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Correlated Inverse Gaussian Frailty Models Based on Reversed Hazard Rate

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Abstract

Frailty models are used in the survival analysis to account for the unobserved heterogeneity in individual risks to disease and death. To analyze the bivariate data on related survival times (*e.g.* matched pairs experiments, twin or family data), the shared frailty models were suggested. Shared frailty models are used despite their limitations. To overcome their disadvantages correlated frailty models may be used. In this paper, we introduce the correlated inverse Gaussian frailty models based on reversed hazard rate with three different baseline distributions namely, the generalized log-logistic type I, the generalized log-logistic type II and the modified inverse Weibull. We introduce the Bayesian estimation procedure using Markov Chain Monte Carlo (MCMC) technique to estimate the parameters involved in these models. We present a simulation study to compare the true values of the parameters with the estimated values. We also apply the proposed models to the Australian twin data set and a better model is suggested. ...

Key words: Australian twin data; Bayesian estimation; Correlated inverse Gaussian frailty; Generalized log-logistic distribution; MCMC; Modified inverse Weibull distribution; Reversed hazard rate.

AMS Subject Classifications: 62F15; 62N01; 62P10

1. Introduction

Frailty models are extensively used in the survival analysis to account for the unobserved heterogeneity in individual risks to disease and death. The frailty model is a random effect model for time to event data which is an extension of the Cox's proportional hazards model. To analyze the bivariate data on related survival times (*e.g.* matched pairs experiments, twin or family data), the shared frailty models were suggested. Bivariate survival data arises whenever each study subjects experience two events. Particular examples include failure times of paired human organs, (*e.g.* kidneys, eyes, lungs, breasts, *etc.*) and the first and the second occurrences of a given disease. In the medical literature, several authors considered paired organs of an individual as a two-component system, which work under interdependency circumstances. In industrial applications, these data may come from systems whose survival depend on the survival of two similar components.

Research on the bivariate survival models has grown rapidly several years in the past. Clayton's (1978) random effect model of the bivariate survival was a key innovation. He introduced the notion of the shared relative risk. This model was further developed by Oakes (1982) to analyze the association between two non-negative random variables. Clayton and Cuzick (1985) added observed covariates to the bivariate survival model with the shared relative risk. Crowder (1985) and Hougaard (1986) proposed the random effect models of the bivariate Weibull distributions. A shared frailty model with a positive stable distribution of frailty was suggested by Hougaard (1987). He also discussed several other bivariate distributions with biomedical and reliability applications. Oakes (1989) developed a shared frailty model related to the "archimedean distributions" studied by Genest and MacKay (1986). He also proposed a local time dependent association measure between bivariate life spans and discussed its use for a large class of bivariate survival functions. Vaupel (1991), Vaupel *et al.* (1991), Nielsen *et al.* (1992) studied genetic and environmental influences on longevity using bivariate survival models.

Hanagal (2006) discussed the gamma frailty regression model in the bivariate survival data and Hanagal (2007) also presented the gamma frailty regression models in the mixture distributions. Hanagal and Dabade (2013), Hanagal and Bhambure (2015, 2016) and Hanagal and Pandey (2014a, 2014b, 2015a, 2015b, 2016, 2017) and Hanagal *et al.* (2017a, 2017b) analyzed kidney infection data and Australian twin data using shared gamma and inverse Gaussian frailty models with different baseline distributions for the multiplicative model. Hanagal and Sharma (2013, 2015a, 2015b, 2015c) analyzed acute leukemia data, kidney infection data and diabetic retinopathy data using shared gamma and inverse Gaussian frailty models for the multiplicative model. Hanagal and Bhambure (2014) developed shared inverse Gaussian frailty model based on the reversed hazard rate for Australian twin data. Hanagal *et al.*(2017b) discussed correlated gamma frailty models for bivariate survival data to analyze kidney infection data and Hanagal and Pandey (2017) proposed correlated gamma frailty models for bivariate survival data based on reversed hazard rate for Australian twin data. Hanagal (2017) gave extensive literature review on different shared frailty models.

Shared frailty explains correlation between subjects within clusters. However, it does have some limitations. Firstly, it forces the unobserved factors to be the same within the cluster, which may not always reflect reality. For example, at times it may be inappropriate to assume that all partners in a cluster share all their unobserved risk factors. Secondly, the dependence between survival times within the cluster is based on marginal distributions of survival times. However, when covariates are present in a proportional hazards model with gamma distributed frailty the dependence parameter and the population heterogeneity are confounded (Clayton and Cuzick, 1985). This implies that the joint distribution can be identified from the marginal distributions (Hougaard, 1986). Thirdly, in most cases, a one-dimensional frailty can only induce positive association within the cluster. However, there are some situations in which the survival times for subjects within the same cluster are negatively associated. For example, in the Stanford Heart Transplantation Study, generally the longer an individual must wait for an available heart, the shorter he or she is likely to survive after the transplantation. Therefore, the waiting time and the survival time afterwards may be negatively associated.

To avoid these limitations, correlated frailty models are being developed for the analysis of multivariate failure time data, in which associated random variables are used to

characterize the frailty effect for each cluster. Correlated frailty models provide not only variance parameters of the frailties as in shared frailty models, but they also contain additional parameter for modeling the correlation between frailties in each group. Frequently one is interested in construction of a bivariate extension of some univariate family distributions (*e.g.*, gamma). For example, for the purpose of genetic analysis of frailty one might be interested in estimation of correlation of frailty. It turns out that it is possible to carry out such extension for the class of infinitely-divisible distributions (Iachine 1995a, 1995b). In this case an additional parameter representing the correlation coefficient of the bivariate frailty distribution is introduced.

2. Reversed Hazard Rate and Correlated Frailty

In many practical situations reversed hazard rate (RHR) is more appropriate to analyze the survival data. Reversed hazard rate was proposed as a dual to the hazard rate by Barlow *et al.* (1963). Shaked and Shantikumar (1994) and Block *et al.* (1998) provided a general definition of reversed hazard rate (RHR) as,

$$m(t) \;=\; \lim_{\triangle t \to 0} P(t - \triangle t < T \leq t | T \leq t) / \triangle t, \quad t > 0. \tag{1}$$

The reversed hazard rate specifies the instantaneous rate of death or failure at time t, given that it failed before time t. Thus in a small interval, $m(t) \triangle t$ is the approximate probability of failure in the interval, given failure until the end of the interval $(t - \triangle t, t]$. In lifetime data analysis, the concepts of reversed hazard rate has potential application when the time elapsed since failure is a quantity of interest in order to predict the time of failure. The reversed hazard rate is more useful in estimating reliability function when the data are left censored or right truncated. Reversed hazard rate plays a vital role in the analysis of parallel systems, in reliability and survival analysis. For example, in certain systems or situations, sometimes the failure is prevented through numerous safety measures.

