Statistics and Applications {ISSN 2454-7395 (online)}
Volume 21, No. 1, 2023 (New Series), pp 27–40

New Intervention Based Exponential Model with Real Life Data Applicability

Vilayat Ali Bhat and Sudesh Pundir
Department of Statistics, Pondicherry University, Kalapet, Puducherry-605014, INDIA.

Received: 07 September 2021; Revised: 03 January 2022; Accepted: 13 February 2022

Abstract

In this paper, the new extension of the extended Exponential model named inverted intervened Exponential distribution has been proposed. To explore the model, the essential statistical properties have been presented in this study, the parametric estimation was also carried out by using the method of maximum likelihood estimation (MLE) technique. Moreover, the reliability characterization has been given which includes the mathematical functions of the reliability, hazard rate, aging intensity, and mean residual life. Also, the Rényi and Shannon entropy measures have been derived. Monte Carlo simulation study by employing the acceptance-rejection algorithm was performed to judge the performance of maximum likelihood estimates (MLE_s) based on the calculated results of absolute average bias ($Abias$) and mean square error (MSE) of the parametric estimates. Lastly, the model applicability checkup was also done by analyzing real data set.

Key words: Intervened model; Entropy; Monte Carlo simulation; Model applicability.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

In literature, the traditional models such as Exponential, Normal, Rayleigh, Gamma, Weibull, etc. are the basic fundamental models in statistical theory. From the past few decades, many developments have been observed in the form of modifications and generalizations to develop more flexible distributions for data analysis purposes. In history, one could observe the most exploiting and frequently used distribution in the field of reliability and survival analysis among them being the Exponential model for reference see Balakrishnan (2019). However, the disadvantage of the Exponential model is meant due to the constant hazard rate, as there arise situations where it is observed the model requirement for increasing, decreasing, bath-tub shaped hazard rate situations as well, to model the failure data. In this context, the successful efforts of the researchers who developed different types of models to cope with these situations to some extent. It gives a clear picture that every new technique has added several types of flexible distributions in statistical theory. Since a few years ago, a new concept intervention was introduced in the distribution theory, and it was Shanmugam (1985) who made a noble attempt to develop a discrete intervention based Poisson model, later which laid to a beginning new intervention-based model development in

Corresponding Author: Sudesh Pundir
E-mail: sudeshpundir19@gmail.com

the statistical literature. A similar attempt on continuous Exponential distribution by Shanmugam *et al.* (2002) developed the intervened Exponential model (I_vED), the outstanding medical applications of the model motivated us to develop a new extension of the model named as inverted intervened Exponential model (II_vD). The cumulative density function (cdf) of the newly developed model along with its probability density function (pdf) are given by:

$$F_{II_vD}(y;\Theta) = \begin{cases} \frac{\rho e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}}{(\rho-1)} & \rho \neq 1 \\ \left(\frac{1-(\delta-\eta)y}{\eta y}\right) e^{-(1-\delta y)/\eta y} & \rho = 1 \end{cases} \tag{1}$$

and,

$$f_{II_vD}(y;\Theta) = \begin{cases} \frac{e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}}{(\rho-1)\eta y^2} & \rho \neq 1 \\ \frac{(1-\delta y)}{\eta^2 y^3} e^{-(1-\delta y)/\eta y} & \rho = 1 \end{cases} \tag{2}$$

where $0 < y < \frac{1}{\delta}$, and the desired parametric space of the model is denoted by $\Theta = \{(\rho, \delta, \eta) : \rho > 0, \delta > 0, \eta > 0\}$ containing η as the rate parameter, ρ being the intervention parameter, and the parameter δ is treated as the truncation point of the model. Further, the graphical illustration of the proposed model based on the desired set of parametric values for pdf is shown below:

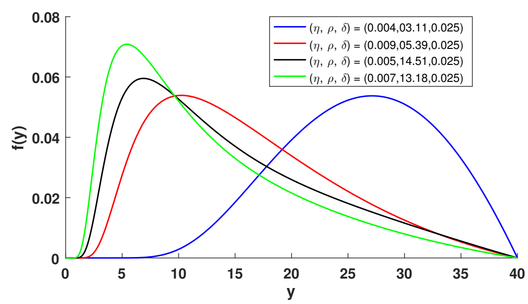

Figure 1: PDF plot

It could be easily predicted from graphical behavior that different shapes for pdf are exhibited on the selected set of parameters.

2. Statistical properties

We attempted in this section, to provide the mathematical derivation of the different statistical properties that would help to understand the nature of II_vD. The mathematical expressions of the obtained results include mean (μ_y), median (M_d), and the variance (σ_y^2) of the model. The other results mentioned in subsections are the mean deviations, r^{th} order moment expressions about the origin, the mean, and the different generating functions for moments. So, to begin this section, the mean of the distribution obtained is as follows:

$$\mu_y = \frac{1}{\rho-1}\left\{e^{\delta/\rho\eta}\Gamma(0,\delta/\rho\eta) - e^{\delta/\rho}\Gamma(0,\delta/\rho)\right\} \tag{3}$$

The variance of the II_vD is given by,

$$\sigma_y^2 = \frac{1}{\rho(\rho-1)\eta}\left\{e^{\delta/\rho\eta}\Gamma(-1,\delta/\rho\eta) - e^{\delta/\rho}\Gamma(-1,\delta/\rho)\right\} - (\mu_y)^2 \tag{4}$$

Now to find the median of II_vD mathematically, we make use of the definition of median as given below:

$$\int_{M_d}^{1/\delta} f_{II_vD}(y;\Theta)dy = 1/2$$

$$\tfrac{1}{(\rho-1)\eta} \int_{M_d}^{1/\delta} \left\{ e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y} \right\} (1/y^2)dy = 1/2$$

$$\rho e^{(\delta M_d-1)/\rho\eta M_d} - e^{(\delta M_d-1)/\eta M_d} = (\rho-1)/2$$

Note: $\Gamma(c,t) = \int_t^{\infty} y^{c-1}e^{-y}dy$ is the upper incomplete gamma function.

2.1. Mean deviations

In statistics, two well-known measures that are used to measure the scatteredness present among the data are called the mean deviations about the mean, and another one is considered as mean deviations about the median. Henceforth, these two measures are represented by D_{μ_y} and D_{M_d} respectively. The mathematical derivation for these two measures is given in the following theorem.

