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Inference Techniques, Properties, and Applications of the T-Marshall-Olkin X Family of Distributions

Meenu Jose[1] and Lishamol Tomy[2]

[1] *Department of Statistics, Carmel College(Autonomous) Mala,Thrissur, Kerala, India*

[2] *Department of Statistics, Deva Matha College, Kuravilangad, Kerala, 686633, India*

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Abstract

In this research, we study and introduce a new family of continuous distributions known as the T-Marshall-Olkin X family. We present some special models and investigate the asymptotic distributions of order statistics of the family half-logistic-Marshall-Olkin X family, which is explored in depth as a specific instance. The half-logistic-Marshall-Olkin Lomax distribution is one unique model in this family that is explored in depth. We list a few of the new distribution's mathematical properties. We use the maximum likelihood method to estimate the model's parameters. The bias and mean square error of the maximum likelihood estimators are examined in a simulation study that is given. Testing the importance of a distribution parameter is done using the likelihood ratio test with a simulation study. The potentiality and flexibility of the new family are illustrated by using two practical data sets.

Key words: T-X family; Marshall-Olkin; Moments; Maximum likelihood estimation; Likelihood ratio test; Applications.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

The statistical literature is rich with different kinds univariate distributions and is still growing rapidly. The classical distributions have various limitations in modelling real-life data. This persuades the statistical researcher to develop methods for generating new classes of distributions starting with a base line distribution.

Marshall and Olkin (1997) proposed a flexible family of distributions by introducing a new shape parameter to the existing family of distributions called the Marshall-Olkin family of distributions. The cumulative density function (CDF) of the Marshall-Olkin (MO) family is given respectively, by $G(x) = \frac{F(x)}{c+(1-c)F(x)}, c > 0,\ x \in \mathbb{R}$, where $F(x)$ is the baseline CDF. This approach produces a stable distribution with broad field behaviour in probability density function (PDF) and hazard rate function (HRF) compared to the baseline distribution. It

Corresponding Author: Meenu Jose
Email: meenusgc@gmail.com

provides a flexible framework for modelling a variety of circumstances and is useful in areas such as reliability, finance, simulation studies, health research, and engineering. Some MO families of distributions are MO-extended Lomax by Ghitany *et al.* (2007), MO-extended Lindley by Ghitany *et al.* (2012), MO-Fréchet by Krishna *et al.* (2013), MO-exponential Weibull by Pogány *et al.* (2015), MO-generalized exponential by Ristić and Kundu (2015), MO-Ikum by Tomy and Gillariose (2018), MO modified Lindley by Gillariose *et al.* (2020), MO Gumbel-Lomax by Nwezza and Ugwuowo (2020) MO-Lindley-Log-logistic by Moakofi *et al.* (2021), MO alpha power inverse exponential by Basheer (2022), MO Inverse log-logistic by Aako *et al.* (2022), MO Extended Gumbel Type-II by Willayat *et al.* (2022), MO extended unit-Gompertz by Opone *et al.* (2022), MO Exponentiated Dagum by Sherwani *et al.* (2023), MO Extended Generalized Exponential by Innocent *et al.* (2023), MO Pranav by Alsultan (2023), MO Exponentiated Fréchet by Niyoyunguruza *et al.* (2023), MO Chris-Jerry by Obulezi *et al.* (2023), MO Pareto type-I by Aldahlan *et al.* (2023), MO Cosine Topp-Leon by Osi *et al.* (2024a), MO Bilal by İrhad *et al.* (2024).

Alzaatreh *et al.* (2013) introduced a powerful method to generate new families of distributions called the transformed-transformer method, and the family is called the T-X family of distributions. This approach extends the beta-G by Eugene *et al.* (2002) and Kumaraswamy-G by Cordeiro and de Castro (2011) families by using any continuous distribution for a random variable T on $[a, b]$. The CDF of the T-X family of distributions is given by $R(x) = \int_a^{W[G(x)]} j(t)dt$, where j(t) is the PDF of a random variable T, $T \in [a, b]$ for $-\infty < a < b < \infty$ and $W[G(x)]$ is a function of the baseline CDF of a random variable X and satisfies three conditions, namely

- $W[G(x)] \in [a, b]$.
- $W[G(x)]$ is differentiable and monotonically non decreasing.
- $W[G(x)] \to a$ as $x \to -\infty$ and $W[G(x)] \to b$ as $x \to \infty$.

