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Bulk Queuing Model with Reneging of Customers and their Retention

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Abstract

In the current competitive scenario, customer satisfaction is a key aspect for any organization. This paper deals with the concept of customer reneging in the system. Due to improper quality of service, customers get dissatisfied, which represents queuing with feedback. Unsatisfied customers, after taking partial service, again put efforts into getting service in case of feedback. A single server Markovian feedback bulk queuing model $M^b/M/1$ (where b is the fixed batch size) is considered with the reneging of customers and their retention. The steady-state solution and various system performance measures are established. Sensitivity analysis of parameters is also performed and the effect on the size of the system is compared with the variation in the probability of retention, which shows that the higher the retention of customers, the larger the queue size in the system. MATLAB software is used to show the results graphically. Some particular cases for the proposed model are also examined.

Key words: Customer retention; Feedback; Bulk queuing model; Steady state solution; Performance measure.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

Nowadays, in this competitive era, businesses and organizations can flourish only if customers are satisfied. The quality of the product as well as quick service by the servers are the demands of every customer. Inefficiency in fulfilling these demands leads to customer dissatisfaction, which results in monetary losses for businesses. Thus, customer satisfaction is the measure of success for any business and reflects the degree to which the organization is able to meet the customer's expectations. The customer enters the system for service, but due to poor quality of service, leaves the system before completion of the service. This process is termed reneging, and the customers who leave the system are called reneged customers. Customer retention is the biggest challenge for organizations, as customer impatience is the

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major cause of this problem. Thus, by incorporating various strategies, an unsatisfied customer is convinced to remain in the system, which is termed retained customers. Customer impatience is categorized into three types:

- Balking is when a customer decides not to join the queue after seeing its size.
- Reneging is when a customer joins the queue for service but leaves the queue after waiting for a long time.
- Jockeying is when a customer switches between the parallel queues because they think that by doing so, they might get quick service.

Haight (1957) and Haight (1959) studied the concept of customer impatience and reneging in queuing theory. The concept of reneging and balking was also studied by Ancker and Gafarian (1963) in the M/M/1/N queuing system and obtained its steady-state solution. Abou-El-Ata and Hariri (1992) studied a multiple-channel truncated queue with balking and reneging and established the steady-state solution and various system performance measures of the proposed queuing model. Choudhury and Medhi (2011) analyzed the Markovian multi-server queuing model with balking and reneging, in which explicit closed forms were presented. Two abandonment scenarios with impatient customers in a single server Markovian queue were studied by Kapodistria (2011) In the first scenario, an existing customer becomes impatient and performs synchronized abandonments, and the customer is excluded from taking service in the second scenario. This work is then extended by him to a multi-server Markovian queue under the second abandonment scenario as well.

Kumar and Sharma (2012a) and Kumar and Sharma (2012b) developed an M/M/1/N queuing model with the reneging of customers and their retention and obtained the steady-state solution and various performance measures of the proposed model. They extended this work and developed an M/M/1/N queuing model using the concept of balking and retention of reneged customers. So, balking is another added concept that they used in their research. VijayaLaxmi and Jyothsna (2013) studied the optimization of reneging and balking queues with vacation interruption under N-policy.

Kumar and Sharma (2013) incorporated the notion of balking and reneging of customers with their retention in the M/M/1 feedback queuing model and developed a steady-state solution. VijayaLaxmi and Kassahun (2018) studied a multi-server Markovian queue with working vacations, reneging of customers, and discouraged arrivals and obtained the steady state and steady probabilities of the system. Kumar and Sharma (2021) discussed a Markovian queuing system with multiple heterogeneous servers, reneging, and retention of reneging customers. They performed transient analysis using a probability-generating function and important performance measures, including the average retention rate. Also, the steady-state solution of the model is obtained. Rimmy and Indra (2022) described the effect of balking and reneging on a two-dimensional state queuing model with multiple servers. They derived the time-dependent probabilities by using Laplace transformations and obtained some measurable outcomes of the system.

In our study, the work of Kumar and Sharma (2013) is extended. They investigated the single server infinite capacity Markovian feedback M/M/I queueing model with retention of reneged customers and balking. In their analysis, they considered a single-server feedback queueing model where one server serves all of the customers who arrive under the presumption that the retention of reneged customers and balking. In this paper web have extended this work to a single server Markovian feedback Mb/M/1 bulk queueing model. The limitations of a single server M/M/1 model are overcome by taking the bulk queueing model into consideration, because many organizations frequently encounter the arrival of customers in batches in real-world settings. In that situation, our study will assist in quickly and successfully resolving their issues. The overhead associated with processing individual requests is reduced in our work by handling requests in batches, which results in greater resource utilization.We also obtained the steady-state solution of the proposed model. Further, several system performance measures and particular cases of the proposed queuing models are obtained.

