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Nonparametric Estimation of Linear Multiplier for Processes Driven by Mixed Fractional Brownian Motion

B. L. S. Prakasa Rao
CR RAO Advanced Institute of Mathematics
Statistics and Computer Science, Hyderabad, India

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Abstract

We study the problem of nonparametric estimation of linear multiplier function $\theta(t)$ for processes satisfying stochastic differential equations of the type

$$dX_t = \theta(t)X_t dt + \epsilon\, d\tilde{W}_t^H, X_0 = x_0, 0 \leq t \leq T$$

where $\{\tilde{W}_t^H, t \geq 0\}$ is a mixed fractional Brownian motion with known Hurst index H and study the asymptotic behaviour of the estimator as $\epsilon \to 0$.

Key words: Nonparametric estimation; Linear multiplier; Mixed Fractional Brownian motion.

AMS Subject Classifications: 62K05

1. Introduction

Professor Aloke Dey and I were colleagues for several years at the Indian Statistical Institute, Delhi Centre until I left due to my superannuation in the year 2004. Prof. Dey's expertise was in the area of optimal designs and my area of interest is in inference for stochastic processes. Even though our areas of research are completely different, we appreciated each others works and had a high regard for each other. I missed his association after I moved to Hyderabad. We did meet once or twice during the last sixteen years after I left New Delhi. I would like to thank Professor Vinod Gupta for inviting me to submit an article for the special issue of this journal dedicated to the memory of Professor Aloke Dey and pay my homage to a great statistician.

Statistical inference for fractional diffusion type processes satisfying stochastic differential equations driven by fractional Brownian motion have been studied earlier and a comprehensive survey of various methods is given in Mishura (2008) and Prakasa Rao (2010). There has been a recent interest to study similar problems for stochastic processes driven by a mixed fractional Brownian motion (mfBm). Existence and uniqueness for solutions of stochastic differential equations driven by a mfBm are investigated in Mishura and Shevchhenko (2012) and Shevchenko (2014) among others. Maximum likelihood estimation for estimation of drift parameter in a linear stochastic differential equations driven by a mfBm

Corresponding Author: B. L. S. Prakasa Rao
Email: blsprao@gmail.com

is investigated in Prakasa Rao (2018). The method of instrumental variable estimation for such parametric models is investigated in Prakasa Rao (2017). Some applications of such models in finance are presented in Prakasa Rao (2015 a,b). For related work on parametric inference for processes driven by mfBm, see Marushkevych (2016), Rudomino-Dusyatska (2003), Song and Liu (2014), Mishra and Prakasa Rao (2017), Prakasa Rao (2009) and Miao (2010) among others. Nonparametric estimation of the trend coefficient in models governed by stochastic differential equations driven by a mixed fractional Brownian motion is investigated in Prakasa Rao (2019).

We now discuss the problem of estimating the function $\theta(t), 0 \leq t \leq T$ (linear multiplier) based on the observations of a process $\{X_t, 0 \leq t \leq T\}$ satisfying the stochastic differential equation

$$dX_t = \theta(t)X_t dt + \epsilon\, d\tilde{W}_t^H, X_0 = x_0, 0 \leq t \leq T$$

where $\{\tilde{W}_t^H, t \geq 0\}$ is a mixed fractional Brownian motion (mfBm) and study the properties of the estimator as $\epsilon \to 0$.

2. Mixed Fractional Brownian Motion

We will now summarize some properties of stochastic processes which are solutions of stochastic differential equations driven by a mixed fractional Brownian motion for completeness.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a stochastic process is understood as the P-completion of the filtration generated by this process. Let $\{W_t, t \geq 0\}$ be a standard Wiener process and $W^H = \{W_t^H, t \geq 0\}$ be an independent normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0. \tag{1}$$

Let

$$\tilde{W}_t^H = W_t + W_t^H, t \geq 0.$$

The process $\{\tilde{W}_t^H, t \geq 0\}$ is called the mixed fractional Brownian motion with Hurst index H. We assume here after that Hurst index H is known. Following the results in Cheridito (2001), it is known that the process $\tilde{W}^H$ is a semimartingale in its own filtration if and only if either $H = 1/2$ or $H \in (\frac{3}{4}, 1]$.