Theorem 1: If a random variable $(r.v.)$ $Y \sim II_vD(\rho,\delta,\eta)$, then the derived expression for D_{μ_y} and D_{M_d} for the proposed model are as:

$$(i) \quad D_{\mu_y} = \left\{ \mu_y F_{II_vD}(\mu_y) - \frac{e^{\delta/\rho\eta}}{(\rho-1)\eta}\Gamma(0,1/\mu_y\rho\eta) + \frac{e^{\delta/\eta}}{(\rho-1)\eta}\Gamma(0,1/\mu_y\eta) \right\}$$

$$(ii) \quad D_{M_d} = (\mu_y - M_d) + 2\left\{ M_d F_{II_vD}(M_d) - \frac{e^{\delta/\rho\eta}}{(\rho-1)\eta}\Gamma(0,1/M_d\rho\eta) + \frac{e^{\delta/\eta}}{(\rho-1)\eta}\Gamma(0,1/M_d\eta) \right\}$$

Proof: (i) For, any $r.v.$ Y the mean deviation about mean is given by

$$\begin{aligned} D_{\mu_y} &= 2\left\{ \mu_y F_{II_vD}(\mu_y) - \int_0^{1/\delta} y f_{II_vD}(y;\Theta)dy \right\} \\ &= 2\left\{ \mu_y F_{II_vD}(\mu_y) - \int_0^{\mu_y} \frac{e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}}{(\rho-1)\eta y}dy \right\} \\ &= \left\{ \mu_y F_{II_vD}(\mu_y) - \frac{e^{\delta/\rho\eta}}{(\rho-1)\eta}\Gamma(0,1/\mu_y\rho\eta) + \frac{e^{\delta/\eta}}{(\rho-1)\eta}\Gamma(0,1/\mu_y\eta) \right\} \end{aligned}$$

Hence, completes proof for part first.

(ii) Again, for a continuous, and non-negative $r.v.$, $Y \sim II_vD(\rho,\delta,\eta)$, we can write the mathematical expression for median deviation as,

$$\begin{aligned} & D_{M_d} = \mu_y - M_d + 2\left\{ M_d F_{II_vD}(M_d) - \int_0^{M_d} y f_{II_vD}(y;\Theta)dy \right\} \\ = \ & (\mu_y - M_d) + 2\left\{ M_d F_{II_vD}(M_d) - \int_0^{M_d} \frac{e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}}{(\rho-1)\eta y}dy \right\} \\ = \ & (\mu_y - M_d) + \left\{ M_d F_{II_vD}(M_d) - \frac{e^{\delta/\rho\eta}}{(\rho-1)\eta}\Gamma(0,1/M_d\rho\eta) + \frac{e^{\delta/\eta}}{(\rho-1)\eta}\Gamma(0,1/M_d\eta) \right\} \end{aligned}$$

This completes proof for part (ii). □

2.2. Moments and moments generating functions

Here in this subsection, we shall derive the expression for r^{th} moments about the origin and the moments about mean, and the generating functions for moments in the following subsequent theorems,

Theorem 2: If Y be any non-negative $r.v.$ possessing II_vD, then the moments about the origin and the mean are given by:

$$(i) \quad \mu_r' = \frac{1}{(\rho-1)\,\eta^r}\left\{\frac{e^{\delta/\rho\eta}}{\rho^r}\Gamma\left(1-r,\delta/\rho\eta\right) - e^{\delta/\eta}\Gamma\left(1-r,\delta/\eta\right)\right\}, \quad r=1,2...n.$$
$$(ii) \quad \mu_r = \frac{1}{(\rho-1)}\sum_{n=0}^{r} {}^rC_n \frac{(-\mu)^{r-n}}{\eta^n}\left\{\frac{e^{\delta/\rho\eta}}{\rho^n}\Gamma\left(1-n,\delta/\rho\eta\right) - e^{\delta/\eta}\Gamma\left(1-n,\delta/\eta\right)\right\}; \quad r=1,2...n.$$

Proof: (i) For a random variable $Y \sim II_vD(\rho,\delta,\eta)$, the expression for r^{th} moment about origin is,

$$\begin{aligned}
\mu_r' &= E(y^r) = \frac{1}{(\rho-1)\eta}\int_0^{1/\delta} y^{r-2}\left\{e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}\right\}dy \\
&= \frac{1}{(\rho-1)\,\eta^r}\left\{\frac{e^{\delta/\rho\eta}}{\rho^r}\Gamma\left(1-r,\delta/\rho\eta\right) - e^{\delta/\eta}\Gamma\left(1-r,\delta/\eta\right)\right\}
\end{aligned}$$

where $r=0,1,...,n$.
(ii) Again, for a random variable $Y \sim II_vD(\rho,\delta,\eta)$, the expression for r^{th} moment about mean is

$$\begin{aligned}
\mu_r &= E(y-\mu_y)^r = \frac{1}{(\rho-1)\eta}\int_0^{1/\delta}(y-\mu_y)^r\frac{\left\{e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}\right\}}{y^2}dy \\
&= \frac{1}{(\rho-1)}\sum_{n=0}^{r} {}^rC_n \frac{(-\mu)^{r-n}}{\eta^n}\left\{\frac{e^{\delta/\rho\eta}}{\rho^n}\Gamma\left(1-n,\delta/\rho\eta\right) - e^{\delta/\eta}\Gamma\left(1-n,\delta/\eta\right)\right\}
\end{aligned}$$

where $r=0,1,...,n$. □

Theorem 3: If Y be any non-negative $r.v.$ possessing II_vD, then the generating functions for moments are given by:
$(i) M_y(t) = \frac{1}{(\rho-1)\eta}\sum_{r=0}^{\infty}\frac{t^r}{\eta^r r!}\left\{\frac{e^{\delta/\rho\eta}}{\rho^r}\Gamma\left(1-r,\delta/\rho\eta\right) - e^{\delta/\eta}\Gamma\left(1-r,\delta/\eta\right)\right\}$,is the moment generating function.
$(ii) \phi_y(t) = \frac{1}{(\rho-1)\eta}\sum_{r=0}^{\infty}\frac{t^r}{\eta^r r!}\left\{\frac{e^{\delta/\rho\eta}}{\rho^r}\Gamma\left(1-r,\delta/\rho\eta\right) - e^{\delta/\eta}\Gamma\left(1-r,\delta/\eta\right)\right\}$, is the characteristic function.
$(iii) K_y(t) = \log\left[\frac{1}{(\rho-1)\eta}\sum_{r=0}^{\infty}\frac{t^r}{\eta^r r!}\left\{\frac{e^{\delta/\rho\eta}}{\rho^r}\Gamma\left(1-r,\delta/\rho\eta\right) - e^{\delta/\eta}\Gamma\left(1-r,\delta/\eta\right)\right\}\right]$, is the cumulant generating function.