Numerous research papers on the T-X family have been published in the literature. The Weibull-Pareto distribution by Alzaatreh *et al.* (2013), Kumaraswamy-Geometric Distribution by Akinsete *et al.* (2014), McDonald quasi Lindley distribution by Merovci *et al.* (2015), Kumaraswamy -Weibull geometric distribution by Rasekhi *et al.* (2018), generalized odd inverted exponential generated family of distributions by Chesneau and Djibrila (2019), Weibull Burr X-G family of distribution by Ishaq *et al.* (2019), weighted odd Weibull generated family of distributions by Mi *et al.* (2021), exponentiated odd Chen-G family of distributions by Eliwa *et al.* (2021), generalized odd linear exponential family of distributions by Jamal *et al.* (2022), Rayleigh-Exponentiated Odd Generalized-Pareto distribution by Yahaya and Doguwa (2022), MO odd power generalized Weibull distribution by Chipepa *et al.* (2022), New Generalized Logarithmic-X family of distributions by Shah *et al.* (2023), New Generalized Odd Fréchet-Exponentiated-G family of distribution by Sadiq *et al.* (2023), new generalized exponentiated Fréchet–Weibull distribution by Klakattawi *et al.* (2023), MO Topp-Leone Half-Logistic-G family of distributions by Sengweni *et al.* (2023), exponentiated Cosine Topp-Leone Generalized family of distributions by Osi *et al.* (2024b) and others are a few examples. A review paper by Tomy *et al.* (2019) provides a detailed account of the T-X family of distributions.

Nowadays, there is a trend toward combining various families of distributions to increase the flexibility and properties of new distributions. Some of them are the beta MO family by Alizadeh *et al.* (2015a), Kumaraswamy MO family by Alizadeh *et al.* (2015b), generalized MO Kumaraswamy-G family by Handique and Chakraborty (2015a), MO-Kumaraswamy-G family by Handique and Chakraborty (2015b), T-transmuted X family by Moolath and Jayakumar (2017), MO Zubair-G family by Nasiru and Abubakari (2022), MO Weibull–Burr XII family by Alsadat *et al.* (2023), type II exponentiated half logistic-MO-G family by Oluyede and Gabanakgosi (2023), new generalized exponentiated Fréchet–Weibull family by Klakattawi *et al.* (2023), new Topp-Leone Kumaraswamy MO generated family by Atchadé *et al.* (2024). The new idea is based on both the MO and T-X families of distributions, combining the MO and T-X families of distributions. The motivations for introducing this new family of distributions are:

1. To generate a new family of distributions that have the properties contained in the MO and T-X families of distributions.

2. The new family of distributions is more adaptable to real-life data than models with same number of parameters and baseline distribution.

3. The desirable characteristics and adaptability provided by this new family of distributions, particularly in terms of the forms of the density and hazard rate functions, have inspired us to create this model, as it proves beneficial for real-life data analysis.

In this chapter, we propose a new extension of the T-X family by considering MO as baseline distribution called the T-Marshall-Olkin X family of distributions. The proposed distribution is well-suited to both biomedical and survival datasets. This study demonstrates that the novel extension of the Lomax distribution provides a better match to the datasets than other well-known distributions (see Section 8). The chapter unfolds as follows: In Section 2, we introduce a new family of distributions called "T-Marshall-Olkin X family" and study its properties. In Section 3, some members of T-Marshall-Olkin X family are identified. The mathematical properties of one of the member of T-Marshall-Olkin X family called, half logistic-Marshall-Olkin X family of distributions are studied in Section 4. In Section 5, we study the half logistic-Marshall-Olkin Lomax distribution and its properties. The maximum likelihood estimator of the unknown parameters with simulation study are discussed in Section 6. The analysis of two real data sets has been presented and illustrating the modelling potential of half logistic-Marshall-Olkin Lomax distribution in Section 8. Finally, the conclusion of the paper appears in Section 9.

2. T-Marshall-Olkin X family of distributions

The CDF of a T-X family of distributions is defined as

$$R(x) = \int_{a}^{W[G(x)]} j(t)dt. \tag{1}$$

Let $[W(G(x)] = -\log(1-G(x))$ and the random variable T be defined on $(0,\infty)$. Then the CDF becomes

$$R(x) = \int_0^{-\log(1-G(x))} j(t)dt. \tag{2}$$

As a special case, we assume $G(x)$ is a MO family of distributions.