Let us consider a stochastic process $X = \{X_t, t \geq 0\}$ defined by the stochastic integral equation

$$X_t = \int_0^t C(s)ds + \tilde{W}_t^H, t \geq 0 \tag{2}$$

where the process $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process. For convenience, we write the above integral equation in the form of a stochastic differential equation

$$dX_t = C(t)dt + d\tilde{W}_t^H, t \geq 0 \tag{3}$$

driven by the mixed fractional Brownian motion $\tilde{W}^H$. Following the recent works by Cai *et al.* (2016) and Chigansky and Kleptsyna (2015), one can construct an integral transformation that transforms the mixed fractional Brownian motion $\tilde{W}^H$ into a martingale M^H. Let $g_H(s,t)$ be the solution of the integro-differential equation

$$g_H(s,t) + H\frac{d}{ds}\int_0^t g_H(r,t)|s-r|^{2H-1}sign(s-r)dr = 1, 0 < s < t. \tag{4}$$

Cai *et al.* (2016) proved that the process

$$M_t^H = \int_0^t g_H(s,t)d\tilde{W}_s^H, t \geq 0 \tag{5}$$

is a Gaussian martingale with quadratic variation

$$< M^H >_t = \int_0^t g_H(s,t)ds, t \geq 0 \tag{6}$$

Furthermore the natural filtration of the martingale M^H coincides with that of the mixed fractional Brownian motion $\tilde{W}^H$. It is clear that the quadratic variation $< M^H >_t$ is differentiable with respect to t. Let $\beta(t)$ denote the derivative of the function $< M^H >_t$ with respect to t. Suppose that, for the martingale M^H defined by the equation (6), the sample paths of the process $\{C(t), t \geq 0\}$ are smooth enough in the sense that the process

$$Q_H(t) = \frac{d}{d < M^H >_t}\int_0^t g_H(s,t)C(s)ds, t \geq 0 \tag{7}$$

is well defined. Define the process

$$Z_t = \int_0^t g_H(s,t)dX_s, t \geq 0. \tag{8}$$

As a consequence of the results in Cai *et al.* (2016), it follows that the process Z is a fundamental semimartingale associated with the process X in the following sense.

Theorem 1: Let $g_H(s,t)$ be the solution of the equation (4). Define the process Z as given in the equation (8). Then the following relations hold.

(i) The process Z is a semimartingale with the decomposition

$$Z_t = \int_0^t Q_H(s)d < M^H >_s + M_t^H, t \geq 0 \tag{9}$$

where M^H is the martingale defined by the equation (5). (ii) The process X admits the representation

$$X_t = \int_0^t \hat{g}_H(s,t)dZ_s, t \geq 0 \tag{10}$$

where

$$\hat{g}_H(s,t) = 1 - \frac{d}{d < M^H >_s} \int_0^t g_H(r,s)dr. \tag{11}$$

(iii) The natural filtrations $(\mathcal{X}_t)$ and $(\mathcal{Z}_t)$ of the processes X and Z respectively coincide.

Applying the Corollary 2.9 in Cai *et al.* (2016), it follows that the probability measures μ_X and $\mu_{\tilde{W}^H}$ generated by the processes X and $\tilde{W}^H$ on an interval $[0,T]$ are absolutely continuous with respect to each other and the Radon-Nikodym derivative is given by

$$\frac{d\mu_X}{d\mu_{\tilde{W}^H}} = \exp[\int_0^T Q_H(s)dZ_s - \frac{1}{2}\int_0^T [Q_H(s)]^2 d < M^H >_s] \tag{12}$$

which is also the likelihood function based on the observation $\{X_s, 0 \leq s \leq T.\}$ Since the filtrations generated by the processes $X$and Z are the same, the information contained in the families of σ-algebras $(\mathcal{X}_t)$ and $(\mathcal{Z}_t)$ is the same and hence the problem of the estimation of the parameters involved based on the observation $\{X_s, 0 \leq s \leq T\}$ and $\{Z_s, 0 \leq s \leq T\}$ are equivalent.

3. Preliminaries

Let $\tilde{W}^H = \{W_t^H, t \geq o\}$ be a mixed fractional Brownian motion with known Hurst parameter $H \in (1/2, 1)$. Consider the problem of estimating the function $\theta(t), 0 \leq t \leq T$ (linear multiplier) from the observations $\{X_t, 0 \leq t \leq T\}$ of process satisfying the stochastic differential equation

$$dX_t = \theta(t)X_t dt + \epsilon\, d\tilde{W}_t^H, X_0 = x_0, 0 \leq t \leq T \tag{13}$$

and study the properties of the estimator as $\epsilon \to 0$. Consider the differential equation in the limiting system of (13), that is , for $\epsilon = 0$, given by

$$dx_t = \theta(t)x_t dt, x_0, 0 \leq t \leq T. \tag{14}$$

Observe that

$$x_t = x_0 \exp\{\int_0^t \theta(s)ds).$$

We assume that the following condition holds:

(A_1): The trend coefficient $\theta(t)$ over the interval $[0,T]$ is bounded by a constant L.