Proof: (i) Let the $r.v.$ $Y \sim II_vD(\rho, \delta, \eta)$, then $M_y(t)$ is derived by

$$\begin{aligned} M_y(t) &= E(e^{ty}) = \int_0^{1/\delta} e^{ty} f_{II_vD}(y;\Theta)dy \\ &= \frac{1}{(\rho-1)\,\eta} \sum_{r=0}^{\infty} \frac{t^r}{\eta^r r!} \left\{ \frac{e^{\delta/\rho\eta}}{\rho^r} \Gamma\left(1-r, \delta/\rho\eta\right) - e^{\delta/\eta} \Gamma\left(1-r, \delta/\eta\right) \right\} \end{aligned}$$

(ii) To prove the characteristic function the same procedure has to be repeated that we used to derive the moment generating function, but the only change is instead of t we have to proceed with ιt.

(iii) Let the $r.v.$ $Y \sim II_vD(\rho, \delta, \eta)$, then $K_y(t)$ is defined by

$$\begin{aligned} K_y(t) &= \log\left\{M_y(t)\right\} \\ &= \log\left[\frac{1}{(\rho-1)\,\eta} \sum_{r=0}^{\infty} \frac{t^r}{\eta^r r!} \left\{ \frac{e^{\delta/\rho\eta}}{\rho^r} \Gamma\left(1-r, \delta/\rho\eta\right) - e^{\delta/\eta} \Gamma\left(1-r, \delta/\eta\right) \right\} \right] \end{aligned}$$

□

3. Reliability properties

The probability measurement of any component or a system, that will not fail before time t to perform its complete operation is called the reliability of the system. Mathematically, it is calculated as:

$$R_{II_vD}(y;\Theta) = Pr.(Y > y) = 1 - Pr.(Y \leq y)$$

Thus, for a $r.v.$ $Y \sim II_vD(\rho, \delta, \eta)$ the derived reliability function is obtained as

$$R_{II_vD}(y;\Theta) = \begin{cases} 1 - \frac{\rho e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}}{(\rho-1)} & \rho \neq 1 \\ 1 - \left(\frac{1-(\delta-\eta)y}{\eta y}\right) e^{-(1-\delta y)/\eta y} & \rho = 1 \end{cases} \tag{5}$$

If $\hat{\rho}$, $\hat{\delta}$ and $\hat{\eta}$ are the MLE_s, then by the in-variance property the reliability estimate are given by

$$\hat{R}_{II_vD}(y;\hat{\Theta}) = \begin{cases} 1 - \frac{\hat{\rho} e^{-(1-\hat{\delta} y)/\hat{\rho}\hat{\eta} y} - e^{-(1-\hat{\delta} y)/\hat{\eta} y}}{(\hat{\rho}-1)} & \rho \neq 1 \\ 1 - \left(\frac{1-(\hat{\delta}-\hat{\eta})y}{\hat{\eta} y}\right) e^{-(1-\hat{\delta} y)/\hat{\eta} y} & \rho = 1 \end{cases} \tag{6}$$

The hazard rate for II_vD, which we will denote by $h_{II_vD}(y)$ is the ratio of pdf and the $R_{II_vD}(y)$ as given below:

$$h_{II_vD}(y;\Theta) = \frac{e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}}{\eta y^2 \left[(\rho-1) - \left\{\rho e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}\right\}\right]} \tag{7}$$

The graphical plot of hazard function for different set of parametric values is shown in Figure 2.

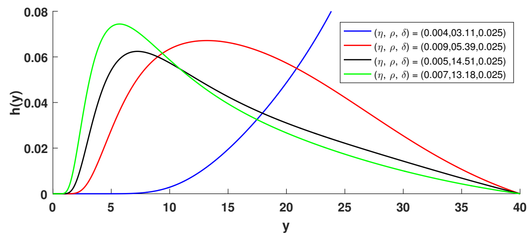

Figure 2: Hazard plot

The hazard rate in the reverse direction of time is called reverse hazard rate which we denote by $h^r_{II_vD}(y)$, this measure is obtained by taking the ratio of pdf and cdf and the obtained expression is

$$h^r_{II_vD}(y;\Theta) = \frac{e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}}{\eta y^2 \left[\rho e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}\right]} \tag{8}$$

The famous reliability measure called aging intensity $(A.I)$ developed by Jiang *et al.* (2003) are used for quantitative aging measurement purposes, as aging representation for the system by uni-modal hazard rate is difficult because of its varying trends observed in the form of constant, increasing and decreasing hazard rates. The $A.I$ for a $r.v.$ $Y \sim II_vD(\rho,\delta,\eta)$, denoted by L_y is give as

$$A.I = \frac{e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}}{\eta y \left[\log(\rho-1) - \log\left\{\rho e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}\right\}\right]\left[\rho e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y}\right]} \tag{9}$$

3.1. Mean residual life function

The mean residual life (MRL) function having a variety of applications in different branches of statistical and applied sciences, to define this measure, suppose a system/component functions without fail up to time $y \geq 0$, then the residual life is counted as the working hours of the system beyond time y until it fails, and the conditional $r.v.$ $Y - y|Y > y$ is used to define this measure Finkelstein (2008).
For non-negative $r.v.$ $Y \sim II_vD(\rho,\delta,\eta)$, the MRL function denoted by $m_{II_vD}(y,\Theta)$ is derived as

$$m_{II_vD}(y;\Theta) = E\left[Y - y|Y > y\right] = \frac{1}{R_{II_vD}(y;\Theta)}\int_y^{1/\delta} R_{II_vD}(y;\Theta)dy.$$

$$= \frac{1}{(\rho-1)R_{II_vD}(y;\Theta)}\int_y^{1/\delta}\left\{(\rho-1) - \rho e^{(1-\delta y)/\rho\eta y} + e^{(1-\delta y)/\eta y}\right\}dy.$$

$$= \frac{1}{(\rho-1)R_{II_vD}(y;\Theta)}\left\{\frac{\left(\rho-\rho e^{\delta/\rho\eta}-e^{\delta/\eta}-1\right)(1-\delta y)}{\delta} - \frac{\left(e^{\delta/\rho\eta}+e^{\delta/\eta}\right)\log(\delta y)}{\eta} + \sum_{r=2}^{\infty}\frac{(-1)^{r+2}}{r!\eta^r(1-r)}\left(\delta^{r-1}-y^{1-r}\right)\left(\rho^{1-r}e^{\delta/\rho\eta}-e^{\delta/\eta}\right)\right\}$$

4. Entropy measures

Entropy measurements are useful to determine, how much the random variable's distribution varies in terms of its level of variations, and the two important measures to address this variation are given by Rényi entropy and Shannon entropy (Refer Rényi, A. (1961, January) and Shannon (1948)).

