0, \alpha, \beta > 0 \text{ and } |\lambda| \leq 1. \tag{3}$$

The corresponding pdf and hazard function (failure rate function) are given respectively

$$g(x) = \alpha\beta x^{-2}(1+\beta x^{-1})^{\alpha-1} e^{1-(1+\beta x^{-1})^{\alpha}}(1+\lambda-\lambda e^{1-(1+\beta x^{-1})^{\alpha}}), \tag{4}$$

and

$$h(x) = \frac{\alpha\beta x^{-2}(1+\beta x^{-1})^{\alpha-1} e^{1-(1+\beta x^{-1})^{\alpha}}(1+\lambda-\lambda e^{1-(1+\beta x^{-1})^{\alpha}})}{1-e^{1-(1+\beta x^{-1})^{\alpha}}(1+\lambda-\lambda e^{1-(1+\beta x^{-1})^{\alpha}})}. \tag{5}$$

The new distribution is flexible to model positive real data sets which display decreasing and upside-down bathtub (UBT) hazard rate shapes. Some plots of density and hazard functions are displayed in Figures (1) and (2) for different values of the parameters. In Figure (1), the plots indicate that the TINH density can be decreasing and unimodal. The plots in Figure (2) show that the TINH hazard function can be decreasing and UBT. The new distribution has no finite moments.

The inverse of the cumulative function (3) yields the following quantile function

$$Q(u) = \beta((1 - ln(\frac{1+\lambda-\sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}))^{1/\alpha} - 1)^{-1}, \quad u \in (0,1) \tag{6}$$

The specification of a distribution through its quantile function takes away the need to describe a distribution through its moments. The following alternative measures in terms of quantiles that reduce the shortcomings of the moment-based ones:
The median as a measure of location is defined by

$$M = Q(0.5) = \beta((1 - ln(\frac{1+\lambda-\sqrt{(1+\lambda)^2 - 2\lambda}}{2\lambda}))^{1/\alpha} - 1)^{-1}.$$

The interquartile range as a measure of dispersion is defined by

$$IQR = Q_3 - Q_1 = Q(0.75) - Q(0.25).$$

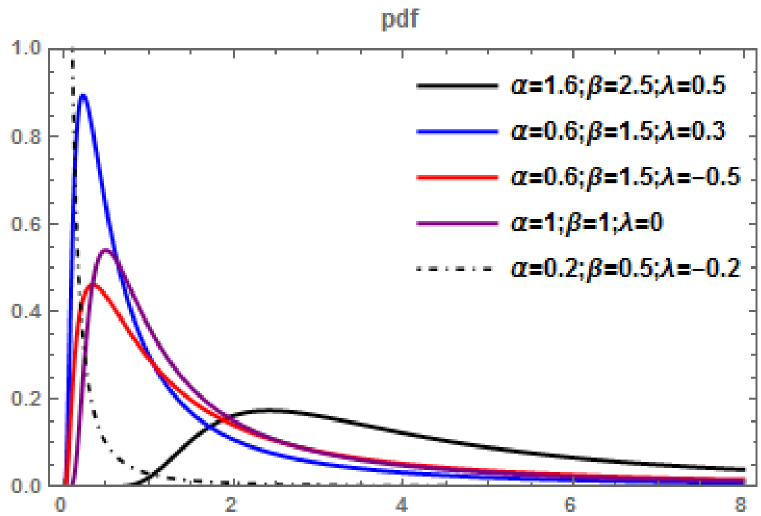

Figure 1: Plots of the TINH density for different parameter values

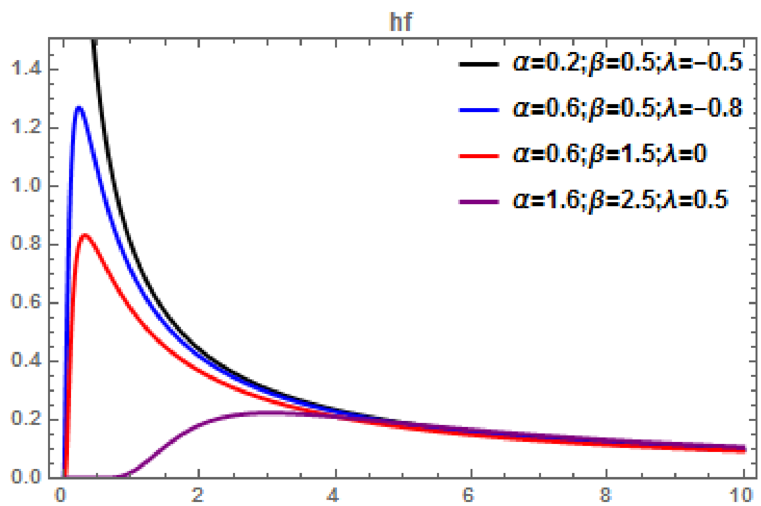

Figure 2: Plots of the TINH hazard rate for different parameter values

Skewness is measured by Galton's coefficient

$$Sk = \frac{Q_3 + Q_1 - 2M}{Q3 - Q1}. \tag{7}$$

Moors, 1988 proposed the measure

$$Ku = (Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125))/IQR \tag{8}$$

as a measure of kurtosis. Given the form of $Q(u)$, the calculations of all the coefficients are very simple, as one needs to only substitute the appropriate fractions for u. For example, one can use formulas (6), (7) and (8) to calculate the skewness and kurtosis for the TINH distribution. Table 1 shows the skewness and kurtosis of the TINH distribution for different values of parameters. One can note that for fixed values of α, the skewness and kurtosis decrease as λ approaching to 1 and for fixed values of λ, the skewness and kurtosis decrease as α increases. Also, for generating random numbers from the TINH distribution, one can use formula (6). Quantile functions have several interesting properties that are not shared by distributions, which makes it more convenient for analysis. For more details see Nair *et al.*,2013.

Table 1: Skewness and Kurtosis of the TINH distribution for some values of parameters

Sk	$\alpha\|\lambda$	-1	-0.8	-0.4	0.4	0.8	1
	0.5	0.4873	0.4889	0.4988	0.4827	0.4171	0.3770
	0.8	0.4788	0.4778	0.4810	0.4518	0.3807	0.3388
	1.0	0.4763	0.4751	0.4770	0.4443	0.3719	0.3295
	1.5	0.4742	0.4725	0.4721	0.4368	0.3629	0.3201
	3.0	0.4729	0.4710	0.4706	0.4322	0.3575	0.3144
	5.0	0.4727	0.4706	0.4700	0.4312	0.3563	0.3132
Ku	$\alpha\|\lambda$	-1	-0.8	-0.4	0.4	0.8	1
	0.5	2.1593	2.1621	2.1766	2.1347	1.8452	1.6651
	0.8	2.1450	2.1445	2.1481	2.0693	1.7814	1.6109
	1.5	2.1417	2.1406	2.1417	2.0547	1.7675	1.5993
	1.5	2.1386	2.1367	2.1354	2.0405	1.7541	1.5882
	3.0	2.1367	2.1344	2.1317	2.0321	1.7463	1.5817
	5.0	2.1363	2.1339	2.1309	2.0304	1.7446	1.5804

Note: neither skewness nor kurtosis of the TINH distribution depends on the value of β.

3. Non-Bayesian Estimation Methods

Here, we use different methods for estimating the parameters of the TINH distribution (α, β and λ). These methods are maximum likelihood estimation (MLE), least squares and weighted least squares estimation, maximum product spacing estimation, AD and RAD estimation and CVM estimation.

3.1. Maximum likelihood estimation (mle)

The mle is the most popular technique for obtaining estimators and it has desirable properties such as constructing confidence intervals. Now, we consider $X_1, X_2, \ldots, X_n$ as a random sample from TINH distribution, defined in (3), with observed values $x_1, x_2, \ldots, x_n$. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (\alpha, \beta, \lambda)^t$ can be expressed by

$$\begin{aligned} l &= nln(\alpha) + nln(\beta) - 2\sum_{i=1}^{n} lnx_i + (\alpha-1)\sum_{i=1}^{n} ln(1+\beta x_i^{-1}) + \sum_{i=1}^{n}(1-(1+\beta x_i^{-1})^{\alpha}) \\ &+ \sum_{i=1}^{n} ln(1+\lambda - 2\lambda e^{(1-(1+\beta x_i^{-1})^{\alpha})}). \end{aligned} \quad (9)$$

The components of the score vector $U(\boldsymbol{\theta})$ take the forms

$$\begin{aligned} U_\alpha &= n/\alpha + \sum_{i=1}^{n} ln(1+\beta x_i^{-1}) + \sum_{i=1}^{n}(1+\beta x_i^{-1})^{\alpha} ln(1+\beta x_i^{-1}) \\ &+ \sum_{i=1}^{n} \frac{2\lambda e^{(1-(1+\beta x_i^{-1})^{\alpha})}(1+\beta x_i^{-1})^{\alpha}) ln(1+\beta x_i^{-1})}{(1+\lambda-2\lambda e^{(1-(1+\beta x_i^{-1})^{\alpha})})}, \\ U_\beta &= n/\beta + (\alpha-1)\sum_{i=1}^{n} x_i^{-1} ln(1+\beta x_i^{-1}) - \alpha \sum_{i=1}^{n} x_i^{-1}(1+\beta x_i^{-1})^{\alpha-1} \qquad (10) \\ &+ \sum_{i=1}^{n} \frac{2\lambda\alpha x_i^{-1} e^{(1-(1+\beta x_i^{-1})^{\alpha})}(1+\beta x_i^{-1})^{\alpha-1}}{(1+\lambda-2\lambda e^{(1-(1+\beta x_i^{-1})^{\alpha})})}, \\ U_\lambda &= \frac{1-2e^{(1-(1+\beta x_i^{-1})^{\alpha})}}{(1+\lambda-2\lambda e^{(1-(1+\beta x_i^{-1})^{\alpha})})}. \end{aligned}$$

Equating formulas in (10) to zero and solving them simultaneously yield the mle estimates of the unknown parameters. To construct confidence interval of the model parameter, this requires the 3×3 observed information matrix $J(\boldsymbol{\theta}) = -J_{sk}$, for $s, k = \alpha, \beta, \lambda$ and $\boldsymbol{\theta} = (\alpha, \beta, \lambda)^t$, whose elements are obtained by taking the second derivative of (9). In the observed information matrix, we replace the model parameters by its mles. Maximum likelihood estimation of the model parameters may be difficult to obtain in certain cases-particularly where the support of the model is unknown. Moreover the mle may not be robust to departures from the assumed model. These considerations motivated the following estimation methods described below.