The correlated frailty model is the important concept in the area of multivariate frailty models. It is a natural extension of the shared frailty approach on the one hand, and of the univariate frailty model on the other. In the correlated frailty model, the frailties of individuals in a cluster are correlated but not necessarily shared. The conditional distribution function in the bivariate case (without observed covariates) is

$$F(t_1, t_2 | Z_1, Z_2) \;=\; S_1(t_1|Z_1) S_2(t_2|Z_2) = e^{-Z_1 M_{01}(t_1)} e^{-Z_2 M_{02}(t2)}, \tag{2}$$

where Z_1 and Z_2 are two correlated frailties and $M_{0i}(t_i) = \int_{t_i}^{\infty} m_{0i}(u) du, (i = 1, 2)$ is cumulative reversed hazard rate. The distribution of the random vector (Z_1, Z_2) needs to be specified and determines the association structure of the event times in the model.

The reversed hazard of the i-th $(i = 1, 2)$ individual of the j-th $(i = j, ..., n)$ pair has the form

$$m(t \mid X_{ij}, Z_{ij}) = Z_{ij} m_{0i}(t) e^{\beta' X_{ij}}, \tag{3}$$

where t denotes age or time, X_{ij} is a vector of observed covariates, β is a vector of regression parameters describing the effect of the covariates X_{ij}, $m_{0i}(.)$ are baseline reversed hazard functions, and Z_{ij} are frailties. Bivariate correlated frailty models are characterized by the joint distribution of a two-dimensional vector of frailties (Z_{1j}, Z_{2j}). If the two frailties are independent, the resulting lifetimes are independent, and no clustering is present in the model. If the two frailties are equal, the shared frailty model is obtained as a special case of the correlated frailty model with correlation one between the frailties.

In order to derive a marginal likelihood function, the assumption of conditional independence of lifespans, given the frailty, is used. Let δ_{ij} be a censoring indicator for individual $i(i = 1, 2)$ in pair $j(j = 1, ..., n)$. Indicator δ_{ij} is 1 if the individual has experienced the event of interest, and 0 otherwise. According to (2), the conditional distribution function of the ith individual in the jth pair is

$$F(t|X_{ij}, Z_{ij}) = e^{-Z_{ij}M_{0i}(t)e^{\beta' X_{ij}}}, \tag{4}$$

with $M_{0i}(t) = \int_t^\infty m_{0i}(u)du$ denoting the cumulative baseline hazard function. Here and in the following, F is used as a generic symbol for a distribution function. The contribution of individual $i(i = 1, 2)$ in pair $j(j = 1, ..., n)$ to the conditional likelihood is given by

$$\left[Z_{ij}m_{0i}(t)e^{\beta' X_{ij}}\right]^{\delta_{ij}} e^{-Z_{ij}M_{0i}(t_{ij})e^{\beta' X_{ij}}}, \tag{5}$$

where t_{ij} stands for observation time of individual i from pair j. Assuming the conditional independence of life spans, given the frailty, and integrating out the frailty, we obtain the marginal likelihood function

$$\prod_{j=1}^{n} \int_{R\times} \int_{R} \left[u_{1j}m_{01}(t_{1j})e^{\beta' X_{1j}}\right]^{\delta_{1j}} e^{-z_{1j}M_{01}(t_{1j})e^{\beta' X_{1j}}}$$
$$\left[u_{2j}m_{02}(t_{2j})e^{\beta' X_{2j}}\right]^{\delta_{2j}} e^{-z_{2j}M_{02}(t_{2j})e^{\beta' X_{2j}}} f(z_{1j}, z_{2j})dz_{1j}dz_{2j} \tag{6}$$

where $f(.,.)$ is the probability density function of the corresponding frailty distribution. All these formulas can be easily extended to the multivariate case, but need a specification of the correlation structure between individuals in a cluster in terms of the multivariate density function, which complicates analysis. For more details see Hanagal(2011) and Hanagal (2019).

3. Correlated Inverse Gaussian Frailty Model

Alternative to the gamma distribution, Hougaard (1984) introduced the inverse Gaussian as a frailty distribution. It provides much flexibility in modeling, when early occurrences of failures are dominant in a life time distribution and its failure rate is expected to be non-monotonic. In such situations, the inverse Gaussian distribution might provide a suitable choice for the lifetime model. Also inverse Gaussian is almost an increasing failure rate distribution when it is slightly skewed and hence is also applicable to describe lifetime distribution which is not dominated by early failures. Secondly, for the inverse Gaussian distribution, the surviving population becomes more homogeneous with respect to time, where as for gamma distribution the relative heterogeneity is constant.

Consider a continuous random variable Z follows inverse Gaussian distribution with parameters μ and σ^2 then density function of Z is,

$$f_Z(z) = \begin{cases} \left[\frac{1}{2\pi\sigma^2}\right]^{\frac{1}{2}} z^{-\frac{3}{2}} e^{-\frac{(z-\mu)^2}{2z\sigma^2\mu^2}} & ;\ z>0, \mu>0, \sigma^2>0 \\ 0 & ;\ otherwise, \end{cases} \tag{7}$$

and the Laplace transform is,

$$L_Z(s) = exp\left[\frac{1}{\mu\sigma^2} - \left(\frac{1}{\sigma^4\mu^2} + \frac{2s}{\sigma^2}\right)^{\frac{1}{2}}\right]. \tag{8}$$

The mean and variance of frailty variable are $E(Z) = \mu$ and $V(Z) = \mu^3\sigma^2$. For identifiability, we assume Z has expected value equal to one i.e. $\mu = 1$. Under this restriction, the density function and the Laplace transformation of the inverse Gaussian distribution reduces to,

$$f_Z(z) = \begin{cases} \left[\frac{1}{2\pi\sigma^2}\right]^{\frac{1}{2}} z^{-\frac{3}{2}} e^{-\frac{(z-1)^2}{2z\sigma^2}} & ;\ z>0, \sigma^2>0 \\ 0 & ;\ otherwise, \end{cases} \tag{9}$$

and the Laplace transform is,

$$L_Z(s) = exp\left[\frac{1-(1+2\sigma^2 s)^{\frac{1}{2}}}{\sigma^2}\right], \tag{10}$$

with variance of Z as σ^2. The frailty variable Z is degenerate at $Z = 1$ when σ^2 tends to zero.

Let Z be an infinitely divisible frailty variable with Laplace transformation $L_Z(s)$ and $\rho \in [0, 1]$, then there exist random variables Z_1, Z_2 each with univariate Laplace transform $L_Z(s)$ such that the Laplace transform of Z_1, Z_2 is given by:

$$L(s_1, s_2) = L_Z^{\rho}(s_1 + s_2) L_Z^{1-\rho}(s_1) L_Z^{1-\rho}(s_2) \tag{11}$$

If Z has a variance the $Corr(Z_1, Z_2) = \rho$.
The respective bivariate survival model is identifiable under mild regularity conditions on Z provided that $\rho > 0$. The case $\rho = 1$ is known as the shared frailty model.

The above equation (11) can be extended to multivariate case ($\rho > 0$) as below.

$$L(s_1, s_2, ..., s_k) \quad = \quad L_Z^{\rho}(s_1, s_2, ..., s_k) L_Z^{1-\rho}(s_1)....L_Z^{1-\rho}(s_k).$$

The case $\rho = 1$ leads to shared frailty. If $\rho = 0$, $Z_1, ...Z_k$ are mutually independent.