The issue of batch arrivals is not addressed in the extensive literature that has been published since 2013, which focuses primarily on a single server queueing model with finite and infinite capacity and some assumption-based research on jockeying, reneging, and balking or on their combinations.

In our study, we took into account a single server Markovian feedback bulk queuing model where customers arrive in predetermined fixed batch size. The bulk queueing model outperforms the preceding single server M/M/1 queueing model by allowing numerous requests to come simultaneously as a batch rather than one at a time. This is accomplished by establishing a fixed batch size. Therefore, in real-world situations, this bulk queueing strategy will boost customer retention, which raises the total number of customers using the system. So, our study plays a pivotal role in the field of queueing theory.

2. Model description

In the study, we consider the single-server Markovian feedback bulk queuing model $\mathrm{M^b/M/1}$ (where b is the fixed batch size of the arrival of the customer) with reneging of the customer. Customers join the system in a Poisson manner with the arrival rate λ and get the service exponentially with the service rate. Due to the concept of reneging, customers join the queue for service and leave the queue after waiting because the queue is too long. Feedback customers are those unsatisfied customers who re-join the system for another regular service after the completion of the previous service.

Let the parameter ξ of reneging time be exponentially distributed. It is found that by incorporating some strategies and schemes, a reneged customer can be convinced to be retained in the system for the service. Let q be the probability with which reneged customers are retained in the system, the probability of non-retention of customers be $p(= 1 - q))$, n be the number of units in the system, $P_n(t)$ be the transient state probability of having n customers in the system at time t, and P_n be the steady state probability of having n customers in the system.

The differential-difference equations of the bulk queuing model $M^b/M/1$ given by Medhi (2001) are:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \tag{1}$$

$$\frac{dP_n(t)}{dt} = -\left[(\lambda+\mu)P_n(t)\right] + \mu P_{n+1}(t) \quad , n < b \quad , \; n = 1, 2, \cdots, b-1 \tag{2}$$

$$\frac{dP_n(t)}{dt} = -\left[\left(\lambda+\mu q+(n-1)\xi p\right)P_n(t)\right] + (\mu q + n\xi p)P_{n+1}(t) + \lambda P_{n-b}(t), n \geq b \tag{3}$$

Equations (1) and (2) were considered from Medhi (2001) and we expanded them to generate equation (3) under the assumptions that the queueing model is a bulk queueing model with a fixed batch size b and reneging and number of customers n are greater than or equal to the batch size b.

In steady state, $\lim_{t\to\infty} P_n(t) = P_n$ and hence $\frac{\mathrm{d}P_n(t)}{dt} = 0$ as $t \to \infty$ and thus equations (1), (2) and (3) gives the difference equations of the model

$$0 = -\lambda P_0 + \mu P_1 \tag{4}$$

$$0 = -\left[(\lambda+\mu)P_n\right] + \mu P_{n+1} \quad , n < b \quad , \; n = 1, 2, \cdots, b-1 \tag{5}$$

$$0 = -\left[\left(\lambda+\mu q+(n-1)\xi p\right)P_n\right] + (\mu q + n\xi p)P_{n+1} + \lambda P_{n-b}, n \geq b \tag{6}$$

Using equation (4), we get

$$P_1 = \frac{\lambda P_0}{\mu} \tag{7}$$

For $n = 1$, equation (5) yields,$(\lambda+\mu)P_1 = \mu P_2$ $\; i.e; \quad P_2 = \frac{(\lambda+\mu)}{\mu}P_1$

$$i.e; \; P_2 = \frac{\lambda(\lambda+\mu)}{\mu^2}P_0$$

For $n = 2$, equation (5) yields, $P_3 = \frac{\lambda(\lambda+\mu)^2}{\mu^3}P_0$

On solving iteratively, we get

$$P_n = \frac{\lambda(\lambda+\mu)^{n-1}}{\mu^n}P_0 \quad , \quad 1 \leq n \leq b \tag{8}$$

For $n > b$, put $n = b$ in equation (6)

$$\left[\left(\lambda+\mu q+(b-1)\xi p\right)P_b\right] = (\mu q + b\xi p)P_{b+1} + \lambda P_0$$

$$P_{b+1} = \frac{\left[\left(\lambda+\mu q+(b-1)\xi p\right)P_b\right] - \lambda P_0}{(\mu q + b\xi p)}$$