3.2. Minimum distance estimation

In this section, we use some methods of estimation based on minimum distance between the cdf of TINH distribution and the empirical cdf. These methods are divided into two approaches; the first group is known as least-square approach; the second group is related to goodness-of-fit statistics.

3.2.1.Least-square approach (LSE)

Swain *et al.* (1988) used least-square approach to parameter estimation to summarize a set of data by a distribution function in Johnson's translation system. They investigated this approach via minimizing the distance between the vector of "uniformized" order statistics and the corresponding vector of expected values. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n with the cdf $F(.)$ in (3) and let $x_{(1:n)} < x_{(2:n)} < \cdots < x_{(n:n)}$ be the ordered observations. The LSEs of α, β and λ, say $\hat{\alpha}_{LSE}, \hat{\beta}_{LSE}$ and $\hat{\lambda}_{LSE}$, can be obtained by minimizing the following formula with respect to α, β, λ.

$$Dls(\alpha, \beta, \lambda) = \sum_{i=1}^{n}(F(x_{(i:n)}; \alpha, \beta, \lambda) - \frac{i}{n+1})^2$$

Also, one can determine these estimators by solving

$$\sum_{i=1}^{n}(F(x_{i:n}; \alpha, \beta, \lambda) - \frac{i}{n+1})\rho_1(x_{i:n}; \alpha, \beta, \lambda) = 0,$$

$$\sum_{i=1}^{n}(F(x_{i:n};\alpha,\beta,\lambda)-\frac{i}{n+1})\rho_2(x_{i:n};\alpha,\beta,\lambda)=0,$$

and

$$\sum_{i=1}^{n}(F(x_{i:n};\alpha,\beta,\lambda)-\frac{i}{n+1})\rho_3(x_{i:n};\alpha,\beta,\lambda)=0,$$

where

$$\begin{aligned}\rho_1(x_{i:n};\alpha,\beta,\lambda) &= (1+\beta x^{-1})^{\alpha} ln(1+\beta x^{-1})e^{1-(1+\beta x^{-1})^{\alpha}}\\ &* (-(1+\lambda)+2\lambda e^{1-(1+\beta x^{-1})^{\alpha}}), \qquad (11)\\ \rho_2(x_{i:n};\alpha,\beta,\lambda) &= \alpha(1+\beta x^{-1})^{\alpha-1}x^{-1}e^{1-(1+\beta x^{-1})^{\alpha}}(-(1+\lambda)+2\lambda e^{1-(1+\beta x^{-1})^{\alpha}}), \qquad (12)\end{aligned}$$

and

$$\rho_3(x_{i:n};\alpha,\beta,\lambda) = e^{1-(1+\beta x^{-1})^{\alpha}}(1-e^{1-(1+\beta x^{-1})^{\alpha}}). \qquad (13)$$

Weighted least-square estimators, $\hat{\alpha}_{WLSE}, \hat{\beta}_{WLSE}$ and $\hat{\lambda}_{WLSE}$, can be determined by minimizing (see Tahir *et al.*, 2018)

$$W(\alpha,\beta,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}(F(x_{(i:n)};\alpha,\beta,\lambda)-\frac{i}{n+1})^2$$

Also, one can obtain these estimators by solving

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}(F(x_{i:n};\alpha,\beta,\lambda)-\frac{i}{n+1})\rho_1(x_{i:n};\alpha,\beta,\lambda)=0,$$

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}(F(x_{i:n};\alpha,\beta,\lambda)-\frac{i}{n+1})\rho_2(x_{i:n};\alpha,\beta,\lambda)=0,$$

and

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}(F(x_{i:n};\alpha,\beta,\lambda)-\frac{i}{n+1})\rho_3(x_{i:n};\alpha,\beta,\lambda)=0,$$

where $\rho_i(.;\alpha,\beta,\lambda), i=1,2,3$ are given by (11)-(13).

3.2.2.The approach based on the goodness-of-fit statistics

Anderson and Darling (1952) proposed a general class of tests for testing the goodness of fit of a sample of n observations to a specified continuous distribution function $F(x)$. Their test was based on the difference between the specified distribution and the empirical distribution $F_n(x)$ of the sample. From the following measure

$$W_n^2 = n\int_{-\infty}^{\infty}(F_n(x)-GF(x))^2\psi(F(x))dF, \qquad (14)$$

where $\psi(t)(\geq 0)$ is some preassigned weight function., when $\psi(t) = 1, W_n^2$ reduces to nw^2, where w^2 is the Cramér-von-Mises test statistic (see Anderson and Darling (1954). Put $\psi(t) = 1/t(1-t)$ in (14) W_n^2 reduces to the statistic A_n^2 which was studied by (Anderson and Darling (1952, 1954)). Here, we estimate the parameters of TINH distribution based on minimization of the goodness-of-fit statistics with respect to α, β and λ. These statistics are Cramér-von-Mises; Anderson-Darling. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n with the cdf G(.) in (3) and let $x_{(1:n)} < x_{(2:n)} < \cdots < x_{(n:n)}$ be the ordered observations. After computing the last integration, the formulae of the two statistics will be obtained in (15) and (16).

Cramér-von Mises (CVM) estimation

The CVM estimators of α, β and λ, say $\hat{\alpha}_{CVM}, \hat{\beta}_{CVM}$ and $\hat{\lambda}_{CVM}$, can be obtained by minimizing the following formula with respect to α, β, λ.(see MacDonald, 1971)

$$CM(\alpha, \beta, \lambda) = \frac{1}{12n} + \sum_{i=1}^{n}(F(x_{(i:n)}; \alpha, \beta, \lambda) - \frac{2i-1}{2n})^2. \tag{15}$$

Also, one can obtain these estimators by solving the following non-linear equations

$$\sum_{i=1}^{n}(F(x_{i:n}; \alpha, \beta, \lambda) - \frac{2i-1}{2n})\rho_1(x_{i:n}; \alpha, \beta, \lambda) = 0,$$

$$\sum_{i=1}^{n}(F(x_{i:n}; \alpha, \beta, \lambda) - \frac{2i-1}{2n})\rho_2(x_{i:n}; \alpha, \beta, \lambda) = 0,$$

and

$$\sum_{i=1}^{n}(F(x_{i:n}; \alpha, \beta, \lambda) - \frac{2i-1}{2n})\rho_3(x_{i:n}; \alpha, \beta, \lambda) = 0,$$

where $\rho_i(.; \alpha, \beta, \lambda), i = 1, 2, 3$ are given by (11)-(13).

Anderson-Darling estimation

The AD estimators of α, β and λ, say $\hat{\alpha}_{AD}, \hat{\beta}_{AD}$ and $\hat{\lambda}_{AD}$, can be obtained by minimizing the following formula with respect to α, β, λ.(see MacDonald, 1971)

$$AD(\alpha, \beta, \lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)(log(F(x_{i:n}; \alpha, \beta, \lambda)) + log(\bar{F}(x_{n+1-i:n}; \alpha, \beta, \lambda)), \tag{16}$$

where $\bar{F}(x) = 1 - F(x)$. Also, one can obtain these estimators by solving the following non-linear equations

$$\sum_{i=1}^{n}(2i-1)(\frac{\rho_1(x_{i:n}; \alpha, \beta, \lambda)}{F(x_{i:n}; \alpha, \beta, \lambda)} - \frac{\rho_1(x_{n+1-i:n}; \alpha, \beta, \lambda)}{\bar{F}(x_{n+1-i:n}; \alpha, \beta, \lambda)}) = 0,$$

$$\sum_{i=1}^{n}(2i-1)(\frac{\rho_2(x_{i:n}; \alpha, \beta, \lambda)}{F(x_{i:n}; \alpha, \beta, \lambda)} - \frac{\rho_2(x_{n+1-i:n}; \alpha, \beta, \lambda)}{\bar{F}(x_{n+1-i:n}; \alpha, \beta, \lambda)}) = 0,$$

and

$$\sum_{i=1}^{n}(2i-1)(\frac{\rho_3(x_{i:n}; \alpha, \beta, \lambda)}{F(x_{i:n}; \alpha, \beta, \lambda)} - \frac{\rho_3(x_{n+1-i:n}; \alpha, \beta, \lambda)}{\bar{F}(x_{n+1-i:n}; \alpha, \beta, \lambda)}) = 0.$$

where $\rho_i(.;\alpha,\beta,\lambda), i = 1,2,3$ are given by (11)-(13).