Let Z_i be the inverse Gaussian distributed with mean 1, variance σ^2, and Laplace transform

$$L(s_i, \sigma^2) = \exp[\frac{1-(1+2\sigma^2 s_i)^{\frac{1}{2}}}{\sigma^2}] \tag{12}$$

The bivariate Laplace transform for the correlated inverse Gaussian frailty model is given by

$$L(s_1, s_2, \sigma^2, \rho) = \exp\left[\rho\frac{1-(1+2\sigma^2(s_1+s_2))^{\frac{1}{2}}}{\sigma^2}\right]\exp\left[(1-\rho)\frac{1-(1+2\sigma^2 s_1)^{\frac{1}{2}}}{\sigma^2}\right] \exp\left[(1-\rho)\frac{1-(1+2\sigma^2 s_2)^{\frac{1}{2}}}{\sigma^2}\right] \tag{13}$$

where $Corr(Z_1, Z_2) = \rho$.

The correlated inverse Gaussian frailty model in the presence of covariates is characterized by the bivariate distribution function of the form:

$$F(t_{1j}, t_{2j}) = \exp\left[\rho\frac{1-(1+2\sigma^2\eta_j(M_{01}(t_{1j})+M_{02}(t_{2j})))^{\frac{1}{2}}}{\sigma^2}\right]\exp\left[(1-\rho)\frac{1-(1+2\sigma^2\eta_j M_{01}(t_{1j}))^{\frac{1}{2}}}{\sigma^2}\right] \exp\left[(1-\rho)\frac{1-(1+2\sigma^2\eta_j M_{02}(t_{2j}))^{\frac{1}{2}}}{\sigma^2}\right] \tag{14}$$

where $M_{01}(t_{1j})$ and $M_{02}(t_{2j})$ are the cumulative baseline hazard functions of the life time random variables T_{1j} and T_{2j} respectively.

The bivariate distribution function in the presence of covariates, when the frailty variable is degenerate is given by

$$F(t_{1j}, t_{2j}) = e^{-[(M_{01}(t_{1j})+M_{02}(t_{2j}))\eta_j]} \tag{15}$$

According to different assumptions on the baseline distributions we get different correlated inverse Gaussian frailty models.

4. Baseline Distributions

We present the modified inverse Weibull distribution, generalized log-logistic type I and generalized log-logistic type II as baseline distribution with the interesting properties.

4.1. Modified Inverse Weibull Distribution

The modified inverse Weibull distribution is more convenient for computational point of view for left censored data. The cumulative distribution function, the reversed hazard rate and the cumulative reversed hazard rate of the modified inverse Weibull are respectively as follows.

$$F(t) = exp\left(-\alpha t^{-\lambda}e^{-\gamma t}\right) \quad ; t > 0, \alpha > 0, \lambda > 0, \gamma > 0, \tag{16}$$

$$m(t) = \alpha e^{-\gamma t}t^{-1-\lambda}(\lambda + \gamma t). \tag{17}$$

$$M(t) = \alpha t^{-\lambda}e^{-\gamma t}, \tag{18}$$

When $\gamma = 0$, this distribution reduces to the inverse Weibull distribution. The reversed hazard rate of the modified inverse Weibull distribution is decreasing function of $t > 0$. For more details see Devendra *et al.* (2011).

4.2. Generalized Log-logistic Distribution

The log-logistic distribution is very useful in a wide variety of applications, especially in the analysis of survival data (O' Quigley and Struthers 1982; Bennett 1983; Cox and Snell 1989). The log-logistic distribution is very similar in shape to the log-normal distribution, however it has the advantage of having simple algebraic expressions for its survivor and hazard functions and a closed form for its distribution function. It is therefore more convenient than the log-normal distribution in handling censored data. However, due to the symmetry of the log-logistic distribution, it may be inappropriate for modeling censored survival data, especially for the cases where the hazard rate is skewed or heavily tailed. In order to overcome this, we use a generalization of the log-logistic distribution and refer to this as the generalized log-logistic distribution given in Mohammed *et al.*(1990). The generalized log-logistic distribution reflects the skewness and the structure of the heavy tail and generally shows some improvement over the log-logistic distribution.

Mohammed *et al.*(1990) show that the distribution function of generalized logistic is given by

$$F(x) = \frac{1}{\beta(m,n)} \int_0^{F_0(x)} u^{m-1}(1-u)^{n-1}du$$

where $\beta(m,n)$ is the complete beta function and

$$F_0(x) = (1+e^{-x})^{-1}, -\infty < x < \infty$$

is the logistic distribution function. We call $F(x)$ the generalized logistic distribution with parameters (m,n), and use the notation $X \sim GLD(m,n)$.
The logarithmic transformation $X = \gamma ln(\lambda T)$ applied to $GLD(m,1)$ to obtain the generalized log-logistic distribution $GLLD(m,1)$. The distribution function of T is

$$F(t) = (1+(\lambda t)^{-\gamma})^{-m}, t, m, \lambda > 0, \gamma \geq 1. \tag{19}$$

Similarly logarithmic transformation $X = \gamma ln(\lambda T)$ applied to $GLD(1,n)$ to obtain the generalized log-logistic distribution $GLLD(1,n)$. The distribution function of T is

$$F(t) = 1-(1+(\lambda t)^{\gamma})^{-n}, t, n, \lambda > 0, \gamma \geq 1. \tag{20}$$

A random variable T with c.d.f. as given by (19) and (20) are generalized log-logistic distribution with parameters $(m,1)$ and $(1,n)$ respectively. We call (19) as generalized log-logistic type I and (20) as generalized log-logistic type II.

Now rearranging the parameters, the cumulative distribution function of the generalized log-logistic distribution type I is

$$F(t) = \left(\frac{(\lambda t)^{\gamma}}{1+(\lambda t)^{\gamma}}\right)^{\alpha}. \tag{21}$$

The corresponding reversed hazard rate and cumulative reversed hazard rate are respectively as follows.

$$m(t) = \frac{\alpha\gamma}{t(1+(\lambda t)^{\gamma})}. \tag{22}$$

$$M(t) = \alpha ln\left(\frac{1+(\lambda t)^{\gamma}}{(\lambda t)^{\gamma}}\right) \tag{23}$$

Now rearranging the parameters, the cumulative distribution function of the generalized log-logistic distribution type II is

$$F(t) = 1 - (1+(\lambda t)^{\gamma})^{-\alpha}. \tag{24}$$

The corresponding reversed hazard rate and cumulative reversed hazard rate are respectively as follows.

$$m(t) = \frac{\alpha\gamma\lambda(\lambda t)^{-1+\gamma}(1+(\lambda t)^{\gamma})^{-1-\alpha}}{1-(1+(\lambda t)^{\gamma})^{-\alpha}}. \tag{25}$$

$$M(t) = -ln(1-(1+(\lambda t)^{\gamma})^{-\alpha}) \tag{26}$$

When $\alpha = 1$, this distribution reduces to log-logistic distribution. The reversed hazard rate of the generalized log-logistic distribution is decreasing function of $t > 0$.