4.1. Reńyi entropy

The Rényi entropy metric for a non-negative $r.v.\ Y \sim II_vD(\rho,\delta,\eta)$ of order ϑ is given by

$$H_R(\vartheta) = \frac{1}{1-\vartheta}\log\left[\int_0^{1/\delta}\left\{f_{II_vD}(y;\Theta)\right\}^{\vartheta}dy\right];\quad \vartheta \geq 0, \vartheta \neq 1 \tag{10}$$

$$= \frac{1}{1-\vartheta}\log\left[\frac{1}{(\rho-1)^{\vartheta}\eta^{\vartheta}}\sum_{r=0}^{\vartheta}\frac{\binom{\vartheta}{r}(-1)^{r+2}e^{\psi}}{r!\psi^{2\vartheta-1}}\Gamma\left(2\vartheta-1,\psi\delta\right)\right]$$

where $\psi = \frac{(\rho-1)}{\rho\eta}\left(r+\frac{\vartheta}{(\rho-1)}\right)$ and $f_{II_vD}(y,\Theta)$ is the pdf given in equation (2), when $\rho \neq 1$.

4.2. Shannon entropy

In this subsection, we will derive the expression for Shannon measure of entropy for a non-negative $r.v.\ Y \sim II_vD(\rho,\delta,\eta)$, the derivation steps for this extend concept of Reńyi entropy are given by

$$H_{II_vD}(y) = -\int_0^{1/\delta} f_{II_vD}(y;\Theta)\log\left\{f_{II_vD}(y;\Theta)\right\}dy.$$

Now, substitute the density function $f_{II_vD}(y;\Theta)$, given in equation (2), when $\rho \neq 1$, and solve the integral we get

$$= \sum_{r=1}^{\infty}\frac{\rho B\left(r+\frac{1}{(\rho-1)},2\right)}{r(\rho-1)^2} + \frac{(\rho+1)}{\rho} - 2\sum_{r=1}^{\infty}\frac{(-1)^{r+1}\eta^r\left(\rho^{r+1}-1\right)\Gamma(r+1)}{r(\rho-1)\delta^r} + \frac{\rho B\left(\frac{1}{(\rho-1)},2\right)\log\{(\rho-1)\eta\}}{(\rho-1)^2} - 2\log\delta$$

5. Order statistics

In the field of reliability, order statistics finds massive applications in life testing experiments for understanding system characterization of system. Let a random sample of size n be taken as $Y = (y_1,\ y_2,\ \ldots\ ,y_n)$ be drawn from $II_vD(\rho,\delta,\eta)$. Then the life of $(n-i+1)$ components out-of-n $i.i.d$ systems based on ordered random sample $y_{(1:n)} \leq y_{(2:n)} \leq \ldots \leq y_{(n:n)}$ are given by $y_{i:n}$; $(i = 1,2,\ldots,n)$. Thus for $II_vD(\rho,\delta,\eta)$ the i^{th} order statistics density function of $y_{(i:n)}$; $1 \leq i \leq n$ are given as

$$f_{i:n}(y,\Theta) = M_1[F_{II_vD}(y)]^{i-1}[1-F_{II_vD}(y)]^{n-i}f_{II_vD}(y). \tag{11}$$

Also, the pdf of $(i,j)^{th}$ order statistics density for $(y_{(i:n)}, y_{(j:n)})$; $1 \leq i \leq j \leq n$ are as

$$f_{i:j:n}(y_i,y_j) = M_2[F_{II_vD}(y_i)]^{i-1}[F_{II_vD}(y_j)-F_{I_vD}(y_i)]^{j-i-1}[1-F_{II_vD}(y_j)]^{n-j}f_{II_vD}(y_i)f_{II_vD}(y_j). \tag{12}$$

where $F(.)$, $f(.)$ is the cdf, pdf of II_vD defined in (1) and (2), and the constants M_1 and M_2 are given by

$$M_1 = \frac{n!}{(i-1)!(n-i)!} \text{ and } M_2 = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

The smallest observation of ordered sample is called first-order statistic given by $y_{(1)} = min.(y_{(1)},\ y_{(2)},\ \ldots\ ,y_{(n)})$, the largest observation is called the n^{th} order statistic, and the middle observation is called the median order given by y_{m+1}

5.1. Order statistic density function of II_vD

Let $y_{(1)}$, $y_{(2)}$, ... ,$y_{(n)}$ be $i.i.d$ ordered random sample from II_vD then according to equations (1) and (2) we can write the first order statistics density ($f_{1:n}(.)$) on substituting $i = 1$, in equation (11), the n^{th} order statistics density ($f_{n:n}(.)$) by substituting $i = n$ in equation (11) and the median order statistics density denoted by ($f_{m+1:n}(.)$);[$m = \frac{n}{2}$] are given below:

$$f_{1:n}(y) = n\left[1 - F_{II_vD}(y_{(1)})\right]^{n-1} f_{II_vD}(y_{(1)})$$

$$f_{1:n}(y) = \frac{n}{\eta(\rho-1)^n y_{(1)}^2}\left[(\rho-1) - \left\{\rho e^{-(1-\delta y_{(1)})/\rho\eta y_{(1)}} - e^{-(1-\delta y_{(1)})/\eta y_{(1)}}\right\}\right]^{n-1}\left[e^{-(1-\delta y_{(1)})/\rho\eta y_{(1)}} - e^{-(1-\delta y_{(1)})/\eta y_{(1)}}\right] \quad (13)$$

Similarly,

$$f_{n:n}(y) = n\left[F_{II_vD}(y)\right]^{n-1} f_{II_vD}(y_{(n)})$$

$$f_{n:n}(y) = \frac{n}{\eta(\rho-1)^n y_{(n)}^2}\left[\rho e^{-(1-\delta y_{(n)})/\rho\eta y_{(n)}} - e^{-(1-\delta y_{(n)})/\eta y_{(n)}}\right]^{n-1}\left[e^{-(1-\delta y_{(n)})/\rho\eta y_{(n)}} - e^{-(1-\delta y_{(n)})/\eta y_{(n)}}\right] \quad (14)$$

and,

$$f_{m+1:n}(y) = \frac{(2m+1)!}{(m!)^2}\left[F_{II_vD}(\bar{y})\right]^m\left[1 - F_{II_vD}(\bar{y})\right]^m f_{II_vD}(\bar{y}). \quad (15)$$