Put the value of P_b for $n = b$ from equation (8), we get

$$P_{b+1} = \frac{\lambda[(\lambda + \mu q + (b-1)\xi p)(\lambda+\mu)^{b-1} - \mu^b]}{\mu^b(\mu q + b\xi p)} P_0$$

Similarly for $n > b$, the steady state probabilities $P_n; n > b+1$ are obtained as

$$P_n = \prod_{k=b+1}^{n} \frac{\lambda[\{\lambda + \mu q + (b-1)\xi p\}(\lambda+\mu)^{b-1} - \mu^b]}{\mu^b(\mu q + kb\xi p)} P_0 \tag{9}$$

For finding the value of P_0 , normalization condition $\sum_{n=0}^{\infty} P_n = 1$ is used and the values of $P_n; n \geq 1$

$$\left[1 + \sum_{n=1}^{b} \frac{\lambda(\lambda+\mu)^{n-1}}{\mu^n} + \sum_{n=b+1}^{\infty} \prod_{k=b+1}^{n} \frac{\lambda\{(\lambda+\mu q+(b-1)\xi p)(\lambda+\mu)^{b-1} - \mu^b\}}{\mu^b(\mu q + kb\xi p)} P_0\right] = 1$$

where

$$P_0 = \frac{1}{1 + \sum_{n=1}^{b} \frac{\lambda(\lambda+\mu)^{n-1}}{\mu^n} + \sum_{n=b+1}^{\infty} \prod_{k=b+1}^{n} \frac{\lambda\{(\lambda+\mu q+(b-1)\xi p)(\lambda+\mu)^{b-1} - \mu^b\}}{\mu^b(\mu q + kb\xi p)}} \tag{10}$$

The steady state probabilities exist if

$$\left[1 + \sum_{n=1}^{b} \frac{\lambda(\lambda+\mu)^{n-1}}{\mu^n} + \sum_{n=b+1}^{\infty} \prod_{k=b+1}^{n} \frac{\lambda\{(\lambda+\mu q+(b-1)\xi p)(\lambda+\mu)^{b-1} - \mu^b\}}{\mu^b(\mu q + kb\xi p)}\right] < \infty$$

3. System performance measures

Now, we derive some common performance measures from the proposed single-server Mb/M/1feedback bulk queuing model, which are useful for investigating the behavior of the system.

3.1. The expected number of customers waiting in the system(L_s)

$$\begin{aligned} L_s &= \sum_{n=0}^{\infty} nP_n \\ &= \left[\sum_{n=1}^{b} \frac{n\lambda(\lambda+\mu)^{n-1}}{\mu^n} + \sum_{n=b+1}^{\infty} n(\prod_{k=b+1}^{\infty} \frac{\lambda\{(\lambda+\mu q+(b-1)\xi p)(\lambda+\mu)^{b-1} - \mu^b\}}{\mu^b(\mu q + kb\xi p)})\right] P_0 \end{aligned} \tag{11}$$

3.2. The expected number of customers waiting in the queue(L_q)

$$\begin{aligned} L_q &= L_s - \frac{\lambda}{\mu} \\ &= \left[\sum_{n=1}^{b} \frac{n\lambda(\lambda+\mu)^{n-1}}{\mu^n} + \sum_{n=b+1}^{\infty} n(\prod_{k=b+1}^{\infty} \frac{\lambda\{(\lambda+\mu q+(b-1)\xi p)(\lambda+\mu)^{b-1} - \mu^b\}}{\mu^b(\mu q + kb\xi p)})\right] P_0 - \frac{\lambda}{\mu} \end{aligned} \tag{12}$$

3.3. The expected waiting time of the customer in the system(W_s)

$$
\begin{aligned}
W_s &= \frac{L_s}{\lambda \mathrm{b}} \\
&= \frac{1}{\lambda \mathrm{b}}\left[\sum_{n=1}^{b}\frac{n\lambda(\lambda+\mu)^{n-1}}{\mu^n}+\sum_{n=b+1}^{\infty}n(\prod_{k=b+1}^{\infty}\frac{\lambda\{(\lambda+\mu q+(b-1)\xi p)(\lambda+\mu)^{b-1}-\mu^b\}}{\mu^b(\mu q+kb\xi p)})\right]P_0
\end{aligned} \tag{13}
$$