5. Proposed Models

Substituting cumulative reversed hazard function for the modified inverse Weibull baseline distribution, generalized log-logistic type I and generalized log-logistic type II, we get following six models.

$$\begin{aligned} F(t_{1j},t_{2j}) &= \exp\left[\rho\frac{1-(1+2\sigma^2\eta_{0j}(\eta_{1j}\alpha_1 t_{1j}^{-\lambda_1}e^{-\gamma_1 t_{1j}}+\eta_{2j}\alpha_2 t_{2j}^{-\lambda_2}e^{-\gamma_2 t_{2j}}))^{\frac{1}{2}}}{\sigma^2}\right] \\ &\quad \exp\left[(1-\rho)\frac{1-(1+2\sigma^2\eta_{0j}\eta_{1j}\alpha_1 t_{1j}^{-\lambda_1}e^{-\gamma_1 t_{1j}})^{\frac{1}{2}}}{\sigma^2}\right] \\ &\quad \exp\left[(1-\rho)\frac{1-(1+2\sigma^2\eta_{0j}\eta_{2j}\alpha_2 t_{2j}^{-\lambda_2}e^{-\gamma_2 t_{2j}})^{\frac{1}{2}}}{\sigma^2}\right] \end{aligned} \tag{27}$$

$$F(t_{1j},t_{2j}) = exp\left(-\eta_{0j}\left\{\eta_{1j}\alpha_1 t_{1j}^{-\lambda_1}e^{-\gamma_1 t_1}+\eta_{2j}\alpha_2 t_{2j}^{-\lambda_2}e^{-\gamma_2 t_2}\right\}\right) \tag{28}$$

$$\begin{aligned} F(t_{1j},t_{2j}) &= \exp\left[\rho\frac{1-(1+2\sigma^2\eta_{0j}(\eta_{1j}\alpha_1\ln(1+1/(\lambda_1 t_{1j})^{\gamma_1})+\eta_{2j}\alpha_2\ln(1+1/(\lambda_2 t_{2j})^{\gamma_2})))^{\frac{1}{2}}}{\sigma^2}\right] \\ &\quad \exp\left[(1-\rho)\frac{1-(1+2\sigma^2\eta_{0j}\eta_{1j}\alpha_1\ln(1+1/(\lambda_1 t_{1j})^{\gamma_1}))^{\frac{1}{2}}}{\sigma^2}\right] \\ &\quad \exp\left[(1-\rho)\frac{1-(1+2\sigma^2\eta_{0j}\eta_{2j}\alpha_2\ln(1+1/(\lambda_2 t_{2j})^{\gamma_2}))^{\frac{1}{2}}}{\sigma^2}\right] \end{aligned} \tag{29}$$

$$F(t_{1j},t_{2j}) = exp\left(-\eta_{0j}\left\{\eta_{1j}\alpha_1\left(ln(\frac{1+(\lambda_1 t_{1j})^{\gamma_1}}{(\lambda_1 t_{1j})^{\gamma_1}})\right)+\eta_{2j}\alpha_2\left(ln(\frac{1+(\lambda_2 t_{2j})^{\gamma_2}}{(\lambda_2 t_{2j})^{\gamma_2}})\right)\right\}\right) \tag{30}$$

$$\begin{aligned} F(t_{1j},t_{2j}) &= \exp\left[\rho\frac{1-(1+2\sigma^2\eta_{0j}(\eta_{1j}\ln(1-(1+1(\lambda_1 t_{1j})^{\gamma_1})^{-\alpha_1})+\eta_{2j}\ln(1-(1+1(\lambda_2 t_{2j})^{\gamma_2})^{-\alpha_2})))^{\frac{1}{2}}}{\sigma^2}\right] \\ &\quad \exp\left[(1-\rho)\frac{1-(1+2\sigma^2\eta_{0j}\eta_{1j}\ln(1-(1+1(\lambda_1 t_{1j})^{\gamma_1})^{-\alpha_1}))^{\frac{1}{2}}}{\sigma^2}\right] \\ &\quad \exp\left[(1-\rho)\frac{1-(1+2\sigma^2\eta_{0j}\eta_{2j}\ln(1-(1+1(\lambda_2 t_{2j})^{\gamma_2})^{-\alpha_2}))^{\frac{1}{2}}}{\sigma^2}\right] \end{aligned} \tag{31}$$

$$F(t_{1j},t_{2j}) = exp\left(\eta_{0j}\left\{\eta_{1j}ln(1-(1+(\lambda_1 t_{1j})^{\gamma_1})^{-\alpha_1})+\eta_{2j}ln(1-(1+(\lambda_2 t_{2j})^{\gamma_2})^{-\alpha_2})\right\}\right) \tag{32}$$

Here onwards we call equations (27), (28), (29), (30), (31), and (32) as Model I, Model II, Model III, Model IV, Model V and Model VI respectively. Model I and Model II are the modified inverse Weibull baseline distribution with and without frailty, Model III and Model IV are the generalized log-logistic baseline distribution type I with and without frailty and likewise Model V and Model VI are the baseline with generalized log-logistic baseline distribution type II with and without frailty.

6. Likelihood Specification and Bayesian Estimation of Parameters

Suppose there are n individuals under study, whose first and second observed failure times are represented by (t_{1j},t_{2j}). Let c_{1j} and c_{2j} be the observed censoring times for the j^{th} individual $(j=1,2,3,...,n)$ for the first and the second recurrence times respectively. We use the left censoring scheme. We assume that the censoring time and the lifetimes of an individual are independently distributed.

The contribution of the bivariate lifetime random variable of the j^{th} individual to likelihood function is given by,

$$L_j(t_{1j},t_{2j}) = \begin{cases} f_1(t_{1j},t_{2j}), & ;\ t_{1j}>c_{1j}, t_{2j}>c_{2j}, \\ f_2(t_{1j},c_{2j}), & ;\ t_{1j}>c_{1j}, t_{2j}<c_{2j}, \\ f_3(c_{1j},t_{2j}), & ;\ t_{1j}<c_{1j}, t_{2j}>c_{2j}, \\ f_4(c_{1j},c_{2j}), & ;\ t_{1j}<c_{1j}, t_{2j}<c_{2j}. \end{cases}$$

and likelihood function is,

$$L(\underline{\psi},\underline{\beta},\theta) = \prod_{j=1}^{n_1} f_1(t_{1j},t_{2j})\prod_{j=1}^{n_2} f_2(t_{1j},c_{2j})\prod_{j=1}^{n_3} f_3(c_{1j},t_{2j})\prod_{j=1}^{n_4} f_4(c_{1j},c_{2j}) \tag{33}$$

where θ, $\underline{\psi}$ and $\underline{\beta}$ are respectively the frailty parameter (σ,ρ), the vector of baseline parameters and the vector of regression coefficients respectively. For without frailty model, likelihood function is

$$L(\underline{\psi},\underline{\beta}) = \prod_{j=1}^{n_1} f_1(t_{1j},t_{2j})\prod_{j=1}^{n_2} f_2(t_{1j},c_{2j})\prod_{j=1}^{n_3} f_3(c_{1j},t_{2j})\prod_{j=1}^{n_4} f_4(c_{1j},c_{2j}) \tag{34}$$