5.2. Joint order statistics density of II_vD

The joint pdf of II_vD is obtained by using the pdf and cdf in (12) as shown below:

$$\begin{aligned} f_{i:j:n}(y_{(i)},\quad y_{(j)}) = {} & \frac{M_2}{\eta^2(\rho-1)^n y^4}\left[\rho e^{-(1-\delta y_{(i)})/\rho\eta y_{(i)}} - e^{-(1-\delta y_{(i)})/\eta y_{(i)}}\right]^{i-1} \\ . \quad & \left[\left\{\rho e^{-(1-\delta y_{(i)})/\rho\eta y_{(i)}} - e^{-(1-\delta y_{(i)})/\eta y_{(i)}}\right\} - \left\{\rho e^{-(1-\delta y_{(j)})/\rho\eta y_{(j)}} - e^{-(1-\delta y_{(j)})/\eta y_{(j)}}\right\}\right]^{j-i-1} \\ . \quad & \left[(\rho-1) - \left\{\rho e^{-(1-\delta y_{(j)})/\rho\eta y_{(j)}} - e^{-(1-\delta y_{(j)})/\eta y_{(j)}}\right\}\right]^{n-j}\left[e^{-(1-\delta y_{(i)})/\rho\eta y_{(i)}} - e^{-(1-\delta y_{(i)})/\eta y_{(i)}}\right] \\ . \quad & \left[e^{-(1-\delta y_{(j)})/\rho\eta y_{(j)}} - e^{-(1-\delta y_{(j)})/\eta y_{(j)}}\right] \end{aligned}$$

6. Stochastic ordering

Stochastic ordering measurement for lifetime distributions has vital importance in reliability theory, nicely discussed by Shaked and Shantikumar (2007). Let the $r.v.'s$ Y_1 and Y_2 possessing the II_vD with pdf's $f_{Y_1}(y)$, $f_{Y_2}(y)$, and cdf's $F_{Y_1}(y)$, $F_{Y_1}(y)$ respectively. Then one would say Y_1 is smaller than Y_2 according to the stochastic ordering measurements given below:

[1] Stochastic order ($Y_1 \leq_{st} Y_2$), if $F_{Y_1}(y) \geq F_{Y_2}(y)$ for all y.
[2] Hazard rate order ($Y_1 \leq_{hr} Y_2$), if $H_{Y_1}(z) \geq H_{Y_2}(y)$ for all y.
[3] Mean residual life order ($Y_1 \leq_{MRL} Y_2$), if $m_{Y_1}(y) \geq m_{Y_2}(y)$ for all y.
[4] Likelihood ratio order ($Y_1 \leq_{LR} Y_2$), if $\frac{f_1(y)}{f_2(y)}$ decreasing in y.

Hence the following implication is revealed according to the above orderings
$Y_1 \leq_{LR} Y_2 \Rightarrow Y_1 \leq_{hr} Y_2 \Rightarrow Y_1 \leq_{MRL} Y_2$ and $Y_1 \leq_{hr} Y_2 \Rightarrow Y_1 \leq_{st} Y_2$.
Following theorem illustrate likelihood ratio ordering for II_vD w.r.t the strongest likelihood.

Theorem 4: Let $Y_1 \sim II_vD(\delta_1, \rho_1, \eta_1)$, and $Y_2 \sim II_vD(\delta_2, \rho_2, \eta_2)$. If $\delta_1 = \delta_2 = \delta$, $(\rho_1 > \rho_2) > 1$, and $(\eta_1 > \eta_2)$ then $(Y_1 \leq_{lr} Y_2)$, $(Y_1 \leq_{st} Y_2)$, $(Y_1 \leq_{hr} Y_2)$, and $(Y_1 \leq_{MRL} Y_2)$.

Proof: To prove the result, the ratio of probability densities is

$$\frac{f_{Y_1}(y;\Theta_1)}{f_{Y_2}(y;\Theta_2)} = \frac{(\rho_2-1)\eta_2}{(\rho_1-1)\eta_1}\frac{\left\{e^{-(1-\delta_1 y)/\rho_1\eta_1 y} - e^{-(1-\delta_1 y)/\eta_1 y}\right\}}{\left\{e^{-(1-\delta_2 y)/\rho_2\eta_2 y} - e^{-(1-\delta_2 y)/\eta_2 y}\right\}}$$

Then,

$$\frac{d}{dy}\log\left\{\frac{f_{Y_1}(y;\Theta_1)}{f_{Y_2}(y;\Theta_2)}\right\} = \frac{\left[\{\eta_2 B\left(A_1-\rho_1 A_2\right)\} - \{\eta_1 A\left(B_1-\rho_2 B_2\right)\}\right]}{y^2 AB}$$

where, $A = \left\{e^{-(1-\delta_1 y)/\rho_1\eta_1 y} - e^{-(1-\delta_1 y)/\eta_1 y}\right\}$, $B = \left\{e^{-(1-\delta_2 y)/\rho_2\eta_2 y} - e^{-(1-\delta_2 y)/\eta_2 y}\right\}$,
$A_1 = e^{-(1-\delta_1 y)/\rho_1\eta_1 y}$, $A_2 = e^{-(1-\delta_1 y)/\eta_1 y}$, $B_1 = e^{-(1-\delta_2 y)/\rho_2\eta_2 y}$, and $B_2 = e^{-(1-\delta_2 y)/\eta_2 y}$.
Hence, If $\delta_1 = \delta_2 = \delta$, $(\rho_1 > \rho_2)$, and $(\eta_1 > \eta_2)$ then $\frac{d}{dy}\log\left\{\frac{f_{Y_1}(y;\Theta_1)}{f_{Y_2}(y;\Theta_2)}\right\} \leq 0$, which implies that $(Y_1 \leq_{lr} Y_2)$, $(Y_1 \leq_{st} Y_2)$, $(Y_1 \leq_{hr} Y_2)$, and $(Y_1 \leq_{MRL} Y_2)$. □

7. Stress-strength reliability

In this section, we study system reliability estimation under stress strength modeling, which possesses a cluster of applications, particularly in engineering statistics. Let Y_1 be the strength of the system subjected to stress Y_2. The system fails, when $Y_2 > Y_1$ (stress > strength), and functions smoothly, when $Y_1 > Y_2$ (stress < strength). Then the system reliability is measured by using the formula $R = Pr.\ (Y_1 > Y_2)$.
For two independent $r.v.'s$, $Y_1 \sim II_vD(\delta, \rho_1, \eta_1)$, and $Y_2 \sim II_vD(\delta, \rho_2, \eta_2)$, having the same parameter δ. For the given, pdf of Y_1 and cdf of Y_2 the stress-strength reliability function R is derived by

$$F_{II_vD}(y;\Theta_2) = \frac{\rho_2 e^{-(1-\delta y)/\rho_2\eta_2 y} - e^{-(1-\delta y)/\eta_2 y}}{(\rho_2-1)} \quad \rho_2 \neq 1 \tag{16}$$

and,

$$f_{II_vD}(y;\Theta_1) = \frac{e^{-(1-\delta y)/\rho_1\eta_1 y} - e^{-(1-\delta y)/\eta_1 y}}{(\rho_1-1)\eta_1 y^2} \qquad \rho_1 \neq 1 \tag{17}$$