Then

$$W(G(x)] = -\log\left\{1 - \frac{F(x)}{c+(1-c)F(x)}\right\} = -\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}.$$

From Equation (2), the CDF of the new family is

$$R(x) = \int_0^{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}} j(t)dt = J\Big\{-ln\{\frac{c(1-F(x))}{c+(1-c)F(x)}\}\Big\}. \tag{3}$$

When considering X as a continuous random variable, the probability density function (PDF) can be generated as follows:

$$\begin{aligned} r(x) &= \frac{d}{dx} J\Big\{-\log\{\frac{c(1-F(x))}{c+(1-c)F(x)}\}\Big\} \\ &= j\Big\{-\log\{\frac{c(1-F(x))}{c+(1-c)F(x)}\}\Big\}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))}; \quad x \in \mathbb{R}. \end{aligned} \tag{4}$$

The corresponding HRF can be found using the formula

$$h_r(x) = \frac{j\Big\{-\log\{\frac{c(1-F(x))}{c+(1-c)F(x)}\}\Big\}f(x)}{[1-F(x)][c+(1-c)F(x)][1-J\{-\log\{\frac{c(1-F(x))}{c+(1-c)F(x)}\}\}]}. \tag{5}$$

The shapes of the PDF and HRF can be enumerated analytically. The critical points of the density function are the roots of the equation:

$$\begin{aligned} \frac{\partial \log[r(x)]}{\partial x} &= \frac{j'\Big\{-\log\{\frac{c(1-F(x))}{c+(1-c)F(x)}\}\Big\}}{j\Big\{-\log\{\frac{c[1-F(x)]}{c+(1-c)F(x)}\}\Big\}}\frac{f(x)}{[1-F(x)][c+(1-c)F(x)]} + \frac{f'(x)}{f(x)} \\ &\quad + \frac{f(x)}{1-F(x)} - \frac{(1-c)f(x)}{c+(1-c)F(x)} = 0. \end{aligned} \tag{6}$$

Equation (6) may have more than one root. If the root of Equation (6) is x = x_0, then it corresponds to a local maximum if $\frac{\partial^2 \log[r(x)]}{\partial x^2} < 0$, a local minimum if $\frac{\partial^2 \log[r(x)]}{\partial x^2} > 0$, and a point of inflection if $\frac{\partial^2 \log[r(x)]}{\partial x^2} = 0$

Similarly, the critical points of $h_r(x)$ are the roots of the equation

$$\begin{aligned}\frac{\partial \log[h_r(x)]}{\partial x} &= \frac{j'\Big\{-\log\big\{\frac{c(1-F(x))}{c+(1-c)F(x)}\big\}\Big\}}{j\Big\{-\log\big\{\frac{c[1-F(x)]}{c+(1-c)F(x)}\big\}\Big\}}\frac{f(x)}{[1-F(x)][c+(1-c)F(x)]}\frac{f'(x)}{f(x)}+\frac{f(x)}{1-F(x)}\\ &\quad -\frac{(1-c)f(x)}{c+(1-c)F(x)}+\frac{j\Big\{-\log\big\{\frac{c[1-F(x)]}{c+(1-c)F(x)}\big\}\Big\}}{1-J\Big\{-\log\big\{\frac{c[1-F(x)]}{c+(1-c)F(x)}\big\}\Big\}}\frac{f(x)}{[1-F(x)][c+(1-c)F(x)]}\\ &= 0. \end{aligned} \tag{7}$$

Equation (7) may have more than one root. If the root of Equation (7) is x = x_0, then it corresponds to a local maximum if $\frac{\partial^2 \log[h_r(x)]}{\partial x^2} < 0$, a local minimum if $\frac{\partial^2 \log[h_r(x)]}{\partial x^2} > 0$, and a point of inflection if $\frac{\partial^2 \log[h_r(x)]}{\partial x^2} = 0$.
Some remarks on the T-Marshall-Olkin X family of distributions:

1. The T-Marshall-Olkin X family of distributions CDF and PDF, which are given in equations Equation (3) and Equation (4), can be as $R(x) = J\Big\{-\log\Big\{1-\frac{F(x)}{c+(1-c)F(x)}\Big\}\Big\} = J(H_g(x))$ and $r(x) = h_g(x)j(H_g(x))$ where $h(x)$ and $H(x)$ are HRF and cumulative HRF of the random variable X with CDF $\Big\{\frac{F(x)}{c+(1-c)F(x)}\Big\}$, ie, the Marshall-Olkin distribution. Hence, the T-Marshall-Olkin X family of distributions can be considered as a family of distributions arising from a weighted hazard function.

2. The random variable T which follows the PDF $j(t)$ and the random variable X following PDF $r(x)$ are related in the following way: $X = F^{-1}\Big\{\frac{c(1-e^{-T})}{1-(1-c)(1-e^{-T})}\Big\}$. This inverse function provides an easy way to simulate the random variable from T-Marshall-Olkin X family of distribution by initially simulating the random variable T and subsequently figuring out $X = F^{-1}\Big\{\frac{c(1-e^{-T})}{1-(1-c)(1-e^{-T})}\Big\}$, which has the CDF $R(x)$. Thus, $E(X) = E\Big\{F^{-1}\Big\{\frac{c(1-e^{-T})}{1-(1-c)(1-e^{-T})}\Big\}\Big\}$.
   The quantile function, $Q_r(u)$, $0 < u < 1$, for the T-Marshall-Olkin X family of distribution likely to be obtained by

$$Q_r(u) = F^{-1}\Big\{\frac{c(1-e^{-J^{-1}(u)})}{1-(1-c)(1-e^{-J^{-1}(u)})}\Big\}.$$