In equation (34) the frailty parameters θ and ρ are absent and in equation (33) they are present. The counts n_1, n_2, n_3 and n_4 be the numbers of individuals for which first and second failure times (t_{1j}, t_{2j}) lie in the ranges $t_{1j} > c_{1j}, t_{2j} > c_{2j}$; $t_{1j} > c_{1j}, t_{2j} < c_{2j}$; $t_{1j} < c_{1j}, t_{2j} > c_{2j}$ and $t_{1j} < c_{1j}, t_{2j} < c_{2j}$ respectively such that $n_1 + n_2 + n_3 + n_4 = n$ and

$$\begin{aligned} f_1(t_{1j}, t_{2j}) &= \frac{\partial^2 F(t_{1j}, t_{2j})}{\partial t_{1j} \partial t_{2j}} \\ f_2(t_{1j}, c_{2j}) &= \frac{\partial F(t_{1j}, c_{2j})}{\partial t_{1j}} \\ f_3(c_{1j}, t_{2j}) &= \frac{\partial F(c_{1j}, t_{2j})}{\partial t_{2j}} \\ f_4(c_{1j}, c_{2j}) &= F(c_{1j}, c_{2j}) \end{aligned} \tag{35}$$

where $\eta_{oj} = e^{(\beta_0 X_0)}, \eta_{1j} = e^{(\beta_1 X_1)}$ and $\eta_{2j} = e^{(\beta_2 X_2)}$. Substituting cumulative reversed hazard rate $M_{01}(t_{1j})$, $M_{02}(t_{2j})$, reversed hazard rate $m_{01}(t_{1j})$, $m_{02}(t_{2j})$ and distribution function $F(c_{1j}, c_{2j})$ for six proposed models into the last relations we get the likelihood function given by equations (33) and (34) for all the six models.

Unfortunately computing the maximum likelihood estimators (MLEs) involves solving a eleven dimensional optimization problem for Model I, Model III and Model V and nine dimensional optimization problem for Model II, Model IV and Model VI. As the method of maximum likelihood fails to estimate the parameters due to convergence problem in the iterative procedure, so we use the Bayesian approach. The traditional maximum likelihood approach to estimation is commonly used in survival analysis, but it can encounter difficulties with frailty models. Moreover, standard maximum likelihood based inference methods may not be suitable for small sample sizes or situations in which there is heavy censoring (see Kheiri *et al.* (2007)). Thus, in our problem a Bayesian approach, which does not suffer from these difficulties, is a natural one, even though it is relatively computationally intensive. To estimate parameters of the model, the Bayesian approach is now popularly used, because computation of the Bayesian analysis become feasible due to advances in computing technology.

To estimate the parameters of the model, the Bayesian approach is now popularly used, because computation of the Bayesian analysis become feasible due to advances in computing technology. Several authors have discussed Bayesian approach for the estimation of parameters of the frailty models. Some of them are, Ibrahim *et al.*(2001) and references their in, Santos and Achcar (2010). Santos and Achcar (2010) considered parametric models with Weibull and generalized gamma distribution as baseline distributions and gamma, log-normal as frailty distributions. Ibrahim *et al.* (2001) and references therein considered Weibull model and piecewise exponential model with gamma frailty. They also considered positive stable frailty models.

The joint posterior density function of parameters for given failure times is obtained as,

$$\begin{aligned} \pi(\alpha_1, \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \theta, \boldsymbol{\beta}) &\propto L(\alpha_1, \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \theta, \boldsymbol{\beta}) \\ &\times g_1(\alpha_1) g_2(\lambda_1) g_3(\gamma_1) g_4(\alpha_2) g_5(\lambda_2) g_6(\gamma_2) g_7(\theta) \prod_{i=1}^{5} p_i(\boldsymbol{\beta}_i) \end{aligned}$$

where $g_i(.)$ $(i = 1, 2, \cdots, 7)$ indicates the prior density function with known hyper parameters of corresponding arguments for baseline parameters and frailty variance; $p_i(.)$ is prior density function for regression coefficient β_i; $\boldsymbol{\beta}_i$ represents a vector of regression coefficients except β_i, $i = 1, 2, \ldots, k$ and likelihood function $L(.)$ is given by equation (33) or (34). Here we assume that all the parameters are independently distributed.

To estimate the parameters of the model, we used Metropolis-Hastings algorithm and Gibbs sampler. We monitored the convergence of a Markov chain to a stationary distribution by Geweke test (Geweke 1992)and Gelman-Rubin Statistics (Gelman and Rubin, 1992). Trace plots, coupling from the past plots and sample autocorrelation plots are used to check the behaviour of the chain, to decide burn-in period and autocorrelation lag respectively.

Algorithm consists in successively obtaining a sample from the conditional distribution of each of the parameter given all other parameters of the model. These distributions are known as full conditional distributions. In our case full conditional distributions are not easy to integrate out. So full conditional distributions are obtained by considering that they are proportional to the joint distribution of the parameters of the model.

We have full conditional distribution of the parameter α_1 with frailty as,

$$\pi_1(\alpha_1 \mid \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \theta, \boldsymbol{\beta}) \quad \propto \quad L(\alpha_1, \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \theta, \boldsymbol{\beta}) \cdot g_1(\alpha_1) \tag{36}$$

We have full conditional distribution of the parameter α_1 without frailty as,

$$\pi_1(\alpha_1 \mid \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \boldsymbol{\beta}) \quad \propto \quad L(\alpha_1, \lambda_1, \gamma_1, \alpha_2, \lambda_2, \gamma_2, \boldsymbol{\beta}) \cdot g_1(\alpha_1) \tag{37}$$

Similarly full conditional distributions for other parameters can be obtained.

To evaluate the performance of the Bayesian estimation procedure we carry out a simulation study. For the simulation purpose we have considered only one covariate X_0 which we assume to follow binomial distribution. The frailty variable Z_1 and Z_2 are assumed to have inverse Gaussian distribution with known variance and correlation ρ . Lifetimes (T_{1j}, T_{2j}) for j^{th} individual are conditionally independent for given frailty $Z_{1j} = z_{1j}$ and $Z_{2j} = z_{2j}$. We assume that $T_{ij}(i = 1, 2.; j = 1, 2, \cdots, n)$ follows one of the baseline distribution modified inverse Weibull distribution, Generalized log-logistic distribution type I and Generalized log-logistic distribution type II. As the Bayesian methods are time consuming, we generate only twenty five pairs of lifetimes.

A widely used prior for frailty parameters σ, are the gamma distribution $G(0.0001, 0.0001)$. In addition, we assume that the regression coefficients are normal with mean zero and large variance say 1000. Similar types of prior distributions are used in Ibrahim *et al.* (2001), Sahu *et al.* (1997) and Santos and Achcar (2010). So in our study we also use same non informative prior for frailty parameters σ, and regression coefficients $\beta_i, i = 1, .., 5$. Since we do not have any prior information about baseline parameters, $\lambda_1, \gamma_1, \alpha_1, \lambda_2, \gamma_2$ and α_2, prior distributions are assumed to be flat. We consider two different non-informative prior distributions for baseline parameters, one is $G(a_1, a_2)$ and another is $U(b_1, b_2)$. All the hyper-parameters a_1, a_2, b_1 and b_2 are known. Here $G(a, b)$ is the gamma distribution with the shape parameter a and the scale parameter b and $U(b_1, b_2)$ represents uniform distribution over the interval (b_1, b_2). For correlation parameter we use uniform distribution $U(0, 1)$. We use different value of baseline parameters for Model I, Model III and Model V, details are given in Table

1, 2 and 3. We assume the value of the hyper-parameters as $a_1 = 1, a_2 = 0.0001, b_1 = 0$ and $b_2 = 100$.