The Right-tail Anderson-Darling (RAD) estimators of α, β and λ, say $\hat{\alpha}_{RAD}, \hat{\beta}_{RAD}$ and $\hat{\lambda}_{RAD}$, can be obtained by minimizing the formula (17) with respect to α, β, λ. (see Tahir *et al.* 2018)

$$RAD(\alpha,\beta,\lambda) = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{i:n};\alpha,\beta,\lambda) - \frac{1}{n}(2i-1)log(\bar{F}(x_{n+1-i:n};\alpha,\beta,\lambda)), \tag{17}$$

where $\bar{F}(x) = 1 - F(x)$. Also, these estimators can be obtained by solving the following non-linear equations

$$-2\sum_{i=1}^{n} \rho_1(x_{i:n};\alpha,\beta,\lambda) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\rho_1(x_{n+1-i:n};\alpha,\beta,\lambda)}{\bar{F}(x_{n+1-i:n};\alpha,\beta,\lambda)} = 0,$$

$$-2\sum_{i=1}^{n} \rho_2(x_{i:n};\alpha,\beta,\lambda) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\rho_2(x_{n+1-i:n};\alpha,\beta,\lambda)}{\bar{F}(x_{n+1-i:n};\alpha,\beta,\lambda)} = 0,$$

and

$$-2\sum_{i=1}^{n} \rho_3(x_{i:n};\alpha,\beta,\lambda) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\rho_3(x_{n+1-i:n};\alpha,\beta,\lambda)}{\bar{F}(x_{n+1-i:n};\alpha,\beta,\lambda)} = 0.$$

where $\rho_i(.;\alpha,\beta,\lambda), i = 1,2,3$ are given by (11)-(13).

3.3. Maximum product of spacing (MPS) estimation

This approach was introduced using two methods. The first was by Cheng and Amin (1983) via the idea of spacings. They proposed it as a general method of estimating parameters in continuous univariate distributions. They studied some properties of their approach such as efficiency; consistency and others. Also, they compared it with the mle method via some examples. The second was introduced by Ranneby (1984) who used an approximation of Kullback-Leibler information like the mle method to derive this approach.
Let $x_1, x_2, \ldots, x_n$ be a random sample of size n with the cdf $G(.)$ in (2.1) and let $x_{(1:n)} < x_{(2:n)} < \cdots < x_{(n:n)}$ be the ordered observations. The uniform spacings of the sample is defined as

$$D_i(\alpha,\beta,\lambda) = G(x_{(i:n)};\alpha,\beta,\lambda) - G(x_{(i-1:n)};\alpha,\beta,\lambda), \quad i = 1,2,\cdots,n, \tag{18}$$

where $G(x_{(0:n)};\alpha,\beta,\lambda) = 0, G(x_{(n+1:n)};\alpha,\beta,\lambda) = 1$ and $\sum_{i=1}^{n+1} D_i(\alpha,\beta,\lambda) = 1$.

The maximum product of spacings estimators $\hat{\alpha}_{MPS}, \hat{\beta}_{MPS},$ and $\hat{\lambda}_{MPS}$ of the parameters α, β and λ are obtained by maximizing the geometric mean of the spacings with respect to α, β and λ , i.e. maximizing $(\prod_{i=1}^{n+1} D_i(\alpha,\beta,\lambda))^{(1/(n+1))}$. Or, equivalently, maximizing the function

$$\mathcal{D}(\alpha,\beta,\lambda) = \frac{1}{n+1}\sum_{i=1}^{n+1} lnD_i(\alpha,\beta,\lambda).$$

The estimators $\hat{\alpha}_{MPS}, \hat{\beta}_{MPS},$ and $\hat{\lambda}_{MPS}$ of the parameters α, β and λ are obtained by solving the following non-linear equations

$$\frac{\partial}{\partial\alpha}\mathcal{D}(\alpha,\beta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta,\lambda)}(\rho_1(x_{i:n};\alpha,\beta,\lambda)-\rho_1(x_{i-1:n};\alpha,\beta,\lambda))=0,$$

$$\frac{\partial}{\partial\beta}\mathcal{D}(\alpha,\beta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta,\lambda)}(\rho_2(x_{i:n};\alpha,\beta,\lambda)-\rho_2(x_{i-1:n};\alpha,\beta,\lambda))=0,$$

and

$$\frac{\partial}{\partial\lambda}\mathcal{D}(\alpha,\beta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta,\lambda)}(\rho_3(x_{i:n};\alpha,\beta,\lambda)-\rho_3(x_{i-1:n};\alpha,\beta,\lambda))=0,$$

where $\rho_i(.;\alpha,\beta,\lambda), i=1,2,3$ are given by (11)-(13).

4. Numerical Study for Different Estimation Methods

Now, a numerical study is carried out to compare the performance of the frequentist estimators discussed above. To do this we compute absolute value of relative bias (ARbias), scaled root mean square error (SRMSE), average absolute (D_{abs}) and maximum absolute (D_{max}) differences between the theoretical and empirical distribution function at the estimate values (see Tahir *et al.*, 2018). The formulas of these statistics take the forms:

$$ARbias(\hat{\theta})=|bias(\hat{\theta})|/\theta, \quad bais(\hat{\theta})=\frac{1}{m}\sum_{i=1}^{m}(\hat{\theta}_i-\theta),$$

$$SRMSE(\hat{\theta})=\sqrt{MSE(\hat{\theta})}/\theta, \quad MSE(\hat{\theta})=\frac{1}{m}\sum_{i=1}^{m}(\hat{\theta}_i-\theta)^2,$$

$$D_{abs}=\frac{1}{nm}\sum_{i=1}^{m}\sum_{j=1}^{n}|F(x_{ij},\theta)-F(x_{ij},\hat{\theta})|, \quad D_{max}=\frac{1}{m}\sum_{i=1}^{m}\max_{j}|F(x_{ij},\theta)-F(x_{ij},\hat{\theta})|,$$

and $\sum Ranks$ gives the partial sum of the ranks. A superscript indicates the rank of each of the estimators for that metric. For example, Table 2 shows the ARbias of the MLE ($\hat{\alpha}$) as 0.072^7 for n=30. This indicates that the ARbias of ($\hat{\alpha}$)obtained using the method of ML ranks 7th among all other estimators. For different sample sizes (n=30, 50, 100, 150), we generate (m=1000) random samples from TINH distribution with parameters $\alpha=2,0.8,\beta=1.5,0.5$ and $\lambda=0.5,-0.5$. The results are reported in Tables 2-5 and one can note that in most cases the ARbias and SRMSE of all estimators decrease when the sample size increases. Also, D_{abs} is smaller than D_{max} for all estimation methods and these statistics are smaller when n increases. According to $\sum Ranks$, CVM and LS are the best compared to the other methods. Although the results are not reported here, we also performed simulation study by taking several different values of λ. The trend of the results are quite similar as reported in Table 2 through 5.

Table 2: Numerical results for $\alpha = 2, \beta = 0.5,$ and $\lambda = 0.5$

n	Estimate	MLE	MPS	CVM	AD	RAD	LS	WLS
30	ARbias(α)	0.0721^{7}	0.0317^{6}	0.0110^{1}	0.0223^{3}	0.0157^{2}	0.0228^{4}	0.0234^{5}
	SRMSE($\hat{\alpha}$)	0.2674^{7}	0.2165^{6}	0.1475^{1}	0.1875^{5}	0.1784^{4}	0.1541^{2}	0.1692^{3}
	ARbias($\hat{\beta}$)	0.0482^{2}	0.1013^{5}	0.0384^{1}	0.1140^{7}	0.1104^{6}	$0.0574^{3.5}$	$0.0574^{3.5}$
	SRMSE($\hat{\beta}$)	0.3528^{7}	0.3312^{6}	0.1951^{1}	0.2923^{5}	0.2700^{4}	0.2089^{2}	0.2142^{3}
	ARbias($\hat{\lambda}$)	0.1519^{6}	0.1671^{7}	0.0569^{3}	0.0846^{4}	0.0980^{5}	0.0401^{2}	0.0055^{1}
	SRMSE($\hat{\lambda}$)	0.5368^{6}	0.6072^{7}	0.4622^{3}	0.4258^{1}	0.4643^{4}	0.4475^{2}	0.4713^{5}
	D_{abs}	0.0364^{7}	0.0193^{4}	0.0141^{3}	0.0215^{5}	0.0232^{6}	0.0096^{2}	0.0096^{1}
	$Dmax$	0.0630^{6}	0.0689^{7}	0.0238^{3}	0.0353^{4}	0.0403^{5}	0.0177^{2}	0.0172^{1}
	$\sum Ranks$	$48^{6.5}$	$48^{6.5}$	16^{1}	34^{4}	36^{5}	19.5^{2}	22.5^{3}
50	ARbias($\hat{\alpha}$)	0.0412^{6}	0.2977^{7}	0.0119^{2}	0.0153^{5}	0.01369^{3}	0.0099^{1}	0.0150^{4}
	SRMSE($\hat{\alpha}$)	0.2311^{7}	0.1874^{6}	01296^{1}	0.1544^{4}	0.15624^{5}	0.1302^{2}	0.1429^{3}
	ARbias($\hat{\beta}$)	0.0605^{6}	0.0881^{7}	0.0175^{1}	0.0774^{2}	0.0793^{3}	$0.0270^{4.5}$	$0.0270^{4.5}$
	SRMSE($\hat{\beta}$)	0.3372^{7}	0.3070^{6}	0.1486^{1}	0.2290^{5}	0.2124^{4}	0.1507^{2}	0.1772^{3}
	ARbias($\hat{\lambda}$)	0.1682^{7}	0.1548^{6}	0.0732^{3}	0.1118^{4}	0.1219^{5}	0.0039^{1}	0.0443^{2}
	SRMSE($\hat{\lambda}$)	0.4760^{6}	0.5103^{7}	0.4140^{5}	0.3786^{1}	0.4139^{4}	0.3931^{2}	0.4130^{3}
	D_{abs}	0.0310^{7}	0.0164^{4}	0.0102^{3}	0.0168^{5}	0.0181^{6}	0.0034^{1}	0.0087^{2}
	$Dmax$	0.0729^{7}	0.0680^{6}	0.0322^{3}	0.0488^{4}	0.0532^{5}	0.0059^{1}	0.0193^{2}
	$\sum Ranks$	53^{7}	49^{6}	19^{2}	30^{4}	35^{5}	14.5^{1}	23.5^{3}
100	ARbias(α)	0.0397^{7}	0.0130^{3}	0.0040^{1}	0.0217^{4}	0.03246	0.0096^{2}	0.0223^{5}
	SRMSE(α)	0.2055^{7}	0.1472^{6}	0.1013^{1}	0.1261^{4}	0.1290^{5}	0.1061^{2}	0.1103^{3}
	ARbias(β)	0.0327^{4}	0.0450^{5}	0.0106^{3}	0.0696^{6}	0.0848^{7}	0.0192^{1}	0.0192^{2}
	SRMSE(β)	0.2938^{7}	0.2233^{6}	0.1132^{1}	0.1978^{5}	0.1792^{4}	0.1227^{2}	0.1408^{3}
	ARbias(λ)	0.1176^{7}	0.0967^{4}	0.0429^{3}	0.1063^{6}	0.1034^{5}	0.0044^{1}	0.0639^{3}
	SRMSE(λ)	0.3691^{6}	0.3979^{7}	0.3383^{5}	0.2991^{1}	0.3294^{4}	0.3243^{2}	0.3282^{3}
	D_{abs}	0.0230^{6}	0.1010^{7}	0.0053^{2}	0.0132^{5}	0.0131^{4}	0.0018^{1}	0.0078^{3}
	$Dmax$	0.0548^{7}	0.0454^{4}	0.0211^{2}	0.0497^{6}	0.0483^{5}	0.0027^{1}	0.0299^{3}
	$\sum Ranks$	51^{7}	42^{6}	17^{2}	37^{4}	40^{5}	12^{1}	25^{3}
150	ARbias(α)	0.0062^{2}	0.0143^{4}	0.0025^{1}	0.0289^{5}	0.0419^{7}	0.0101^{3}	0.0343^{6}
	SRMSE(α)	0.1871^{7}	0.1256^{6}	0.0856^{2}	0.1105^{4}	0.1130^{5}	0.0031^{1}	0.0989^{3}
	ARbias(β)	0.0670^{5}	0.0377^{4}	0.0129^{1}	0.0729^{6}	0.0932^{7}	$0.0161^{2.5}$	$0.0161^{2.5}$
	SRMSE(β)	0.2859^{7}	0.1943^{5}	0.1030^{1}	0.1948^{6}	0.1664^{4}	0.1062^{2}	0.1285^{3}
	ARbias(λ)	0.0987^{7}	0.0566^{4}	0.0097^{1}	0.0862^{5}	0.0901^{6}	0.0160^{2}	0.0544^{3}
	SRMSE(λ)	0.3151^{6}	0.3302^{7}	0.2798^{5}	0.2327^{1}	0.2716^{4}	0.2737^{3}	0.2615^{2}
	D_{abs}	0.0204^{7}	0.0066^{3}	0.0026^{2}	0.0110^{5}	0.0115^{6}	0.0022^{1}	0.0070^{4}
	$Dmax$	0.0468^{7}	0.0270^{4}	0.0046^{1}	0.0411^{5}	0.0429^{6}	0.0077^{2}	0.0259^{3}
	$\sum Ranks$	48^{7}	$37^{4.5}$	14^{1}	$37^{4.5}$	45^{6}	16.5^{2}	26.5^{3}