Therefore, possible derived cases are given by:
Case (i): when $\rho_1 \neq 1$, and $\rho_2 \neq 1$.

$$\begin{aligned} R &= \int_0^{1/\delta}\left\{\int_0^y f_{y_2}(y)dy\right\} f_{y_1}(y)dy = \int_0^{1/\delta} F_{Y_2}(y) f_{Y_1}(y)dy \\ &= \int_0^{1/\delta}\left\{\frac{\rho_2 e^{-(1-\delta y)/\rho_2\eta_2 y} - e^{-(1-\delta y)/\eta_2 y}}{(\rho_2-1)}\right\}\left\{\frac{e^{-(1-\delta y)/\rho_1\eta_1 y} - e^{-(1-\delta y)/\eta_1 y}}{(\rho_1-1)\eta_1 y^2}\right\} dy \\ &= \frac{\eta_2^2}{\rho_2-1}\left\{\frac{\rho_2^3}{(\rho_1\eta_1+\rho_2\eta_2)(\eta_1+\rho_2\eta_2)} - \frac{1}{(\rho_1\eta_1+\eta_2)(\eta_1+\eta_2)}\right\} \end{aligned}$$

Case (ii): when $\rho_1 \neq 1$, and $\rho_2 = 1$.

$$R = \frac{\eta_2}{(\rho_1 - 1)} \left\{ \frac{\rho_1^2 \eta_1}{(\rho_1\eta_1 + \eta_2)^2} + \frac{\rho_1}{(\rho_1\eta_1 + \eta_2)} - \frac{\eta_1}{(\eta_1 + \eta_2)^2} - 1 \right\}$$

Case (iii): when $\rho_1 = 1$, and $\rho_2 \neq 1$.

$$R = \frac{\eta_2^2}{(\rho_2 - 1)} \left\{ \frac{\rho_2^3}{(\eta_1 + \rho_2\eta_2)^2} - \frac{1}{(\eta_1 + \eta_2)^2} \right\}$$

Case (iv): when $\rho_1 = 1$, and $\rho_2 = 1$.

$$R = \frac{\eta_2^2 (3\eta_1 + \eta_2)}{(\eta_1 + \eta_2)^3}$$

8. Estimation of the parameters

Let us consider a random sample of n observations, say y_1, y_2, ... ,y_n drawn from II_vD with desired defined parametric space $\Theta = (\rho, \delta, \eta)^T$ consisting $k \times 1$ vector of parameters. Then the completer data log-likelihood of the model when $\rho \neq 1$ is given by

$$\log L = \sum_{i=1}^{n} \log \left\{ e^{-(1-\delta y)/\rho\eta y} - e^{-(1-\delta y)/\eta y} \right\} - n\log(\rho - 1) - n\log(\eta) - \sum_{i=1}^{n} \log(y_i^2)$$

Let us take, $V_1 = e^{-(1-\delta y)/\rho\eta y}$, and $V_2 = e^{-(1-\delta y)/\eta y}$, then we re-write the above equation as

$$\log L = \sum_{i=1}^{n} \log\{V_1 - V_2\} - n\log(\rho - 1) - n\log(\eta) - \sum_{i=1}^{n} \log(y_i^2) \tag{18}$$

Now, the partial derivative for the above equation (18) with respect to the parameters ρ, δ, and η are obtained as:

$$\frac{\partial \log L}{\partial \rho} = \sum_{i=1}^{n} \frac{[1 - \delta y_i] V_1}{[V_1 - V_2] \rho^2 \eta y_i} - \frac{n}{(\rho - 1)} \tag{19}$$

$$\frac{\partial \log L}{\partial \delta} = \sum_{i=1}^{n} \frac{V_1 - \rho V_2}{\rho\eta [V_1 - V_2]} - 0 - 0 \tag{20}$$

$$\frac{\partial \log L}{\partial \eta} = \sum_{i=1}^{n} \frac{(1 - \delta y_i)[V_1 - \rho V_2]}{[V_1 - V_2] \rho\eta^2 y_i} - \frac{n}{\eta} \tag{21}$$

Equating the partial derivatives given in equations (19), (20), and (21) to zero, i,e $\frac{\partial \log L}{\partial \rho} = 0$, $\frac{\partial \log L}{\partial \delta} = 0$, and $\frac{\partial \log L}{\partial \eta} = 0$, we get $\hat{\rho}$, $\hat{\delta}$, and $\hat{\eta}$ as the MLE_s of the parameters space $\Theta = \{(\rho, \delta, \eta) > 0\}$. Since the equation (19), (20) and (21) does not reveal the explicit solution, to get the parametric solution of the equations, one can counter this situation by employing the Newton Raphson algorithm. However, log-likelihood maximization could be done by using *nlm* or *optim* function in R-software.

The first-order derivatives of the log-likelihood equation of $II_vD(\rho,\delta,\eta)$ are defined in equations (19), (20), and, (21). The continuity of these partial derivatives reflects the second order partial derivatives of the log-likelihood equation does exist. If we denote the MLE_s of the parametric space, $\Theta = \{(\rho,\delta,\eta) > 0\}$ by $\hat{\Theta} = \left\{(\hat{\rho},\hat{\delta},\hat{\eta}) > 0\right\}$, then the Fisher information matrix is given by

$$I(\Theta) = -E\begin{bmatrix} \frac{\partial^2 \log L}{\partial\rho^2} & \frac{\partial^2 \log L}{\partial\rho\partial\delta} & \frac{\partial^2 \log L}{\partial\rho\partial\eta} \\ \frac{\partial^2 \log L}{\partial\delta\partial\rho} & \frac{\partial^2 \log L}{\partial\delta^2} & \frac{\partial^2 \log L}{\partial\delta\partial\eta} \\ \frac{\partial^2 \log L}{\partial\eta\partial\rho} & \frac{\partial^2 \log L}{\partial\eta\partial\delta} & \frac{\partial^2 \log L}{\partial\eta^2} \end{bmatrix} \tag{22}$$