The condition (A_1) will ensure the existence and uniqueness of the solution of the equation (13).

Lemma 1: Let the condition (A_1) hold and $\{X_t, 0 \leq t \leq T\}$ and $\{x_t, 0 \leq t \leq T\}$ be the solutions of the equations (13) and (14) respectively. Then, with probability one,

$$(a)|X_t - x_t| < e^{Lt}\epsilon \sup_{0 \leq s \leq t} |\tilde{W}_s^H| \tag{15}$$

and

$$(b) \sup_{0\leq t\leq T} E(X_t - x_t)^2 \leq 4e^{2LT}\epsilon^2(T^{2H}+T). \tag{16}$$

Proof of (a): Let $u_t = |X_t - x_t|$. Then by (A_1); we have,

$$\begin{aligned} u_t &\leq \int_0^t |\theta(v)(X_v - x_v)|dv + \epsilon|\tilde{W}_t^H| \\ &\leq L\int_0^t u_v dv + \epsilon \sup_{0\leq s\leq t}|\tilde{W}_s^H|. \end{aligned} \tag{17}$$

Applying the Gronwall's lemma (cf. Lemma 1.12, Kutoyants (1994), p.26), it follows that

$$u_t \leq \epsilon \sup_{0\leq s\leq t}|\tilde{W}_s^H|e^{Lt}. \tag{18}$$

Proof of (b): From the equation (15), we have

$$\begin{aligned} E(X_t - x_t)^2 &\leq e^{2Lt}\epsilon^2 E[(\sup|\tilde{W}_s^H|)^2] \\ &\leq 4e^{2Lt}\epsilon^2(t^{2H}+t) \end{aligned} \tag{19}$$

from the fact that the mixed fractional Brownian motion W^H is a sum of a Wiener process and fractional Brownian motion and from the maximal inequalities for a Wiener process and a fractional Brownian motion (cf. Muneya and Shieh (2009), Prakasa Rao (2014)). Hence

$$\sup_{0\leq t\leq T} E(X_t - x_t)^2 \leq 4e^{2LT}\epsilon^2(T^{2H}+T). \tag{20}$$

This completes the proof of the lemma.

Define

$$\begin{aligned} Q^*_{H,\theta}(t) &= \frac{d}{d<M^H>_t}\int_0^t g_H(t,s)\theta(s)x(s)ds \\ &= \frac{d}{d<M^H>_t}\int_0^t g_H(t,s)\theta(s)[x_0\exp(\int_0^s\theta(u)du)]ds \end{aligned} \tag{21}$$

by using the equation (14). Here after, we consider the problem of nonparametric estimation of the function $Q^*_{H,\theta}(t)$ instead of the function $\theta(t)$. We assume that the function $\theta(.)$ belongs to a class of functions Θ uniformly bounded by a constant L and the following condition holds:

(A_2): Differentiation under the integral sign is valid in the equation (21) and the function $\beta(t)Q^*_{H,\theta}(t)$is Lipschitz of order γ in the sense that

$$|\beta(t)Q^*_{H,\theta}(t) - \beta(s)Q^*_{H,\theta}(s)| \leq C|t-s|^\gamma$$

for some constant $C>0$ and $\gamma>0$ uniformly for $\theta(.)\in\Theta$.

Instead of estimating the function $\theta(.)$, we consider the problem of estimating the function $Q^*_{H,\theta}(.)$ defined via the equation (21). This is justified by the observation that the processes $\{X_t, 0 \leq t \leq T\}$ governed by the stochastic differential equation (13) and the corresponding related process $\{Z_t, 0 \leq t \leq T\}$ as defined by (8) have the same filtrations by the results in Cai *et al.* (2016).

Consider the kernel type estimator defined by

$$\begin{aligned} \widehat{Q}_{H,\theta}(t) &= \frac{1}{h_\epsilon}\int_0^T G\left(\frac{s-t}{h_\epsilon}\right) dZ_s \\ &= \frac{1}{h_\epsilon}\int_0^T G\left(\frac{s-t}{h_\epsilon}\right)\left(Q_{H,\theta}(s)d < M^H >_s + \epsilon\, dM^H_s\right) \\ &= \frac{1}{h_\epsilon}\int_0^T G\left(\frac{s-t}{h_\epsilon}\right)\left(Q_{H,\theta}(s)\beta(s)ds + \epsilon\, dM^H_s\right) \end{aligned} \tag{22}$$

by using the equation (9) where $G(u)$ is a bounded function with finite support $[A, B]$ satisfying the condition

(A_3):$G(u) = 0$ for $u < A, u > B$, $\int_A^B |G(u)|du < \infty$ and $\int_A^B G(u)du = 1$;

Consider a normalizing function $h_\epsilon \to 0$ as $\epsilon \to 0$. In addition, suppose that $\epsilon^2 h_\epsilon^{-3/2} \to 0$ as $\epsilon \to 0$.