Table 3: Numerical results for $\alpha = 2, \beta = 1.5,$ and $\lambda = 0.5$

n	Estimate	MLE	MPS	CVM	AD	RAD	LS	WLS
30	ARbias(α)	0.0715^{7}	0.0370^{2}	0.0208^{1}	0.0487^{4}	0.0473^{3}	0.0610^{6}	0.0526^{5}
	SRMSE(α)	0.2985^{7}	0.2263^{6}	0.1587^{1}	0.2058^{5}	0.1885^{3}	0.1723^{2}	0.1898^{4}
	ARbias(β)	0.0691^{1}	0.1050^{3}	0.0818^{2}	0.1471^{7}	0.1466^{6}	0.1200^{5}	0.1073^{4}
	SRMSE(β)	0.4113^{7}	0.3474^{6}	0.2173^{1}	0.2966^{5}	0.2919^{4}	0.2309^{2}	0.2584^{3}
	ARbias(λ)	0.1613^{6}	0.1702^{7}	0.0995^{4}	0.0887^{3}	0.1195^{5}	0.0135^{2}	0.0131^{1}
	SRMSE(λ)	0.5427^{6}	0.5944^{7}	0.4704^{4}	0.4399^{1}	0.4948^{5}	0.4582^{2}	0.4685^{3}
	D_{abs}	0.0409^{7}	0.0184^{4}	0.0158^{3}	0.0203^{5}	0.0217^{6}	0.0064^{1}	0.0096^{2}
	$Dmax$	0.0676^{6}	0.0700^{7}	0.0367^{4}	0.0360^{3}	0.0483^{5}	0.0108^{1}	0.0162^{2}
	$\sum Ranks$	47^{7}	42^{6}	20^{1}	33^{4}	37^{5}	21^{2}	24^{3}
50	ARbias(α)	0.0602^{7}	0.0367^{2}	0.0277^{1}	0.0468^{4}	0.0499^{5}	0.0576^{6}	0.0414^{3}
	SRMSE(α)	0.2629^{7}	0.2018^{6}	0.1483^{1}	0.1808^{5}	0.1739^{4}	0.1575^{2}	0.1658^{3}
	ARbias(β)	0.0426^{1}	0.0972^{5}	0.0751^{2}	0.1237^{7}	0.1226^{6}	0.09713.5	$0.0971^{3.5}$
	SRMSE(β)	0.3647^{7}	0.3227^{6}	0.1934^{1}	0.2652^{5}	0.2584^{4}	0.2074^{2}	0.2296^{3}
	ARbias(λ)	0.1070^{6}	0.1248^{7}	0.0835^{4}	0.0842^{5}	0.0791^{3}	0.0096^{1}	0.0313^{2}
	SRMSE(λ)	0.4719^{6}	0.5223^{7}	0.4079^{3}	0.3880^{1}	0.4234^{5}	0.3962^{2}	0.4124^{4}
	D_{abs}	0.0304^{7}	0.0153^{5}	0.0118^{3}	0.0161^{6}	0.0147^{4}	0.0046^{1}	0.0087^{2}
	$Dmax$	0.0482^{6}	0.0549^{7}	0.0367^{5}	0.0365^{4}	0.0344^{3}	0.0074^{1}	0.0137^{2}
	$\sum Ranks$	47^{7}	45^{6}	20^{2}	37^{5}	34^{4}	18.5^{1}	22.5^{3}
100	ARbias(α)	0.0387^{2}	0.0428^{3}	0.0342^{1}	0.0487^{6}	0.0516^{7}	0.0482^{5}	0.0448^{4}
	SRMSE(α)	0.2100^{7}	0.1646^{6}	0.1224^{1}	0.1467^{5}	0.1463^{4}	0.1268^{2}	0.1335^{3}
	ARbias(β)	0.0460^{1}	0.0982^{5}	0.0794^{2}	0.1147^{7}	0.1142^{6}	$0.0874^{3.5}$	$0.0874^{3.5}$
	SRMSE(β)	0.3992^{7}	0.2718^{6}	0.1698^{1}	0.2150^{4}	0.2226^{5}	0.1749^{2}	0.1955^{3}
	ARbias(λ)	0.0802^{4}	0.0918^{6}	0.0841^{5}	0.0922^{7}	0.0776^{3}	0.0413^{1}	0.0588^{2}
	SRMSE(λ)	0.3705^{6}	0.4097^{7}	0.3270^{4}	0.3073^{1}	0.3270^{5}	0.3137^{2}	0.3232^{3}
	D_{abs}	0.0221^{7}	0.0124^{4}	0.0108^{3}	0.0140^{6}	0.0125^{5}	0.0066^{1}	0.0095^{2}
	$Dmax$	0.0373^{4}	0.0429^{6}	0.0394^{5}	0.0430^{7}	0.0361^{3}	0.0192^{1}	0.0274^{2}
	$\sum Ranks$	$38^{4.5}$	$43^{6.5}$	22^{2}	$43^{6.5}$	$38^{4.5}$	17.5^{1}	22.5^{3}
150	ARbias(α)	0.0106^{1}	0.0375^{2}	0.0401^{3}	0.0571^{6}	0.0584^{7}	0.0496^{5}	0.0475^{4}
	SRMSE(α)	0.1955^{7}	0.1594^{6}	0.1144^{1}	0.1391^{5}	0.1350^{4}	0.1185^{2}	0.1225^{3}
	ARbias(β)	0.0639^{1}	0.0895^{5}	0.0807^{2}	0.1215^{7}	0.1193^{6}	$0.0860^{3.5}$	$0.0860^{3.5}$
	SRMSE(β)	0.3025^{7}	0.2604^{7}	0.1612^{1}	0.2035^{4}	0.2117^{5}	0.1655^{2}	0.1861^{3}
	ARbias(λ)	0.0601^{5}	0.0554^{4}	0.0491^{3}	0.0740^{7}	0.0622^{6}	0.0187^{1}	0.0425^{2}
	SRMSE(λ)	0.3234^{6}	0.3413^{7}	0.2784^{4}	0.2640^{1}	0.2886^{5}	0.2697^{2}	0.2778^{3}
	D_{abs}	0.0195^{7}	0.0108^{4}	0.0079^{2}	0.0121^{6}	0.0108^{5}	0.0050^{1}	0.0081^{3}
	$Dmax$	0.0297^{6}	0.0260^{4}	0.0233^{3}	0.0351^{7}	0.0295^{5}	0.0088^{1}	0.0201^{2}
	$\sum Ranks$	40^{5}	38^{4}	19^{2}	$43^{6.5}$	$43^{6.5}$	17.5^{1}	23.5^{3}