We run two parallel chains for model one using two sets of prior distributions with the different starting points using Metropolis-Hastings algorithm and Gibbs sampler based on normal transition kernels. We iterate both the chains for 100000 times. There is no effect of prior distribution on posterior summaries because the estimates of parameters are nearly the same and the convergence rate of Gibbs sampler for both the prior sets is almost the same. Also for both the chains the results were somewhat similar. For all models, the trace plots, the coupling from the past plots, the running mean plots and the sample autocorrelation plots for the simulation study are not provided due to lack of space. Table 1, 2 and 3 presents the estimates, the credible intervals of the parameters for the Model I, Model III and Model V based on the simulation study. These also contains the Gelman-Rubin (Gelman and Rubin, 1992) convergence statistic and the Geweke test (Geweke, 1992) for all the parameters of the Model I, Model III and Model V based on the simulation study. The Gelman-Rubin convergence statistic values are nearly equal to one and also the Geweke test values are quite small and the corresponding p-values are large enough to say that the chain attains stationary distribution. Simulated values of the parameters have the autocorrelation of lag k. So that every k^{th} iteration is selected as a sample from the posterior distribution.

7. Analysis of Australian Twin data

Duffy *et al.* (1990) considered Australian twin data which consist of information about the age at appendectomy of monozygotic (MZ) and dizygotic (DZ) twins. There were some pairs with missing age at onset and those are the left censored observations. Duffy *et al.* (1990) excluded these left censored observations in the analysis. It is therefore, appropriate to model common random effect by including those left censored observations, which can be done by developing frailty models using RHR. Accordingly, Sankaran and Gleeja (2011) introduced frailty as a common random effect that acts multiplicatively on reversed hazard rates, which is useful for the analysis of left censored data.

Now we apply the all six models to the Australian twin data given in Duffy *et a1.* (1990). The data consists of six zygote categories. We consider the subset of the data with zygote category 4. The data consists of males gender only and consist if 350 pair of twins with 9 and 11 censored in twin 1 and twin 2 respectively. An individual having age at onset less than 11 are considered as left censored observations. The data has information on the age at onset at appendectomy of twins. The genetic effect involved in the risk of appendectomy is the frailty variable. Here there is a common covariate age of twins for both T_1 and T_2 and one covariate each for T_1, T_2, i.e., presence or absence of appendectomy. To check goodness of fit of Australian twin data set, We obtain Kolmogorov-Smirnov(K-S) statistics and their p-values for T_1 and T_2. For Model I, Model III and Model V p-values of observe that p-values for Kolmogorov-Smirnov (K-S) statistics are provided in Table 4. Thus from p values of K-S test are quite high. We can say that there is no statistical evidence to the reject the hypothesis that data are from these three models.

As in case of simulation, here also we assume the same set of prior distributions. We run two parallel chains for all models using two sets of prior distributions with the different starting points using the Metropolis-Hastings algorithm and the Gibbs sampler based on

normal transition kernels. We iterate both the chains for 100000 times. As seen in simulation study here also we got nearly same estimates of parameters for both the set of priors, so estimates are not dependent on the different prior distributions. Convergence rates of Gibbs sampler for both the prior sets are almost the same. Also both the chains show somewhat similar results, so we present here the analysis for only one chain with $G(1, 0.0001)$ as prior for baseline parameters and $G(0.0001, 0.0001)$ as the prior for the frailty parameter σ^2. Due to lack of space we are presenting only for model one(trace plots and coupling from the past plots) for the parameters. Trace plots for all the parameters shows zigzag pattern which indicates that parameters move and mix more freely. Thus, it seems that the Markov chain has reached the stationary state. Burn in period is decided by using coupling from the past plot. However, a sequence of draws after burn-in period may have autocorrelation. Because of autocorrelation consecutive draws may not be random, but values at widely separated time points are approximately independent. So, a pseudo random sample from the posterior distribution can be found by taking values from a single run of the Markov chain at widely spaced time points (autocorrelation lag) after burn-in period. The autocorrelation of parameters become almost negligible after the certain lag.

The Gelman-Rubin convergence statistic values are nearly equal to one and the Geweke test statistic values are quite small and corresponding p-values are large enough to say the chains attains stationary distribution. The posterior mean and standard error with 95% credible intervals for baseline parameters, frailty parameter and regression coefficients are presented in Tables 5-10. The posterior summery of the Model I, Model II, Model III, Model IV, Model V and Model VI are given in Tables 5, 6, 7, 8, 9 and 10. Tables 5, 6, 7, 8, 9 and 10 presents estimates, credible intervals, Geweke test and Gelman-Rubin statistics for all the parameters of the Model I, Model II, Model III, Model IV, Model V and Model VI respectively, based on data. For Model I, Model III and Model V the estimates of frailty parameter σ are respectively 5.6081, 5.4875 and 4.7686. This shows that there is a heterogeneity between the pairs of twins. Bayes factor for Model I with Model II is 32.80, for Model III and Model IV is 298.41 and Model V with Model VI is 1704.12. This is also a Bayesian test based on Bayes factor for testing $\sigma^2 = 0$ against $\sigma^2 > 0$ and which supports the alternative hypothesis, i.e., models with frailty fits better. The credible interval of regression coefficient β_0 does not contain zero for all models except, Model VI. The credible interval of regression coefficient β_1 contain zero for all models except, Model III and Model V. The credible interval of regression coefficient β_2 contain zero for all models. Hence age is the significant covariate for Model I, Model II, Model III, Model IV and Model V. The convergence rate of Gibbs sampling algorithm does not depend on these choices of prior distributions in our proposed model for Australian twin data. The Geweke test values are near to zero and corresponding p-values are quite high and the Gelman-Rubin Statistics for all the parameters of all six models based on data are very close to one.

To compare six models we first use Aikaike information criteria (AIC), Bayesian information criteria (BIC) and deviance information criteria (DIC) values which are given in Table 11 and Bayes factor in Table 12. The AIC, BIC and DIC values for Model V is least among all six models. On the basis of AIC, BIC and DIC values Model V is the best among all six models. Similarly the Bayes factor show that models with frailty (Model I, Model III and Model V) are better than the models without frailty and Model V, the correlated inverse Gaussian frailty based on reversed hazard rate with generalised log-logistic type II baseline is the best and the frailty is significant.

8. Conclusions

Our main aim of the study is to examine the role of the bivariate correlated frailty model based on the reversed hazard rate in survival studies. For this we used the correlated inverse Gaussian frailty model with the modified inverse Weibull distribution, generalized log-logistic type I and generalized log-logistic type II as a baseline distribution and these models are compared with their baseline model based on reversed hazard rate. We also found that the correlated inverse Gaussian frailty models are better models as compared to their baseline model on the basis of AIC, BIC and DIC values for Australian twin data set. Bayes factor support the correlated frailty models.

Initially we thought to use the method of maximum likelihood to estimate the parameters but likelihood equations do not converge and the method of maximum likelihood fails to estimate the parameters so we used the Bayesian approach. In this study, the model is specified in a Bayesian framework and estimated with the MCMC algorithms. The estimates of the parameters are not dependent on the different prior distributions.