4. Main Results

Theorem 2 : Suppose the conditions $(A_1), (A_2)$ and (A_3) are satisfied. Then the estimator $\widehat{Q}_{H,\theta}(t)$ is uniformly consistent, that is,

$$\lim_{\epsilon\to 0}\sup_{\theta(.)\in\Theta}\sup_{0\leq t\leq T} E_\theta(|\widehat{Q}_{H,\theta}(t) - \beta(t)Q^*_{H,\theta}(t)|^2) = 0. \tag{23}$$

Proof: From (9), we have,

$$\begin{aligned} &E_\theta|\widehat{Q}_{H,\theta}(t) - \beta(t)Q^*_{H,\theta}(t)|^2 \\ &= E\left|\frac{1}{h_\epsilon}\int_0^T G\left(\frac{s-t}{h_\epsilon}\right)(Q_{H,\theta}(s)\beta(s)ds + \epsilon dM^H_s) - \beta(t)Q^*_{H,\theta}(t)\right|^2 \\ &= E_\theta|\frac{1}{h_\epsilon}\int_0^T G\left(\frac{s-t}{h_\epsilon}\right)(Q_{H,\theta}(s) - Q^*_{H,\theta}(s))\beta(s)ds \\ &\quad + \frac{1}{h_\epsilon}\int_0^T G\left(\frac{s-t}{h_\epsilon}\right)(Q^*_{H,\theta}(s)\beta(s) - Q^*_{H,\theta}(t)\beta(t))ds \\ &\quad + \frac{\epsilon}{h_\epsilon}\int_0^T G\left(\frac{s-t}{h_\epsilon}\right) dM^H_s|^2 \end{aligned} \tag{24}$$

$$
\begin{aligned}
&= E_\theta[I_1+I_2+I_3]^2 \ \text{ (denoting the three integrals as } I_1, I_2 \text{ and } I_3 \text{ respectively)}\\
&\leq 3\ E(I_1^2)+3\ E(I_2^2)+3\ E(I_3^2).
\end{aligned}
$$

Now

$$
\begin{aligned}
3\ E_\theta[I_1^2] &= 3\ E_\theta\left|\frac{1}{h_\epsilon}\int_0^T G\left(\frac{s-t}{h_\epsilon}\right)(Q_{H,\theta}(t)-Q^*_{H,\theta}(s))\beta(s)ds\right|^2 \qquad (25)\\
&\leq \frac{3}{h_\epsilon^2}[\int_0^T G^2\left(\frac{s-t}{h_\epsilon}\right)ds][E\int_0^T \beta^2(s)(Q_{H,\theta}(s)-Q^*_{H,\theta}(s))^2 ds].
\end{aligned}
$$

Note that

$$
\begin{aligned}
&E_\theta\int_0^T \beta^2(s)(Q_{H,\theta}(s)-Q^*_{H,\theta}(s))^2 d<M^H>_s \qquad (26)\\
&= \int_0^T \beta^2(s)E_\theta\left[\frac{d}{d<M^H>_s}\int_0^s g_H(s,v)\theta(v)(X(v)-x(v))dv\right]^2 d<M^H>_s\\
&\leq C_1\int_0^T E_\theta\left[\int_0^s \frac{\partial g_H(s,v)}{\partial s}\theta(v)(X(v)-x(v))dv\right]^2 ds\\
&\leq C_2\int_0^T\{\int_0^s\left(\frac{\partial g_H(s,v)}{\partial s}\right)^2\theta^2(v)dv\int_0^s E(X(v)-x(v))^2dv\}ds
\end{aligned}
$$

for some positive constant C_2 depending on T and H. Furthermore

$$E_\theta(X_v-x_v)^2 \leq 4e^{2Lv}\epsilon^2(v^{2H}+v) \ \text{ (by Lemma 1).}$$

Hence, from the equation (26) and the condition (A_3), we get that

$$
\begin{aligned}
3E_\theta[I_1^2] &\leq C\frac{1}{h_\epsilon^2}\left\{\int_{-\infty}^{\infty}G^2(\frac{s-t}{h_\epsilon})\beta(s)ds\right\}\epsilon^2 h_\epsilon \qquad (27)\\
&\quad\times\int_0^T\beta^2(s)\left\{\int_0^s e^{2Lv}(v^{2H}+v)dv\right\}\left\{\int_0^s\left(\frac{\partial g_H(s,v)}{\partial s}\right)^2 dv\right\}ds\\
&\leq C_3\epsilon^2 h_\epsilon^{-1}
\end{aligned}
$$

for some positive constant C_3 depending on T and H and the last term tends to zero as $\epsilon\to 0$.