3.4. The expected waiting time of the customer in the queue(W_q)

$$
\begin{aligned}
W_q &= W_s-\frac{1}{\mu} \\
&= \left[\sum_{n=1}^{b}\frac{n\lambda(\lambda+\mu)^{n-1}}{\mu^n}+\sum_{n=b+1}^{\infty}n(\prod_{k=b+1}^{\infty}\frac{\lambda\{(\lambda+\mu q+(b-1)\xi p)(\lambda+\mu)^{b-1}-\mu^b\}}{\mu^b(\mu q+kb\xi p)})x\right]P_0-\frac{1}{\mu}
\end{aligned} \tag{14}
$$

4. Sensitivity analysis

Sensitivity analysis evaluates the responsiveness of a model to the changes in various controllable parameters. In this section, we evaluate the sensitivity of the proposed model for different values of various parameters.

For a fixed value of n and for different values of λ, μ, ξ, q, we calculate the variations in the expected number of customers waiting in the system(L_s) by using equation (11) and discuss their effects graphically.

Case I. Effect on the size of the system with the variation in arrival rate

For $n = 4$, $\lambda = 2.0, 2.1, 2.2, 2.3, 2.4$, $\mu = 3$, $\xi = 0.1$, $q = 0.6$, p=0.4, b=3, we substitute these values in (11), we have

Table 1: Effect on the size of the system with the variation in Arrival rate

S.No.	λ	P_0	L_s
1.	2.0	0.115	2.85
2.	2.1	0.103	2.92
3.	2.2	0.093	2.97
4.	2.3	0.084	3.03
5.	2.4	0.076	3.08

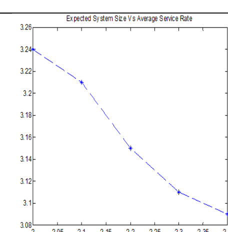

Figure 1
X- Axis : Average arrival rate,
Y-Axis: Expected system size

From Table 1 and Figure 1 above, we observe that the size of the system is directly proportional to the arrival rate, *i.e.* more the arrival of customers, larger the size of the system and vice-versa.

Case II. Effect on the size of the system with the variation in service rate

For $n = 4$, $\lambda = 2$, $\mu = 2.0, 2.1, 2.2, 2.3, 2.4$, $\xi = 0.1$, $q = 0.6$, $p = 0.4$, $b = 3$, we substitute these values in (11), we have

Table 2: Effect on the size of the system with the variation in service rate

S.No.	μ	P_0	L_s
1.	2.0	0.046	3.24
2.	2.1	0.052	3.21
3.	2.2	0.058	3.15
4.	2.3	0.064	3.11
5.	2.4	0.071	3.09

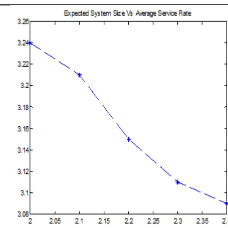

Figure 2
X- Axis : Average service,
Y-Axis: Expected system size

From Table 2 and Figure 2 above, we observe that as the average service rate increases, the size of the system decreases.

Case III. Effect on the size of the system with the variation in average reneging rate

For $n = 4$, $\lambda = 2$, $\mu = 3$, $\xi = 0.01, 0.02, 0.03, 0.04, 0.05$, $q = 0.6$, $p = 0.4$, $b = 3$ we substitute these values in (11), we have

Table 3 : Effect on the size of the system with the variation in average reneging rate

S.No.	ξ	P_0	L_s
1.	0.01	0.102	2.98
2.	0.02	0.104	2.97
3.	0.03	0.105	2.94
4.	0.04	0.107	2.94
5.	0.05	0.108	2.91

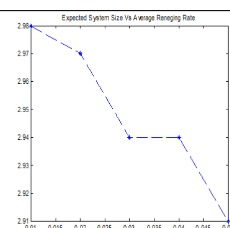

Figure 3
X- Axis : Average reneging rate,
Y-Axis: Expected system size

From Table 3 and Figure 3 above, we observe that as the average reneging rate increases, the size of the system decreases.

Case IV. Effect on the size of the system with the variation in retention probability

For $n = 4$, $\lambda = 2$, $\mu = 3$, $\xi = 0.1$, q=0.1, 0.2, 0.3, 0.4, 0.5, $b = 3$ we substitute these values in (11), we have

Table 4 : Effect on the size of the system with the variation in retention probability

S.No.	**q**	**p(=1-q)**	P_0	L_s
1.	0.1	0.9	0.120	2.63
2.	0.2	0.8	0.122	2.76
3.	0.3	0.7	0.120	2.80
4.	0.4	0.6	0.118	2.82
5.	0.5	0.5	0.116	2.83

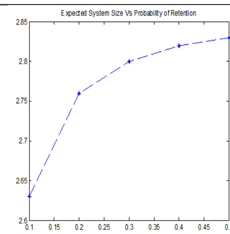

Figure 4
X- Axis : Probability of retention,
Y-Axis: Expected system size

From Table 4 and Figure 4 above, it is observed that the higher the retention of customers from reneging, the larger the size of the system.