Table 4: Numerical results for $\alpha = 0.8, \beta = 1.5,$ and$\lambda = 0.5$

n	Estimate	MLE	MPS	CVM	AD	RAD	LS	WLS
30	ARbias(α)	0.0841^{7}	0.0240^{2}	0.0113^{1}	0.0365^{4}	0.0306^{3}	0.0547^{6}	0.0424^{5}
	SRMSE(α)	0.2615^{7}	0.2171^{6}	0.1645^{1}	0.1778^{4}	0.1914^{5}	0.1692^{3}	0.1671^{2}
	ARbias(β)	0.0506^{1}	0.1462^{5}	0.0948^{2}	0.1741^{7}	0.1649^{6}	$0.1292^{3.5}$	$0.1292^{3.5}$
	SRMSE(β)	0.4542^{7}	0.4360^{6}	0.2716^{1}	0.3205^{4}	0.3506^{5}	0.2864^{2}	0.2917^{3}
	ARbias(λ)	0.1479^{6}	0.2263^{7}	0.0943^{3}	0.1258^{5}	0.1156^{4}	0.0251^{2}	0.0228^{1}
	SRMSE(λ)	0.6135^{6}	0.6522^{7}	0.5121^{2}	0.4906^{1}	0.5163^{3}	0.5212^{4}	0.5313^{5}
	D_{abs}	0.0369^{7}	0.0239^{6}	0.0157^{3}	0.0210^{4}	0.0211^{5}	0.0067^{1}	0.0087^{2}
	$Dmax$	0.0612^{6}	0.08894^{7}	0.0376^{3}	0.0495^{5}	0.0454^{4}	0.0118^{1}	0.0135^{2}
	$\sum Ranks$	47^{7}	46^{6}	16^{1}	34^{4}	35^{5}	22.5^{2}	23.5^{3}
50	ARbias(α)	0.0498^{7}	0.0257^{2}	0.0238^{1}	0.0410^{5}	0.0386^{4}	0.0491^{6}	0.0365^{3}
	SRMSE(α)	0.2123^{7}	0.1810^{6}	0.1371^{2}	0.1508^{4}	0.1598^{5}	0.1423^{3}	0.1357^{1}
	ARbias(β)	0.0678^{1}	0.1309^{5}	0.1056^{2}	0.1731^{7}	0.1494^{6}	$0.1216^{3.5}$	$0.1216^{3.5}$
	SRMSE(β)	0.4271^{7}	0.3980^{6}	0.2470^{1}	0.3016^{4}	0.3055^{5}	0.2530^{2}	0.25784^{3}
	ARbias(λ)	0.0980^{5}	0.1297^{7}	0.0837^{4}	0.1207^{6}	0.0750^{3}	0.0001^{1}	0.0269^{2}
	SRMSE(λ)	0.5355^{6}	0.5842^{7}	0.4386^{3}	0.4344^{2}	0.4395^{4}	0.4304^{1}	0.4648^{5}
	D_{abs}	0.0289^{7}	0.0183^{5}	0.0134^{3}	0.0195^{6}	0.0152^{4}	0.0054^{1}	0.0083^{2}
	$Dmax$	0.0445^{5}	0.0563^{7}	0.0363^{4}	0.0519^{6}	0.0320^{3}	0.0103^{1}	0.0124^{2}
	$\sum Ranks$	$45^{6.5}$	$45^{6.5}$	20^{2}	40^{5}	34^{4}	18.5^{1}	21.5^{3}
100	ARbias(α)	0.0247^{1}	0.0309^{2}	0.0339^{3}	0.0448^{6}	0.0427^{5}	0.0477^{7}	0.0361^{4}
	SRMSE(α)	0.1632^{7}	0.1518^{6}	0.1078^{2}	0.1167^{4}	0.1264^{5}	0.1115^{3}	0.1030^{1}
	ARbias(β)	0.0659^{1}	0.1458^{6}	0.1129^{2}	0.1576^{7}	0.1375^{5}	$0.1225^{3.5}$	$0.1225^{3.5}$
	SRMSE(β)	0.3705^{6}	0.3946^{7}	0.2186^{1}	0.2538^{4}	0.2689^{5}	0.2236^{2}	0.2400^{3}
	ARbias(λ)	0.0980^{3}	0.1401^{6}	0.1192^{5}	0.1447^{7}	0.1014^{4}	0.0762^{1}	0.0871^{2}
	SRMSE(λ)	0.4657^{6}	0.5196^{7}	0.3696^{2}	0.3869^{4}	0.3858^{3}	0.3686^{1}	0.4070^{5}
	D_{abs}	0.0216^{7}	0.0192^{6}	0.0127^{3}	0.0169^{5}	0.0131^{4}	0.0085^{1}	0.0104^{2}
	$Dmax$	0.0452^{3}	0.0648^{6}	0.0554^{5}	0.0670^{7}	0.0469^{4}	0.0353^{1}	0.0404^{2}
	$\sum Ranks$	34^{4}	46^{7}	23^{3}	44^{6}	35^{5}	19.5^{1}	22.5^{2}
150	ARbias(α)	0.0021^{1}	0.0426^{3}	0.0451^{4}	0.0574^{7}	0.0454^{5}	0.0536^{6}	0.0412^{2}
	SRMSE(α)	0.1380^{6}	0.1402^{7}	0.1017^{2}	0.1109^{5}	0.1089^{4}	0.1043^{3}	0.0917^{1}
	ARbias(β)	0.1010^{1}	0.1664^{6}	0.1394^{3}	0.1854^{7}	0.1344^{2}	$0.1439^{4.5}$	$0.1439^{4.5}$
	SRMSE(β)	0.3635^{6}	0.3843^{7}	0.2292^{1}	0.2412^{4}	0.2395^{3}	0.2322^{2}	0.2435^{5}
	ARbias(λ)	0.0972^{3}	0.1316^{5}	0.1324^{6}	0.1597^{7}	0.0971^{2}	0.1010^{4}	0.0917^{1}
	SRMSE(λ)	0.4207^{6}	0.4562^{7}	0.3144^{2}	0.3365^{4}	0.3298^{3}	0.3084^{1}	0.3445^{5}
	D_{abs}	0.0194^{7}	0.0183^{6}	0.0138^{4}	0.0174^{5}	0.0117^{3}	0.0108^{2}	0.0106^{1}
	$Dmax$	0.0460^{2}	0.0623^{5}	0.0630^{6}	0.0758^{7}	0.0461^{3}	0.0480^{4}	0.0436^{1}
	$\sum Ranks$	32^{5}	$46^{6.5}$	28^{4}	$46^{6.5}$	25^{2}	26.5^{3}	20.5^{1}