In addition,

$$
\begin{aligned}
I_2^2 &= 3\left\{\frac{1}{h_\epsilon}\int_0^T G\left(\frac{s-t}{h_\epsilon}\right)(Q^*_{H,\theta}(s)ds-Q^*_{H,\theta}(t))d<M^H>_s\right\}^2 \qquad (28)\\
&= \left\{\frac{1}{h_\epsilon}\int_0^T G\left(\frac{s-t}{h_\epsilon}\right)(Q^*_{H,\theta}(s)\beta(s)-Q^*_{H,\theta}(t)\beta(t))ds\right\}^2\\
&= 3\left\{\int_{-\infty}^{\infty}G(u)(Q^*_{H,\theta}(t+h_\epsilon u)\beta(t+h_\epsilon u)-Q^*_{H,\theta}(t)\beta(t))du\right\}^2 \quad (\text{by}(A_2))
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_4\left\{\int_{-\infty}^{\infty} G(u)|h_\epsilon u|^\gamma du\right\}^2 \quad (\text{by}(A_2))\\
&\leq C_4 h_\epsilon^{2\gamma}\left(\int_{-\infty}^{\infty} G(u)|u|^\gamma du\right)^2\\
&\leq C_5 h_\epsilon^{2\gamma} \ \text{by}(A_3))
\end{aligned}
$$

for some positive constant C_5 depending on T and $H$and the last term tends to zero as $\epsilon \to 0$. Furthermore

$$
\begin{aligned}
I_3^2 &= \frac{3\epsilon^2}{h_\epsilon^2} E\left(\int_0^T G\left(\frac{s-t}{h_\epsilon}\right) dM_s^H\right)^2 \qquad (29)\\
&= \frac{3\epsilon^2}{h_\epsilon^2}\int_0^T G^2\left(\frac{s-t}{h_\epsilon}\right)\beta(s)ds\\
&\leq \frac{3\epsilon^2}{h_\epsilon^2}\left\{\int_0^T G^2\left(\frac{s-t}{h_\epsilon}\right)ds\int_0^T \beta^2(s)ds\right\}^{\frac{1}{2}}\\
&\leq C_6\frac{3\epsilon^2}{h_\epsilon^2}\left\{h_\epsilon(\int_{-\infty}^{\infty} G^2(u)\ du)\right\}^{\frac{1}{2}}\\
&\leq C_7\epsilon^2 h_\epsilon^{-3/2}.
\end{aligned}
$$

for some positive constants C_7 depending on T and H. The result follows from the equations (27), (28) and (29).

Corollary 1: Under the conditions $(A_1), (A_2)$ and (A_3),

$$\lim_{\epsilon\to 0}\sup_{\theta(.)\in\Theta} E\left\{\widehat{Q}_{H,\theta}(t) - \beta(t)Q^*_{H,\theta}(t)\right\}^2 \epsilon^{\frac{8\gamma}{4\gamma+3}} < \infty.$$

Proof: From the inequalities derived in (27), (28) and (29), we get that there exist positive constants D_1, D_2 and D_3 depending on T and H such that

$$\sup_{\theta(.)\in\Theta} E\left\{\widehat{Q}_{H,\theta}(t) - \beta(t)Q^*_{H,\theta}(t)\right\}^2 \leq D_1\epsilon^2 h_\epsilon^{-1} + D_2 h_\epsilon^{2\gamma} + D_3\epsilon^2 h_\epsilon^{-\frac{3}{2}}. \qquad (30)$$

Let $h_\epsilon = \epsilon^\beta, 0 < \beta < \frac{4}{3}$. Then the condition $h_\epsilon^{2\gamma} = \epsilon^2 h_\epsilon^{-3/2}$ leads to the choice $\beta = \frac{4}{4\gamma+3}$ and we get an optimum bound in (30) and hence

$$\lim_{\epsilon\to 0}\sup_{\theta(.)\in\Theta} E\left[\widehat{Q}_{H,\theta}(t) - \beta(t)Q^*_{H,\theta}(t)\right]^2 \epsilon^{-\frac{8\gamma}{4\gamma+3}} \leq C \qquad (31)$$

for some positive constant C which implies the result.

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