The second order partial derivatives of $I(\Theta)$ are given by

$$\frac{\partial^2 \log L}{\partial\rho^2} = \sum_{i=1}^{n} \frac{(1-\delta y_i)^2 [V_1 - V_2] V_1 - (1-\delta y_i) V_1 \{2\rho\eta [V_1 - V_2] y_i + (1-\delta y_i) V_1\}}{[(V_1 - V_2)\rho^2\eta y_i]^2} + \frac{n}{(\rho-1)^2} \tag{23}$$

$$\frac{\partial^2 \log L}{\partial\delta^2} = \sum_{i=1}^{n} \frac{[V_1 - V_2][V_1 - \rho^2 V_2] - [V_1 - \rho V_2]^2}{[(V_1 - V_2)\rho\eta]^2} \tag{24}$$

$$\frac{\partial^2 \log L}{\partial\eta^2} = \sum_{i=1}^{n} \frac{(1-\delta y_i)^2 [V_1 - V_2]\left[V_1 - \rho^2 V_2\right] - (1-\delta y_i)[V_1 - \rho V_2]\{2\rho\eta [V_1 - V_2] y_i + (1-\delta y_i)[V_1 - \rho V_2]\}}{[(V_1 - V_2)\rho\eta^2 y_i]^2} - \frac{n}{\eta^2} \tag{25}$$

$$\frac{\partial^2 \log L}{\partial\delta\partial\rho} = \sum_{i=1}^{n} \frac{V_1 [V_1 - V_2]\{(1-\delta y_i) - \rho\eta y_i\} - V_1 [V_1 - \rho V_2](1-\delta y_i)}{[(V_1 - V_2)\eta]^2 \rho^3 y_i} \tag{26}$$

$$\frac{\partial^2 \log L}{\partial\eta\partial\rho} = \sum_{i=1}^{n} \frac{V_1 [V_1 - V_2](1-\delta y_i)^2 - V_1 (1-\delta y_i)\{\rho\eta [V_1 - V_2] y_i + (1-\delta y_i)[V_1 - \rho]\}}{[(V_1 - V_2) y_i]^2 (\rho\eta)^3} \tag{27}$$

$$\frac{\partial^2 \log L}{\partial\eta\partial\delta} = \sum_{i=1}^{n} \frac{\eta [V_1 - V_2][V_1 - \rho^2 V_2](1-\delta y_i) - [V_1 - \rho V_2]\{\rho\eta^2 [V_1 - V_2] y_i - [V_1 - \rho V_2]\}}{[(V_1 - V_2)\rho\eta^2]^2 y_i} \tag{28}$$

It is difficult to obtain the expectation of second-order partial derivative expressions. Thus, in this situation, one can use the alternative measure called observed Fisher information matrix given by

$$I\left(\hat{\Theta}\right) = -\begin{bmatrix} \frac{\partial^2 \log L}{\partial\rho^2} & \frac{\partial^2 \log L}{\partial\rho\partial\delta} & \frac{\partial^2 \log L}{\partial\rho\partial\eta} \\ \frac{\partial^2 \log L}{\partial\delta\partial\rho} & \frac{\partial^2 \log L}{\partial\delta^2} & \frac{\partial^2 \log L}{\partial\delta\partial\eta} \\ \frac{\partial^2 \log L}{\partial\eta\partial\rho} & \frac{\partial^2 \log L}{\partial\eta\partial\delta} & \frac{\partial^2 \log L}{\partial\eta^2} \end{bmatrix}_{(\rho,\delta,\eta)=(\hat{\rho},\hat{\delta},\hat{\eta})} \tag{29}$$

The inverse of the observed Fisher information matrix $I\left(\hat{\Theta}\right)$, will give diagonal elements as variances whereas the off-diagonal elements represents the co-variances of the matrix. The approximate $(1-\sigma)$ 100% confidence intervals for all the three parameters of II_vD i,e ρ, δ, and η are $\hat{\rho} \pm \psi_{\sigma/2}\sqrt{V(\hat{\rho})}$, $\hat{\delta} \pm \psi_{\sigma/2}\sqrt{V(\hat{\delta})}$, and $\hat{\eta} \pm \psi_{\sigma/2}\sqrt{V(\hat{\eta})}$ respectively. where, $V(\hat{\rho})$, $V(\hat{\delta})$, and $V(\hat{\eta})$ are variances given in diagonal elements of $I(\Theta)^{-1}$ and the upper $(\sigma/2)$ percentile of a standard normal distribution is denoted by $\psi_{\sigma/2}$.

9. Simulation

In this section, a Monte Carlo simulation study with 1000 repetitions has been performed through R-software, to illustrate the theoretical findings of the proposed model. Since data generation has been done by employing the acceptance-rejection algorithm due to the complexity of the quantile function. The performance of the parametric space Θ with different sample sizes $n = (25, 75, 125, 175, 250, 400)$ are checked by observing the calculated $AAbias$ and the MSE of the estimated parameters. The output result of the simulation are summarized in Table 1, given below:

Table 1: Simulated results of parameters for different sample sizes

(δ, η, ρ)	n	$AAbias$			MSE		
		$\hat{\delta}$	$\hat{\eta}$	$\hat{\rho}$	$\hat{\delta}$	$\hat{\eta}$	$\hat{\rho}$
$(0.53, 0.92, 0.94)$	025	0.13926	0.36362	36.61545	0.01939	0.13222	1340.691
	075	0.05185	0.25571	02.96885	0.00269	0.06539	08.81406
	125	0.03573	0.22752	01.26163	0.00128	0.05177	01.59170
	175	0.02620	0.20220	00.74678	0.00069	0.04088	00.55769
	250	0.02049	0.18039	00.59632	0.00042	0.03254	00.35560
	400	0.01488	0.16094	00.50596	0.00022	0.02590	00.25599
$(1.53, 0.92, 1.04)$	025	0.13811	0.32654	21.63234	0.01907	0.10663	467.9582
	075	0.05757	0.21819	02.32618	0.00331	0.04761	05.41112
	125	0.03954	0.19292	01.01383	0.00156	0.03722	01.02784
	175	0.02690	0.16254	00.61841	0.00072	0.02642	00.38243
	250	0.02142	0.13128	00.46323	0.00046	0.01724	00.21458
	400	0.01614	0.12249	00.40460	0.00026	0.01500	00.16370
$(0.53, 1.92, 0.94)$	025	0.30424	0.77793	37.09144	0.09256	0.60518	1375.775
	075	0.11171	0.52213	03.76642	0.01248	0.27262	14.18596
	125	0.07156	0.44599	01.42634	0.00512	0.19891	02.03445
	175	0.05285	0.39587	00.70007	0.00279	0.15671	00.49010
	250	0.04294	0.36643	00.59156	0.00184	0.13427	00.34995
	400	0.02577	0.31186	00.45712	0.00066	0.09726	00.20896

It is easily noticed in Table 1 while increasing the sample size the $AAbias$ and MSE are reducing. Hence, this admits the consistency property of the parametric space of our model.