Table 5: Numerical results for $\alpha = 0.8, \beta = 0.5,$ and $\lambda = 0.5$

n	Estimate	MLE	MPS	CVM	AD	RAD	LS	WLS
30	ARbias(α)	0.0603^{7}	0.0356^{5}	0.0027^{1}	0.0271^{3}	0.0223^{2}	0.0413^{6}	0.0314^{4}
	SRMSE(α)	0.2334^{7}	0.2203^{6}	0.1472^{1}	0.1656^{4}	0.1684^{5}	0.1561^{3}	0.1479^{2}
	ARbias(β)	0.0849^{2}	0.2042^{7}	0.0725^{1}	0.1555^{6}	0.1477^{5}	$0.0989^{3.5}$	$0.0989^{3.5}$
	SRMSE(β)	0.4387^{6}	0.5016^{7}	0.2511^{1}	0.3126^{4}	0.3248^{5}	0.2635^{3}	0.2542^{2}
	ARbias(λ)	0.1630^{6}	0.2335^{7}	0.0405^{2}	0.0890^{5}	0.0867^{4}	0.0829^{3}	0.0356^{1}
	SRMSE(λ)	0.5989^{6}	0.6852^{7}	0.4927^{2}	0.4623^{1}	0.5096^{4}	0.4946^{3}	0.5343^{5}
	D_{abs}	0.0365^{7}	0.0291^{6}	0.0128^{3}	0.0200^{4}	0.0202^{5}	0.0111^{2}	0.0087^{1}
	$Dmax$	0.0658^{6}	0.0932^{7}	0.0187^{2}	0.0353^{5}	0.0347^{3}	0.0353^{4}	0.0160^{1}
	$\sum Ranks$	47^{6}	51^{7}	13^{1}	32^{4}	33^{5}	27.5^{3}	19.5^{2}
50	ARbias(α)	0.0428^{7}	0.0385^{6}	0.0128^{1}	0.03106^{4}	0.0274^{3}	0.0361^{5}	0.0244^{2}
	SRMSE(α)	0.1874^{7}	0.1817^{6}	0.1245^{2}	0.1369^{4}	0.1474^{5}	0.1301^{3}	0.1130^{1}
	ARbias(β)	0.0533^{1}	0.1757^{7}	0.0665^{2}	0.1343^{6}	0.1115^{5}	$0.0781^{3.5}$	$0.0781^{3.5}$
	SRMSE(β)	0.3883^{6}	0.4497^{7}	0.2145^{3}	0.2809^{5}	0.2639^{4}	0.2139^{2}	0.2128^{1}
	ARbias(λ)	0.1125^{5}	0.1795^{7}	0.0626^{3}	0.1131^{6}	0.0720^{4}	0.0183^{2}	0.0089^{1}
	SRMSE(λ)	0.5420^{6}	0.6263^{7}	0.43226^{2}	0.4320^{1}	0.4612^{4}	0.4346^{3}	0.4817^{5}
	D_{abs}	0.02515^{7}	0.02252^{6}	0.009755^{3}	0.01655^{5}	0.01277^{4}	0.0039^{2}	0.0038^{1}
	$Dmax$	0.0490^{4}	0.0781^{7}	0.0274^{2}	0.0492^{5}	0.0311^{3}	0.0087^{1}	0.0613^{6}
	$\sum Ranks$	43^{6}	53^{7}	18^{1}	36^{5}	32^{4}	21.5^{3}	20.5^{2}
100	ARbias(α)	0.0076^{1}	0.0486^{6}	0.0244^{2}	0.0494^{7}	0.0393^{5}	0.0354^{4}	0.0325^{3}
	SRMSE(α)	0.1515^{7}	0.1456^{6}	0.0977^{2}	0.1154^{5}	0.1052^{4}	0.1005^{3}	0.0887^{1}
	ARbias(β)	0.1026^{4}	0.1863^{7}	0.0911^{1}	0.1809^{6}	0.1332^{5}	$0.0948^{2.5}$	$0.0948^{2.5}$
	SRMSE(β)	0.3691^{6}	0.3912^{7}	0.1885^{1}	0.2554^{5}	0.2314^{4}	0.1886^{2}	0.1943^{3}
	ARbias(λ)	0.1147^{5}	0.1591^{7}	0.07464^{3}	0.15804^{6}	0.1005^{4}	0.0288^{1}	0.0571^{2}
	SRMSE(λ)	0.4450^{6}	0.5113^{7}	0.3426^{2}	0.3454^{3}	0.3645^{4}	0.3372^{1}	0.3783^{5}
	D_{abs}	0.0231^{7}	0.0203^{6}	0.0106^{3}	0.0193^{5}	0.0135^{4}	0.0062^{1}	0.0086^{2}
	$Dmax$	0.0528^{4}	0.0734^{7}	0.0345^{3}	0.0730^{6}	0.0464^{5}	0.0131^{1}	0.0263^{2}
	$\sum Ranks$	40^{5}	53^{7}	20^{2}	43^{6}	35^{4}	15.5^{1}	20.5^{3}
150	ARbias(α)	0.0028^{1}	0.0367^{4}	0.0303^{3}	0.0544^{7}	0.0416^{6}	0.0384^{5}	0.0294^{2}
	SRMSE(α)	0.1347^{7}	0.1326^{6}	0.0902^{2}	0.1079^{5}	0.0923^{3}	0.0949^{4}	0.0775^{1}
	ARbias(β)	0.1020^{2}	0.1439^{5}	0.0997^{1}	0.1844^{6}	0.1243^{4}	$0.1050^{3.5}$	$0.1050^{3.5}$
	SRMSE(β)	0.3407^{6}	0.3449^{7}	0.1846^{1}	0.2442^{5}	0.2099^{4}	0.1917^{2}	0.1918^{3}
	ARbias(λ)	0.1186^{5}	0.1258^{6}	0.1009^{3}	0.1782^{7}	0.1060^{4}	0.0717^{2}	0.0690^{1}
	SRMSE(λ)	0.4072^{6}	0.4384^{7}	0.2982^{2}	0.3103^{3}	0.3188^{4}	0.2905^{1}	0.3281^{5}
	D_{abs}	0.0201^{7}	0.0167^{5}	0.0112^{3}	0.0189^{6}	0.0117^{4}	0.0085^{2}	0.0081^{1}
	$Dmax$	0.0563^{5}	0.0600^{6}	0.0481^{3}	0.0847^{7}	0.0505^{4}	0.0341^{2}	0.0329^{1}
	$\sum Ranks$	39^{5}	$46^{6.5}$	18^{2}	$46^{6.5}$	33^{4}	21.5^{3}	17.5^{1}

5. Bayesian Estimation

In this section, Bayesian estimation of the three unknown parameters of the TINH distribution will be discussed. Approximate Bayes estimates are computed using the Gibbs sampling procedure with generating samples from the posterior distributions. This requires prior density functions of the unknown parameters (α, β and λ). Here, we assume that α, β and λ

are independent random variables. The parameters α and β have gamma distributions while λ follows uniform distribution. Their pdfs, respectively are $g_1(\alpha) \propto \alpha^{(a-1)}e^{(-b\alpha)}$, $g_2(\beta) \propto \beta^{(c-1)}e^{(-d\beta)}$ and $g_3(\lambda)$ = constant. The hyper-parameters a, b, c and d are assumed to be known.

The joint prior distribution for (α, β and λ) takes the form $g(\alpha, \beta, \lambda) \propto \alpha^{(a-1)}\beta^{(c-1)}e^{(-b\alpha-d\beta)}$ and the likelihood function is given by

$$\begin{aligned} L(\boldsymbol{x}; \alpha, \beta, \lambda) &\propto \alpha^n \beta^n e^{-\alpha \sum_{i=1}^{n} ln(1+\beta x_i^{-1})^{-1}} e^{-\sum_{i=1}^{n}(1+\beta x_i^{-1})} e^{-\alpha \sum_{i=1}^{n}(1+\beta x_i^{-1})} \\ &* \; e^{-\sum_{i=1}^{n} ln(1+\lambda-2\lambda e^{(1-(1+\beta x_i^{-1})^{\alpha})})^{-1}}. \end{aligned} \tag{19}$$

Then the joint posterior is given by

$$\begin{aligned} g(\alpha, \beta, \lambda|\boldsymbol{x}) &\propto \alpha^{n+a-1}\beta^{n+c-1} e^{-\alpha(b+\sum_{i=1}^{n} ln(1+\beta x_i^{-1})^{-1})} e^{-(d\beta+\sum_{i=1}^{n}(1+\beta x_i^{-1}))} e^{-\alpha \sum_{i=1}^{n}(1+\beta x_i^{-1})} \\ &* \; e^{-\sum_{i=1}^{n} ln(1+\lambda-2\lambda e^{(1-(1+\beta x_i^{-1})^{\alpha})})^{-1}}. \end{aligned} \tag{20}$$

The conditional posterior distributions used in the Gibbs sampling algorithm are given by

$$\begin{aligned} g(\alpha|\beta, \lambda, \boldsymbol{x}) &\propto \alpha^{n+a-1} e^{-\alpha(b+\sum_{i=1}^{n} ln(1+\beta x_i^{-1})^{-1})} e^{-\alpha \sum_{i=1}^{n}(1+\beta x_i^{-1})} \\ &* \; e^{-\sum_{i=1}^{n} ln(1+\lambda-2\lambda e^{(1-(1+\beta x_i^{-1})^{\alpha})})^{-1}}, \end{aligned} \tag{21}$$

$$\begin{aligned} g(\beta|\alpha, \lambda, \boldsymbol{x}) &\propto \beta^{n+c-1} e^{-\alpha(b+\sum_{i=1}^{n} ln(1+\beta x_i^{-1})^{-1})} e^{-(d\beta+\sum_{i=1}^{n}(1+\beta x_i^{-1}))} e^{-\alpha \sum_{i=1}^{n}(1+\beta x_i^{-1})} \\ &* \; e^{-\sum_{i=1}^{n} ln(1+\lambda-2\lambda e^{(1-(1+\beta x_i^{-1})^{\alpha})})^{-1}}, \end{aligned} \tag{22}$$

and

$$g(\lambda|\alpha, \beta, \boldsymbol{x}) \propto e^{-\sum_{i=1}^{n} ln(1+\lambda-2\lambda e^{(1-(1+\beta x_i^{-1})^{\alpha})})^{-1}}. \tag{23}$$

The computation can be achieved using the WinBUGS software which requires only the specification of the joint distribution for the data and the prior distributions for the model parameters. Gibbs sampling algorithm works as follows

1. Specify the size of the samples we wish to generate, say m.
2. Choose an initial value of $\boldsymbol{\theta}$, say $\boldsymbol{\theta}_0$.
3. For iteration i from 1 to m, generate $\theta_j^{(i)}$ from $g(\theta_j|\theta_1^{(i)}, ..., \theta_{(j-1)}^{(i)}, \theta_{(j+1)}^{(i-1)}, ..., \theta_p^{(i-1)})$, for j from 1 to p.
4. Return the values $\boldsymbol{\theta}^{(0)}, \boldsymbol{\theta}^{(1)}, ..., \boldsymbol{\theta}^{(m)}$.

Discarding the early m_0 number of burn-in draws and using the remaining $m - m_0$, $\boldsymbol{\theta}^{(m_0+1)}, \boldsymbol{\theta}^{(m_0+2)}, \ldots, \boldsymbol{\theta}^{(m)}$, as the chosen draws from the joint posterior distribution, the Bayes estimate of θ_j is

$$\hat{\theta}_j = \frac{\sum_{i=m_0+1}^{m-m_0} \theta_j^{(i)}}{m - m_0}, j = 1, 2, 3.$$

Furthermore, the lower and upper bounds of the $100(1-\nu)\%, 0<\nu<1$, Bayesian probability interval of θ_j can be obtained using $\nu/2$ 100 th and $(1-\nu/2)$ 100 th percentiles of the sequence of the $m-m_0$ draws; $\theta^{(m_0+1)}, \theta^{(m_0+2)}, \ldots, \theta^{(m)}$.
Here, we generate 10,000 samples of α, β and λ, after a "burn-in-sample" of size 1000 and the approximate Bayes estimates with some posterior summaries, such as MC error, 95% credible interval, median, are given in Table 6. Table 7 has the results of Bayesian estimation for real data set (mentioned in the section 5) and the graphical representation of the marginal posteriors of α, β and λ are displayed in Figure 3. One can note that the posteriors of α and β are approximately normal while skewed for λ. Another MCMC method called Metropolis-Hastings algorithm is used to generate random draws from the joint posterior distribution without deriving its explicit form. Metropolis-Hastings algorithm unlike Gibbs-sampling, it requires a proposal distribution and a common choice of it is the multivariate normal distribution. Metropolis-Hastings algorithm steps are

1. Set the size of the random draws we wish to generate, say m.
2. Choose an initial value of $\boldsymbol{\theta}$, say $\boldsymbol{\theta}^{(0)}$.
3. For $i=1,2,\ldots,m$, repeat the following steps:
   (a) Set $\theta^{(i)}=\theta^{(i-1)}$.
   (b) Generate a candidate value θ^{*} from a proposal distribution $p(\theta^{(*)}|\theta^{(i)})$.
   (c) Calculate the ratio $\kappa=min(1, \frac{g(\boldsymbol{\theta}^{(*)}|data)/p(\boldsymbol{\theta}^{(*)}|\boldsymbol{\theta}^{(i)}}{g(\boldsymbol{\theta}^{(i)}|data)/p(\boldsymbol{\theta}^{(i)}|\boldsymbol{\theta}^{(*)})})$.
   (d) Generate a random value u from uniform distribution on (0, 1).
   (e) Put $\boldsymbol{\theta}^{(i)}=\boldsymbol{\theta}^{*},$ if $\kappa\geq u,$ otherwise put $\theta^{(i)}=\theta^{(i-1)}$.
4. Return the values $\boldsymbol{\theta}^{(0)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(m)}$.