5. Particular cases of the model

In this section, some particular cases of the proposed model are derived.

5.1. When retention probability of reneged customers is zero

If retention of reneged customers is zero, then $q = 1 - p = 0$. In this case, proposed model becomes $\mathbf{M^b/M/1}$ feedback bulk queuing model with reneging and we get

$$P_n = \frac{\lambda(\lambda+\mu)^{n-1}}{\mu^n} P_0 \quad , \quad 1 \le n \le b \tag{15}$$

$$P_n = \prod_{k=b+1}^{n} \frac{\lambda[(\lambda+(b-1)\xi p)(\lambda+\mu)^{b-1} - \mu^b]}{\mu^b(kb\xi p)} P_0, \ n > b \tag{16}$$

where $P_0 = \frac{1}{1+\sum_{n=1}^{b}\frac{\lambda(\lambda+\mu)^{n-1}}{\mu^n}+\sum_{n=b+1}^{\infty}\prod_{k=b+1}^{n}\frac{\lambda\{(\lambda+(b-1)\xi p)(\lambda+\mu)^{b-1}-\mu^b\}}{\mu^b(\mathrm{kb}\xi\mathrm{p})}}.$

5.2. When no reneging in the system

If there is no reneging in the system, then $\xi = 0$. In this case proposed model reduces to simple $\mathrm{M^b/M/1}$ queue model and we get

$$P_n = \frac{\lambda(\lambda+\mu)^{n-1}}{\mu^n} P_0 \quad , \quad 1 \le n \le b$$

$$P_n = \prod_{k=b+1}^{n} \frac{\lambda[(\lambda+\mu q)(\lambda+\mu)^{b-1} - \mu^b]}{\mu^b(\mu q)} P_0, \ n > b \tag{17}$$

where $P_0 = \frac{1}{1+\sum_{n=1}^{b}\frac{\lambda(\lambda+\mu)^{n-1}}{\mu^n}+\sum_{n=b+1}^{\infty}\prod_{k=b+1}^{n}\frac{\lambda\{(\lambda+\mu q)(\lambda+\mu)^{b-1}-\mu^b\}}{\mu^b(\mu\ \mathrm{q})}}.$

5.3. When the system is of finite capacity

If system capacity is finite, say N, then proposed model reduces to $\mathrm{M^b/M/1/N}$ feedback queuing model with retention of reneged customers and

$$P_n = \frac{\lambda(\lambda+\mu)^{n-1}}{\mu^n} P_0 \quad , \quad 1 \le n \le b$$

$$P_n = \prod_{k=b+1}^{n} \frac{\lambda[(\lambda+\mu q+(b-1)\xi p)(\lambda+\mu)^{b-1}-\mu^b]}{\mu^b(\mu q+kb\xi p)} P_0,\ b+1 \le n \le N \tag{18}$$

where $P_0 = \frac{1}{1+\sum_{n=1}^{b}\frac{\lambda(\lambda+\mu)^{n-1}}{\mu^n}+\sum_{n=b+1}^{N}\prod_{k=b+1}^{n}\frac{\lambda\{(\lambda+\mu q+(b-1)\xi p)(\lambda+\mu)^{b-1}-\mu^b\}}{\mu^b(\mu q+kb\xi p)}}$

6. Conclusion

In this paper, a single-server $M^b/M/1$ feedback bulk queuing model with reneged customers and their retention is discussed. The steady-state solution and various system performance measures are also derived for the proposed model. The sensitivity analysis of the proposed model is performed, and the effect of variation in the retention probability on the size of the system is discussed. From the results obtained, we concluded that the higher the retention of customers, the larger the size of the system. Thus, the study suggests any organization employ more strategies to retain customers for maximum profit.However, under some unusual circumstances, like epidemics or catastrophic events, this conclusion may not be true since customer retention will decrease due to impatience if the arrival of customers in batches is exponentially increasing and rises to be extremely large. Numerical results are analyzed by graphical representation using MATLAB software. Further, some particular cases of the proposed model are also discussed, and for different cases, we obtained some more queuing models with feedback. These extensions of models and their comparisons can be explored in future work.

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Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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