Two different chains were run for the proposed models from different starting points using the Metropolis-Hastings algorithm within Gibbs sampler. We have provided 100,000 iterations to perform the simulation study. Estimates were calculated after discarding a burn-in interval for each chain. The quality of convergence was checked by Gelman-Rubin statistics. The values of the Gelman-Rubin statistics in this case are quite close to one and also the Geweke test values are small with large p-values. Thus the sample can be considered to have arisen from stationary distribution and descriptive statistics can be seen as valid estimates of unknown parameters. The simulation results indicate that the performance of the Bayesian estimation method is quite satisfactory. Bayes factor is used to test the frailty parameter $\sigma^2 = 0$ and it is observed that the frailty parameter is highly significant in all frailty models. From Table 12 it is clear that the models with frailty fit better than without frailty models and Model V is best among the all six models. Age is the significant for all the models except Model VI.

The choice of the best model for Australian twin data is based on AIC, BIC, DIC and Bayes factor values. We found that Model V is a best Model on the basis of AIC, BIC, DIC and Bayes factor values. The age is the significant covariate for all models except Model IV. Correlated inverse Gaussian frailty models(Model I, Model III and Model V) are better than their baseline model. We also compare with correlated gamma frailty models suggested by Hanagal and Pandey (2017) and observe that correlated inverse Gaussian frailty based on reversed hazard rate with generalized log-logistic type II baseline performs better and more suitable than the correlated gamma frailty models proposed by Hanagal and Pandey (2017) for Australian twin data set, with left censored observations. The methods discussed in this paper may be extended into other frailty models and correlated frailty models with different baseline distributions, using the Bayesian approach, provided the models fit to the data.

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ANNEXURE

Table 1: Baseline Distribution Modified inverse Weibull Distribution Model I with Correlated Inverse Gaussian Frailty (Simulation for Model I)

Parameter (value)	Estimate	Standard Error	Lower Credible Limit	Upper Credible Limit	Geweke values	p values	Gelman & Rubin values
burn in period = 2500; autocorrelation lag = 250							
α_1 (2.0)	1.9894	0.1319	1.7806	2.2365	-0.00670	0.4973	0.9999
λ_1 (1.5)	1.5197	0.0188	1.4725	1.5510	-0.00085	0.4996	1.0003
γ_1 (2.5)	2.5306	0.1482	2.2461	2.7781	0.00052	0.5002	1.0020
α_2 (2.2)	2.2126	0.0370	2.1228	2.2726	-0.00321	0.4987	1.0007
λ_2 (2.5)	2.5197	0.0186	2.4721	2.5474	-0.02074	0.4917	1.0332
γ_2 (3.0)	3.0537	0.1413	2.7424	3.2960	-0.00574	0.4977	1.0006
σ (2.0)	2.0660	0.0544	1.8817	2.1360	0.00084	0.5003	1.0031
ρ (0.7)	0.7349	0.0324	0.6458	0.7785	-0.00578	0.4976	1.0047
β (0.50)	0.5131	0.0343	0.4157	0.5812	-0.00229	0.4990	1.0059

Table 2: Baseline Distribution Generalized Logistic Distribution Type I Model-III with Correlated Inverse Gaussian Frailty(Simulation for Model III)

Parameter (value)	Estimate	Standard Error	Lower Credible Limit	Upper Credible Limit	Geweke values	p values	Gelman & Rubin values
burn in period = 7000; autocorrelation lag = 350							
α_1 (2.0)	1.8692	0.0528	1.7779	2.0937	0.00261	0.5010	0.9999
λ_1 (1.5)	1.5621	0.0211	1.5032	1.5786	0.00171	0.5006	1.0031
γ_1 (2.5)	2.3454	0.0836	2.2347	2.6174	-0.01263	0.4949	1.0099
α_2 (2.2)	2.0152	0.0472	1.9713	2.2819	0.00531	0.5021	1.0138
λ_2 (2.5)	2.5667	0.0425	2.4663	2.6287	0.00511	0.5020	1.0043
γ_2 (2.5)	2.4307	0.1117	2.2773	2.6411	-0.00644	0.4974	1.0015
σ (0.20)	0.2304	0.0085	0.2091	0.2396	0.01255	0.5050	1.0071
ρ (0.7)	0.7686	0.0745	0.6106	0.8548	0.00563	0.5022	1.0001
β (0.50)	0.4879	0.0149	0.4702	0.5105	-0.02020	0.4919	1.0005

Table 3: Baseline Distribution Generalized Logistic Distribution Type II Model-V with Correlated Inverse Gaussian Frailty(Simulation for Model V)

Parameter (value)	Estimate	Standard Error	Lower Credible Limit	Upper Credible Limit	Geweke values	p values	Gelman & Rubin values
burn in period = 7000; autocorrelation lag = 250							
α_1 (2.0)	2.0512	0.1433	1.7726	2.2631	-0.00394	0.4984	1.0012
λ_1 (1.5)	1.5015	0.0218	1.4812	1.5492	0.00149	0.5005	1.0027
γ_1 (3.5)	3.5009	0.1216	3.3806	3.6401	0.00061	0.5002	1.0009
α_2 (2.2)	2.1858	0.1226	1.9576	2.4178	-0.01473	0.4941	0.9999
λ_2 (2.5)	2.5055	0.0212	2.4715	2.5318	0.00056	0.5002	1.0004
γ_2 (3.5)	3.4905	0.1519	3.4567	3.6781	-0.00213	0.4991	1.0097
σ (0.2)	0.21811	0.0198	0.1801	0.2191	-0.00431	0.4982	1.0047
ρ (0.7)	0.7125	0.1011	0.6128	0.7867	-0.00015	0.4999	1.0055
β (0.50)	0.4888	0.0402	0.4557	0.5549	-0.00171	0.4993	1.0000

Table 4: p-values of K-S Statistics for Goodness of Fit Test for Australian Twin Data Set

Distribution	Recurrence time First	Second
Model I	0.57402	0.59688
Model III	0.85443	0.7794
Model V	0.99977	0.99787

Table 5: Posterior Summary for Australian Twin Data Set (Model I)

Parameter (value)	Estimate	Standard Error	Lower Credible Limit	Upper Credible Limit	Geweke values	p values	Gelman & Rubin values
burn in period = 7500; autocorrelation lag = 1300							
α_1	44.1762	2.7887	39.12322	48.7418	0.01771	0.5070	1.0068
λ_1	0.4632	0.0311	0.4111	0.5071	-0.00162	0.4993	1.0341
γ_1	0.1227	0.0042	0.1106	0.1312	-0.00224	0.4991	1.0034
α_2	41.1201	2.9285	35.816	46.3785	0.00857	0.5034	1.0192
λ_2	0.4574	0.0217	0.4161	0.4989	0.00243	0.5009	1.0041
γ_2	0.2011	0.0035	0.1913	0.2112	-0.01221	0.4951	1.0066
ρ	0.9294	0.0414	0.8424	0.9978	-0.00069	0.4997	1.0099
σ	5.6081	0.0651	5.4172	5.7571	-0.01036	0.4958	1.0054
β_0	0.0209	0.0023	0.0133	0.0304	-0.01873	0.4925	1.0000
β_1	-0.0742	0.0641	-0.2287	0.1041	-0.00320	0.4987	1.0086
β_2	-0.0312	0.0204	-0.0564	0.0161	-0.00684	0.4972	1.0045