The lower and upper bounds of the $100(1-\nu)\%$ Bayesian probability interval of θ_j as given above. The computations are carried out using R software. We use the two previously MCMC methods to analyze the same real dataset. We generate 10,000 samples of α, β and λ, after a "burn-in-sample" of size 1000 with assuming gamma priors for α and β and uniform prior for λ. The results of Bayesian estimation for real dataset are given in Tables 7-8. Table 7 has the results of Bayesian estimation for real data set (mentioned in the section 5) and the graphical representation of the marginal posteriors of α, β and λ are displayed in Figure 5. One can note that the posteriors of α and β are approximately normal while skewed for λ. Table 8 displays the posterior mean, median, standard deviation and the limits of a 95% credible interval of each parameter. Figure 6 shows The approximated marginal Posterior density functions of α, β and λ.

Table 6: Summary results for the posterior parameters in the case of the TINH model

Parameter	n	Estimate	SD	MC error	95% Credible Interval	Median
$\alpha = 2$	30	2.0140	0.4468	0.013930	(1.2990, 3.0340)	1.9660
	50	2.3050	0.4715	0.017700	(1.5210,3.3660)	2.2540
	100	1.9060	0.4109	0.021700	(1.2910,2.8780)	1.8400
	150	1.7690	0.3769	0.022720	(1.2200,2.6930)	1.7090
$\beta = 0.5$	30	0.5695	0.1794	0.005541	(0.2963,0.0986)	0.5428
	50	0.5178	0.1397	0.005364	(0.3031,0.8414)	0.5000
	100	0.5614	0.1701	0.008805	(0.2924,0.9551)	0.5417
	150	0.5597	0.1729	0.010350	(0.2776,0.9335)	0.5384
$\lambda = 0.5$	30	0.4009	0.2585	0.004148	(0.0172,0.9304)	0.3726
	50	0.8234	0.1485	0.002417	(0.4459,0.9933)	0.8612
	100	0.3069	0.1925	0.005165	(0.0181,0.7155)	0.2857
	150	0.4439	0.1966	0.007221	(0.0585,0.8048)	0.4533
$\alpha = 2$	30	1.9590	0.3896	0.011880	(1.3330, 2.8670)	1.9090
	50	2.1540	0.3982	0.013320	(1.5060,3.0490)	2.1120
	100	2.2250	0.4046	0.017630	(1.5400,3.1500)	2.1760
	150	2.1170	0.3882	0.019640	(1.4970,2.9960)	2.0550
$\beta = 1.5$	30	1.5260	0.3962	0.012380	(1.4820,2.4280)	2.4280
	50	1.7240	0.4165	0.014080	(1.0370,2.6490)	1.6800
	100	1.4540	0.3508	0.014830	(0.8763,2.2910)	1.4230
	150	1.4660	0.3605	0.018850	(0.8729,2.2600)	1.4400
$\lambda = 0.5$	30	0.7376	0.2166	0.002997	(0.1965,0.9917)	0.9917
	50	0.7676	0.1897	0.003344	(0.2849,0.9909)	0.8152
	100	0.6903	0.1674	0.002574	(0.3211,0.9637)	0.7069
	150	0.3957	0.1627	0.003313	(0.0792,0.7070)	0.3990
$\alpha = 0.8$	30	0.6644	0.1216	0.003085	(0.4721, 0.9483)	0.6484
	50	0.7987	0.1368	0.003655	(0.5742,1.0200)	0.7822
	100	0.7989	0.1256	0.005275	(0.5978,1.0800)	0.7839
	150	0.8240	0.1186	0.005274	(0.6321,1.0950)	0.8093
$\beta = 0.5$	30	0.5976	0.2113	0.005306	(0.2687,1.0880)	0.5706
	50	0.5635	0.1921	0.005415	(0.2774,1.0300)	0.5332
	100	0.5212	0.1750	0.007392	(0.2570,0.9230)	0.4954
	150	0.4990	0.1490	0.006292	(0.2595,0.8474)	0.4822
$\lambda = 0.5$	30	0.2354	0.1876	0.003029	(0.00653,0.6865)	0.1903
	50	0.3402	0.2279	0.004328	(0.01394,0.8417)	0.3101
	100	0.4408	0.2238	0.006032	(0.03971,0.8596)	0.4437
	150	0.4422	0.1905	0.005595	(0.05382,0.7761)	0.4587
$\alpha = 0.8$	30	0.9075	0.16500	0.004467	(0.6420, 1.2850)	0.8889
	50	0.6691	0.09102	0.002224	(0.5147,0.8685)	0.6598
	100	0.9733	0.14330	0.005941	(0.7385,1.2970)	0.9598
	150	0.8045	0.10710	0.004466	(0.6268,1.0430)	0.7936

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Table 6 – *Continued from previous page*

Parameter	n	Estimate	SD	MC error	95% Credible Interval	Median
$\beta = 1.5$	30	1.2040	0.3524	0.009401	(0.6320,2.0060)	1.1590
	50	1.8200	0.4637	0.011370	(1.0370,2.8300)	1.7790
	100	1.3500	0.3489	0.014150	(0.7682,2.1310)	1.3140
	150	1.4280	0.4045	0.016750	(0.7979,2.3730)	1.3710
$\lambda = 0.5$	30	0.6908	0.2128	0.002818	(0.1850,0.9612)	0.7307
	50	0.1735	0.1394	0.002083	(0.00468,0.5163)	0.1409
	100	0.7858	0.1563	0.003428	(0.4087,0.9894)	0.8163
	150	0.3297	0.1892	0.005103	(0.02239,0.7159)	0.3160

Table 7: Summary results for the posterior parameters in the case of the TINH model based on 128 bladder cancer patients (Gibbs sampling)

Parameter	n	Estimate	SD	MC error	95% Credible Interval	Median
α	128	0.6774	0.04840	9.059E-4	(0.5871, 0.77580)	0.67620
β		5.1520	0.75810	0.01403	(3.8190,6.78600)	5.10800
λ		0.05183	0.05317	8.52E-4	(2.815E-4,0.1935)	0.03513

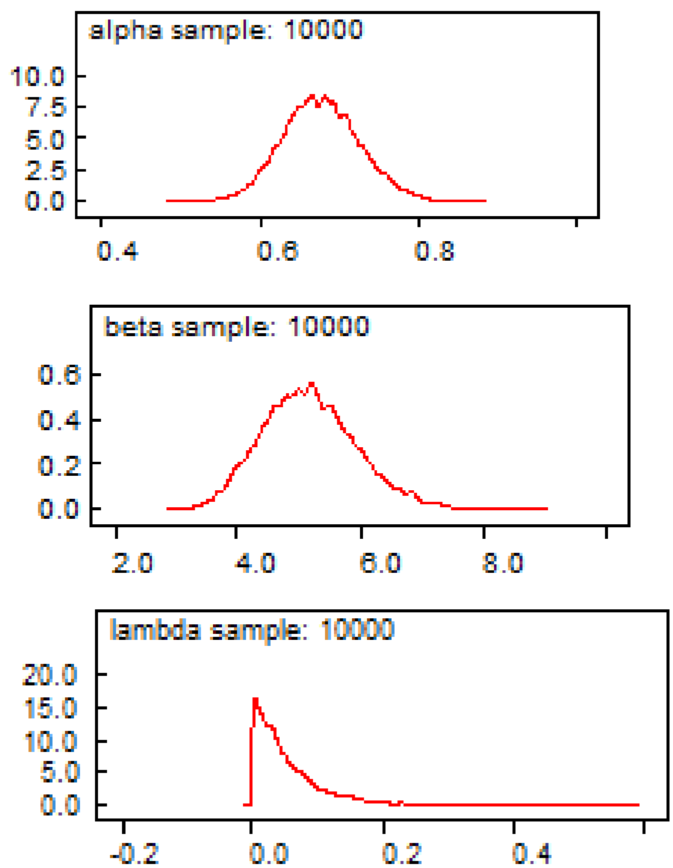

Figure 3: Posteriors of α, β and λ using Gibbs sampling for real data set

Table 8: Summary results for the posterior parameters in the case of the TINH model based on 128 bladder cancer patients (Metropolis-Hasting algorithm)

Parameter	n	Estimate	SD	95% Credible Interval	Median
α	128	0.6481	0.03593	(0.5834, 0.7227)	0.64583
β		0.0640	0.38463	(4.6554, 5.9891)	5.6639
λ		0.05183	0.04823	(0.01154, 0.1956)	0.05007

From Tables 7-8, one can note that all Bayesian point estimates are close however the Metropolis-Hastings provides narrower credible intervals.