10. Applications

This section is about the model applicability checkup on a real-life data basis. The data set has been analyzed. In this study, the performance of newly developed II_vD is compared with existing distributions like the Exponential distribution (ED), Inverse Exponential distribution (IED), generalized Exponential distribution (GED), and the generalized inverse Exponential distribution ($GIED$). The best model is chosen having minimum value of Akaike information criteria (defined as, $AIC =$ - $2 \log L(y, \Theta) + 2k$), Bayesian information criteria (defined as, $BIC =$ - $2 \log L(y, \Theta) + k \log(n)$), Hannan Quinn information criteria (defined as, $HQIC =$ - $2 \log L(y, \Theta) + 2k \log[\log(n)]$), and the goodness of fit tests, that includes Cramér-von Mises (C_{vm}) test, Anderson Darling (A_n) test, Kolmogorov Smirnov (KS) statistic respectively. The constant k denotes the number of parameters in the model.

For a given real-life data set, the results of different information criteria, the goodness of fit measures, and the *p-value* are reported in Table 2. The II_vD is compared with existing models whose probability density functions are given by

$$ED = f(y;\delta) = \delta e^{-\delta y} \tag{30}$$

$$IED = f(y;\delta) = \frac{\delta}{y^2} e^{-\delta/y} \tag{31}$$

$$GED = f(y;\delta,\eta) = \delta\eta e^{-\delta y}\left(1-e^{-\delta y}\right)^{\eta-1} \tag{32}$$

$$GIED = f(y;\delta,\eta) = \frac{\delta\eta}{y^2} e^{-\delta/y}\left(1-e^{-\delta/y}\right)^{\eta-1} \tag{33}$$

The given data set is taken from the paper published by Ahmed M. A. (2021), which represents the lifetime (in hours) of traditional lights, for 50 devices. The data are: 0.913, 0.786, 0.860, 0.904, 0.971, 0.616, 0.961, 0.789, 0.817, 0.722, 0.956, 0.835, 0.853, 0.692, 0.850, 0.677, 0.898, 0.965, 0.820, 0.964, 0.865, 0.947, 0.798, 0.746, 0.926, 0.709, 0.615, 0.747, 0.931, 0.913, 0.895, 0.745, 0.839, 0.766, 0.690, 0.531, 0.838, 0.846, 0.876, 0.817, 0.719, 0.907, 0.915, 0.879, 0.890, 0.865, 0.869, 0.772, 0.933, 0.875.

Table 2: Results of information measures and goodness of fit tests

$Models$	$Part-I$						
	$\hat{\delta}$	$\hat{\eta}$	$\hat{\rho}$	$\log L$	AIC	BIC	$HQIC$
II_vD	1.02014	0.02359	7.68817	50.175	-94.35043	-88.61436	-92.16610
$GIED$	7.68390	7145.98	-	45.414	-86.82726	-83.00322	-85.37104
GED	8.67100	838.120	-	34.645	-65.29067	-61.46663	-63.83445
IED	0.81630	-	-	-40.742	83.48415	85.39617	84.21226
ED	1.20440	-	-	-40.699	83.39836	85.31039	84.12647
$Models$	$Part-II$						
	$\hat{\delta}$	$\hat{\eta}$	$\hat{\rho}$	C_{vm}	A_n	KS	*p-value*
II_vD	1.02014	0.02359	7.68817	0.02439	0.19464	0.05978	0.9941
$GIED$	7.68390	7145.98	-	0.09202	0.57741	0.11345	0.5405
GED	8.67100	838.120	-	0.34397	2.05467	0.16876	0.1159
IED	0.81630	-	-	0.24371	1.47237	0.56858	1.82×10^{-14}
ED	1.20440	-	-	0.18330	1.11543	0.50322	2.01×10^{-11}

From Table 2, it is well observed that the II_vD fits best as it has a minimum value for all the information criteria (AIC, BIC, and $HQIC$) as well as the goodness of fit tests, and a higher *p-value*.

11. Conclusion

This manuscript presents an intervention-based model called inverted intervened Exponential distribution. The graphical plots based on a different set of parameters for pdf and hazard rate are shown, pdf having different shapes where the hazard rate function has upside down and exponentially increasing shapes, that could be useful to model different types of failure data. The essential statistical and reliability properties are derived. The parameters have been estimated by using the method of maximum likelihood estimation. A Monte Carlo simulation study has been done, where it is observed that both bias and mean square error for all the parameter decreases while increasing the sample size. The real-life data set have been analyzed and it is predicted that the values of all the information measures and the different goodness of fit tests for the proposed distribution are very less, with a higher *p-value* as compared to the existing models, which ensures the real-life applicability of the model.

Acknowledgements

The first author is very grateful to Pondicherry University for providing him with a research fellowship to carry out this work. Also, the authors are grateful to the editor of the journal and the anonymous reviewers for their valuable suggestions.

References

Ahmed, M. A. (2021). On the identified power Topp-Leone distribution: properties, simulation, and applications. *Thailand Statistician*, **19**, 838-854.

Balakrishnan, K. (2019). *Exponential Distribution: Theory, Methods and Applications*. CRC Press. url: https://books.google.co.in/books?id=aTmDDwAAQBAJ

Finkelstein, M. (2008). *Failure Rate Modelling for Reliability and Risk*. Springer Science & Business Media. doi: 10.1007/978-1-84800-986-8

Jiang, R. Ji, P. and Xiao, X. (2003). Aging property of unimodal failure rate models. *Reliability Engineering & System Safety*, **79**, 113-116. doi:10.1016/S0951-8320(02)00175-8

Rényi, A. (1961, January). On measures of entropy and information. *In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics* (pp. 547-561). University of California Press.

Shaked, M. and Shanthikumar, J. G. (2007). Stochastic orders. *Springer*, New York. doi: 10.1007/978-0-387-34675-5

Shanmugam, R. (1985). An intervened Poisson distribution and Its medical application. *Biometrics*, **41**, 1025-1029. doi:10.2307/2530973

Shanmugam, R. Bartolucci, A. A. and Singh, K. P. (2002). The analysis of neurologic studies using an extended exponential model. *Biometrics*, **59**, 81-85. doi: 10.1016/S0378-4754(01)00395-0

Shannon, C. E. (1948). A mathematical theory of communication. *The Bell System Technical Journal*, **27**, 379-423. doi:10.1002/j.1538-7305.1948.tb01338.x