Table 6: Posterior Summary for Australian Twin Data Set (Model II)

Parameter (value)	Estimate	Standard Error	Lower Credible Limit	Upper Credible Limit	Geweke values	p values	Gelman & Rubin values
burn in period = 12000; autocorrelation lag = 180							
α_1	11.44908	0.5554	10.47060	12.34491	-0.00330	0.4986	1.0075
λ_1	0.06949	0.0112	0.04948	0.08727	0.01716	0.5068	1.0043
γ_1	0.10227	0.0036	0.09499	0.10941	-0.00179	0.4992	1.0065
α_2	10.43929	0.5275	9.44735	11.34131	-0.00460	0.4981	1.0008
λ_2	0.07101	0.0106	0.05109	0.08880	-0.00714	0.4971	1.0031
γ_2	0.09919	0.0038	0.09192	0.10693	9.27e-05	0.5001	0.9999
β_0	0.00575	0.0020	0.00152	0.00950	0.003835	0.5015	1.0000
β_1	-0.01649	0.0715	-0.14075	0.12942	-0.007237	0.4971	1.0008
β_2	0.06323	0.1238	-0.17651	0.29295	0.004567	0.5018	1.0000

Table 7: Posterior Summary for Australian Twin Data Set (Model III)

Parameter (value)	Estimate	Standard Error	Lower Credible Limit	Upper Credible Limit	Geweke values	p values	Gelman & Rubin values
burn in period = 7500; autocorrelation lag = 1100							
α_1	18.5175	0.2150	18.0612	18.8641	0.0015	0.5006	1.0004
λ_1	0.0701	0.0012	0.0676	0.0722	-0.0259	0.4896	1.0095
γ_1	23.1861	0.2119	22.6789	23.4689	-0.0056	0.4977	1.0000
α_2	18.0016	0.5215	17.1468	18.1801	0.0022	0.5008	1.0003
λ_2	0.0801	0.0012	0.0771	0.0823	-0.0254	0.4898	1.0149
γ_2	24.1014	0.2182	23.6952	24.4461	-0.0043	0.4982	1.0026
ρ	0.8941	0.0151	0.8721	0.9078	0.0026	0.5010	0.9999
σ	5.7845	0.1155	5.5526	5.9101	-0.0061	0.4975	1.0039
β_0	0.8465	0.0290	0.8161	0.8722	-0.0235	0.4906	1.0083
β_1	-0.0507	0.0277	-0.0971	-0.0052	0.0042	0.5017	1.0134
β_2	-0.0143	0.0314	-0.0426	0.0413	0.0113	0.5045	1.0321

Table 8: Posterior Summary for Australian Twin Data Set (Model IV)

Parameter (value)	Estimate	Standard Error	Lower Credible Limit	Upper Credible Limit	Geweke values	p values	Gelman & Rubin values
burn in period = 6500; autocorrelation lag = 200							
α_1	1.4806	0.2172	1.08298	1.87900	-0.00716	0.4971	1.0091
λ_1	0.0475	0.0031	0.04162	0.05341	-0.00838	0.4966	1.0008
γ_1	1.3022	0.1518	3.29435	3.93994	0.00624	0.5024	1.0017
α_2	0.0456	0.0026	1.00322	1.56740	-0.00976	0.4961	1.0096
λ_2	3.6265	0.1957	0.04062	0.05151	-0.00932	0.4962	1.0088
γ_2	3.6135	0.1680	3.25814	3.99208	0.01088	0.5043	1.0018
β_0	0.0059	0.0026	0.00068	0.01158	-0.00174	0.4993	1.0005
β_1	8.9e-06	0.0024	-0.00437	0.00442	-0.00050	0.4997	1.0054
β_2	0.0592	0.1285	-0.20118	0.29163	-7.76e-05	0.4999	1.0134

Table 9: Posterior Summary for Australian Twin Data Set (Model V)

Parameter (value)	Estimate	Standard Error	Lower Credible Limit	Upper Credible Limit	Geweke values	p values	Gelman & Rubin values
burn in period = 6500; autocorrelation lag = 300							
α_1	0.2817	0.0032	0.2781	0.2877	-0.00066	0.4997	1.0002
λ_1	0.0701	0.0025	0.0585	0.0724	-0.00274	0.4989	1.0004
γ_1	55.4383	1.1412	52.9768	57.2222	0.00761	0.5030	1.0026
α_2	0.1051	0.0031	0.0891	0.1108	0.00678	0.5027	1.0069
λ_2	0.0706	0.0012	0.0687	0.0728	-0.00166	0.4993	1.0012
γ_2	58.6274	1.6105	55.7728	61.0344	-0.00435	0.4983	1.0057
ρ	0.8824	0.0242	0.8461	0.9165	0.00041	0.5001	0.9999
σ	4.7686	0.0505	4.5495	4.8869	0.00315	0.5012	0.9999
β_0	0.0785	0.0057	0.0751	0.0903	0.00317	0.5012	1.0002
β_1	-0.0412	0.0202	-0.0819	-0.0051	0.01608	0.5064	1.0018
β_2	-0.0214	0.0247	-0.0615	0.0221	-0.00370	0.4985	1.0136

Table 10: Posterior Summary for Australian Twin Data Set (Model VI)

Parameter (value)	**Estimate**	**Standard Error**	**Lower Credible Limit**	**Upper Credible Limit**	**Geweke values**	**p values**	**Gelman & Rubin values**
burn in period = 6500; autocorrelation lag = 200							
α_1	0.6047	0.1027	0.4205	0.8341	0.01169	0.5047	1.00
λ_1	0.0475	0.0026	0.0428	0.0526	-0.00679	0.4972	1.03
γ_1	5.1896	0.5964	4.1191	6.4377	-0.01175	0.4953	1.00
α_2	0.6736	0.0885	0.5048	0.8412	0.01642	0.5065	1.00
λ_2	0.0463	0.0029	0.0406	0.0524	-0.00860	0.4965	1.00
γ_2	4.7336	0.4456	3.9453	5.7410	-0.01335	0.4946	1.01
β_0	0.0042	0.0041	-0.0041	0.0119	0.00415	0.5016	1.00
β_1	-0.0013	0.0239	-0.0441	0.0452	0.01221	0.5048	1.00
β_2	0.0481	0.1225	-0.1985	0.2838	-0.00571	0.4977	1.01

Table 11: AIC, BIC and DIC Comparison

Model	**AIC**	**BIC**	**DIC**
Model- I	5155.713	5188.813	5113.985
Model- II	5384.161	5426.847	5375.313
Model- III	5071.699	5082.212	5057.809
Model- IV	5355.809	5396.766	5351.894
Model- V	5016.714	5018.908	5003.065
Model- VI	5781.328	5901.931	5935.093

Table 12: Bayes Factors for Four Models

-	M12	M31	M14	M51	M16	M32	M42	M52
Bayes Factor	32.80	302.08	5.59	336.81	1274.5	301.2	27.40	338.49
-	M26	M34	M53	M36	M54	M46	M56	-
Bayes Factor	1268.83	298.41	32.79	1582.82	316.8	1271.11	1704.12	-

$$Mij = 2 * ln(\frac{Ii}{Ij})$$