6. Applications

Now, to illustrate the potentiality of the TINH distribution, we use a real data set and show that the new distribution is fit to this data set. The data set represents the remission times (in months) of a random sample of 128 bladder cancer patients. Bladder cancer is a disease in which abnormal cells multiply without control in the bladder. The most common type of bladder cancer recapitulates the normal histology of the urothelium and is known as transitional cell carcinoma. The data are as follows: 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05. These data were studied by Zea *et al.* (2012), among others. According to real data set the maximum likelihood estimates are obtained for the TINH distribution as follow:
$\hat{\alpha} = 3.15, \hat{\beta} = 1.8$ and $\hat{\lambda} = 0.85$.
Given the cumulative distribution function $F_0(x)$ of the hypothesized distribution (here TINH distribution) and the empirical distribution function Fdata (x) of the observed data, the popular Kolmogorov-Smirnov goodness of fit test (K-S) was carried out at 5% level of significance. The test statistic is given by:$D = \sup_x |F_0(x) - F_{data}(x)|$.For above data set, K-S statistic $D = 0.117$ with p-value $0.1 > 0.05$.

In many applications, there is qualitative information about the hazard rate shape, which can help with selecting a particular model. The empirical scaled TTT transform (Aarset, 1987) can be used to identify the shape of the hazard function. The scaled TTT transform is convex (concave) if the hazard rate is decreasing (increasing), and for bathtub (unimodal) hazard rates, the scaled TTT transform is first convex (concave) and then concave (convex). The TTT plot for complete data is the plot of $(i/n, G(i/n))$, where $G(\frac{i}{n}) = \frac{\sum_{j=1}^{i} T_{j:n} + (n-i)T_{i:n}}{\sum_{j=1}^{n} T_{j:n}}$ for $i = 1, 2, \ldots, n$, $\sum_{j=1}^{i} T_{j:n} + (n-i)T_{i:n}$ is the total time on test at the ith failure for $i = 1, 2, \ldots, n$ and $T_{(j:n)}, j = 1, 2, \ldots, n$, are the order statistics of the sample. Figure 4 presents TTT of complete data. As displayed in Figure 4: the TTT plot has first

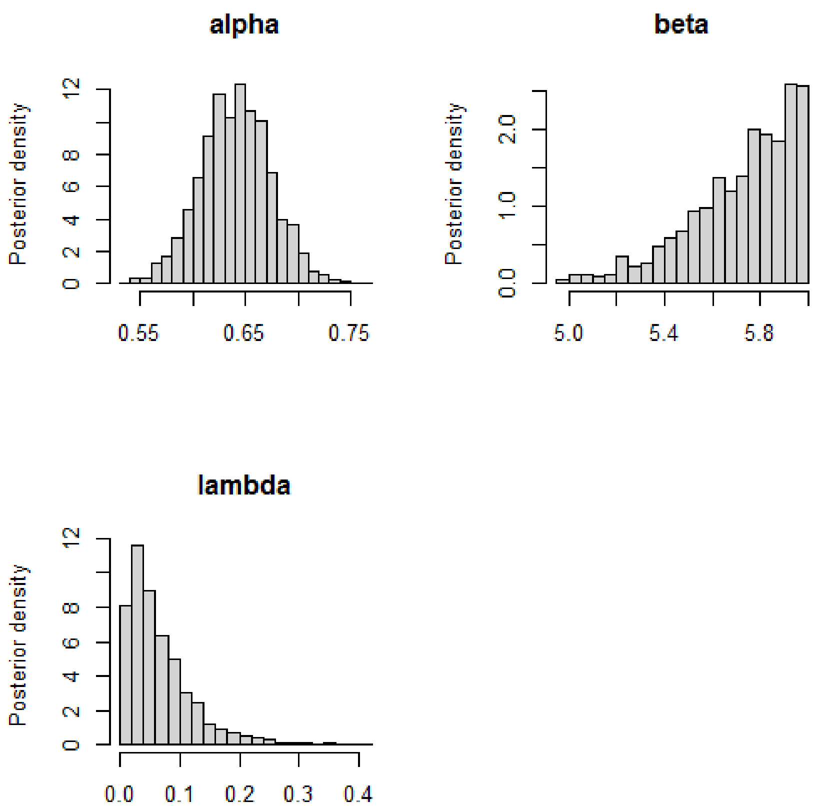

Figure 4: The approximated marginal Posterior density functions of α, β and λ using Metropolis-Hastings for real data set

a concave shape and then a convex shape. It depicts a unimodal shaped failure rate which agrees with the estimated parameters.

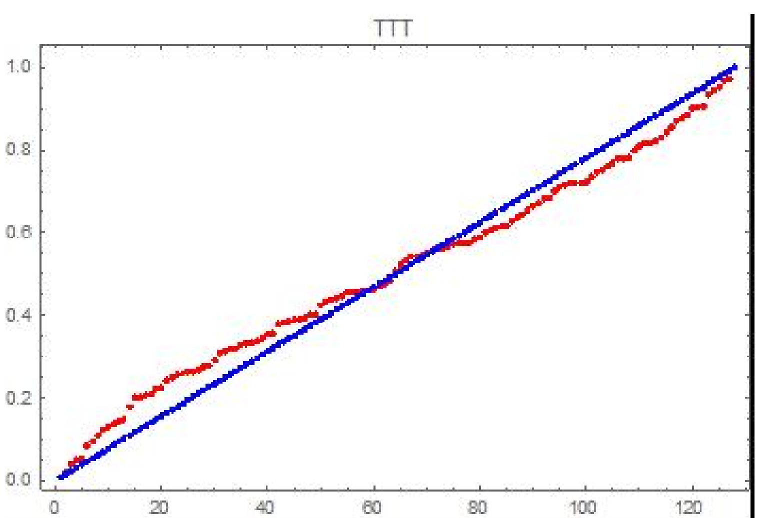

Figure 5: the TTT for real data set

Now we will compare the fits of the TINH, TMIW, TMIR, TMIE and MIW models by mean of another real data set to illustrate the potentiality of the TINH model. The cdfs associated with the competitive models are given by:

$$\begin{aligned}
F_{TMIW}(x) &= e^{-\gamma x^{-1}-\beta x^{-\alpha}}(1+\lambda-\lambda e^{-\gamma x^{-1}-\beta x^{-\alpha}}), \quad z>0,\alpha,\beta,\gamma>0 \text{ and } |\lambda|\le 1,\\
F_{TMIR}(x) &= e^{-\gamma x^{-1}-\beta x^{-2}}(1+\lambda-\lambda e^{-\gamma x^{-1}-\beta x^{-2}}), \quad z>0,\beta,\gamma>0 \text{ and } |\lambda|\le 1,\\
F_{TMIE}(x) &= e^{-(\gamma+\beta)x^{-1}}(1+\lambda-\lambda e^{-(\gamma+\beta)x^{-1}}), \quad z>0,\beta,\gamma>0 \text{ and } |\lambda|\le 1,\\
F_{MIW}(x) &= e^{-\gamma x^{-1}-\beta x^{-\alpha}}, \quad z>0,\alpha,\beta>0, \text{ and } \gamma>0.
\end{aligned}$$

The following data represents a complete data with the exact times of failure. This data is considered a data set of the life of fatigue fracture of Kevlar 373/*epoxy* that are subject to constant pressure at the 90% stress level until all had failed. The data are: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960. This data is considered by Ogunde *et al.* (2017). For model comparison, we consider some well-known measures such as the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC) and the Hannan-Quinn information criterion (HQIC). These criterions are defined by:

$$\begin{aligned}
AIC &= -2l(\hat{\boldsymbol{\theta}})+2p;\\
BIC &= -2l(\hat{\boldsymbol{\theta}})+plog(n);\\
CAIC &= -2l(\hat{\boldsymbol{\theta}})+\frac{2pn}{n-p-1};\\
HQIC &= -2l(\hat{\boldsymbol{\theta}})+2log(log(n)).
\end{aligned}$$

where $l(\hat{\boldsymbol{\theta}})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates for parameters $\boldsymbol{\theta}$, p is the number of parameters and n is the sample size. The model with minimum AIC (or BIC, CAIC and HQIC) value is chosen as the best model to fit the data. Also we consider the statistics AD (A*) and CVM (W*) to compare the models, where lower values of these statistics indicate a good fit. Table 9 lists the mles of the model parameters, the values of the measures AIC, BIC, CAIC and HQIC and from this table one can conclude that the TINH model provides a better fit to the current data than the other models. Furthermore, the values of the statistics in Table 10 indicate the TINH model provides the best fit compared to the other models.

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Table 9: The MLEs and some measures for the fitted models

Model	Estimates	$l(\hat{\boldsymbol{\theta}})$	AIC	BIC	CAIC	HQIC
TINH	$\hat{\alpha}=0.60, \hat{\beta}=1.99, \hat{\lambda}=.08$	-145.6	297.13	304.12	297.46	299.92
TMIW	$\hat{\alpha}=0.69, \hat{\beta}=1.09, \hat{\lambda}=0.97, \hat{\gamma}=0.17$	-150.4	308.73	318.05	309.29	312.45
TMIR	$\hat{\beta}=0.01, \hat{\lambda}=0.57, \hat{\gamma}=0.68$	-189.3	384.69	391.68	385.02	387.48
TMIE	$\hat{\beta}=0.50, \hat{\lambda}=0.02, \hat{\gamma}=0.12$	-163.5	332.92	339.91	333.25	335.71
MIW	$\hat{\alpha}=0.71, \hat{\beta}=0.74, \hat{\gamma}=0.14$	-155.4	316.7	323.7	317.1	319.5

Table 10: Statistics A* and W*

Distribution	A*	W*
TINH	4.120	0.693
TMIW	4.983	0.851
TMIR	8.529	1.542
TMIE	6.851	1.206
MIW	5.720	0.988

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