3. If X is a discrete random variable with probability mass function (PMF) $f(x)$. Then the PMF of the T-Marshall-Olkin X family of discrete distributions can be exhibited as

$$r(x) = R(x)-R(x-1) = J\Big\{-\log\big\{\frac{c(1-F(x))}{c+(1-c)F(x)}\big\}\Big\}-J\Big\{-\log\big\{\frac{c(1-F(x-1))}{c+(1-c)F(x-1)}\big\}\Big\}.$$

   In this article, the situation in which X is a continuous random variable will be covered.

4. when $c = 1$, the T-Marshall-Olkin X family of distributions reduces to the T-X family of distributions.

3. Some members of T-Marshall-Olkin X family of distributions

Several families of distributions can be derived from the T-Marshall-Olkin X family for different choices of $j(t)$. For various T distributions, Table 1 lists a few members of the T-Marshall-Olkin X family.

Some characteristics of the T-Marshall-Olkin X family for various T distributions will be dealt in the remaining areas of this section.

3.1. Exponential-Marshall-Olkin X family of distributions

In the instance when the random variable T follows the exponential distribution with parameter λ then $j(t) = \lambda e^{-\lambda t};\ t > 0,\ \lambda > 0$. Based on the Equation (4), the PDF of the exponential-Marshall-Olkin X family is.

$$r(x) = \lambda\left\{\frac{c(1 - F(x))}{c + (1 - c)F(x)}\right\}^{\lambda} \frac{f(x)}{(1 - F(x))(c + (1 - c)F(x))}; \quad c, \lambda > 0. \tag{8}$$

The CDF of the exponential distribution is $J(t) = 1 - e^{-\lambda x}$ and from Equation (3) the CDF of the exponential-Marshall-Olkin X family is

$$R(x) = 1 - \left\{\frac{c(1 - F(x))}{c + (1 - c)F(x)}\right\}^{\lambda}. \tag{9}$$

The corresponding HRF is illustrated as

$$h_r(x) = \frac{\lambda f(x)}{(1 - F(x))(c + (1 - c)F(x))} = \frac{\lambda h_f(x)}{c + (1 - c)F(x)} = \lambda h_g(x), \tag{10}$$

where $h_f(x)$ and $h_g(x)$ are the HRF of the distribution with PDF $f(x)$ and $g(x)$.
Thus

$$\lim_{x\to-\infty} h_r(x) = \lim_{x\to-\infty} \frac{\lambda h_f(x)}{c} = \lim_{x\to-\infty} \lambda h_g(x)$$
$$\lim_{x\to\infty} h_r(x) = \lim_{x\to\infty} \lambda h_f(x) = \lim_{x\to\infty} \lambda h_g(x).$$

It follows from Equation (10) that

$$\frac{\lambda h_f(x)}{c} \le h_r(x) \le \lambda h_f(x) \qquad (-\infty < x < \infty, \lambda \le c)$$

$$\lambda h_f(x) \le h_r(x) \le \frac{\lambda h_f(x)}{c} \qquad (-\infty < x < \infty, \lambda \ge c).$$

Again, Equation (10) shows that $\frac{h_r(x)}{h_f(x)}$ is increasing in x for$c \ge 1$ and dereasing for $0 < c \le 1$. Some unique instances of exponential-Marshall-Olkin X family are illustrated below

1. When $c = 1$, the exponential-Marshall-Olkin X family reduces to exponential-X family of distribution.

Table 1: Some members of T-Marshall-Olkin X family of distributions for different T distributions

Name	The density of T	The density of the family $r(x)$
Exponential	$\lambda e^{-\lambda x}$	$\lambda\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\lambda}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\quad c,\lambda>0$
Half logistic	$\frac{2\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^2}$	$\frac{2\lambda\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\lambda}}{\left\{1+\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\lambda}\right\}^2}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\quad c,\lambda>0$
Half Normal	$\frac{1}{\sigma}(\frac{2}{\pi})^{\frac{1}{2}}e^{-\frac{t^2}{2\sigma^2}}$	$\frac{1}{\sigma}(\frac{2}{\pi})^{\frac{1}{2}}e^{-\frac{\left\{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}^2}{2\sigma^2}}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\quad \sigma>0$
Type 2 Gumbel	$abx^{(-a-1)}e^{(-bx^{-a})}$	$ab\left\{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}^{(-a-1)}e^{\left\{-b\left\{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}^{-a}\right\}}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\quad a,b,c>0$
Gamma	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{\frac{-x}{\beta}}$	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\left\{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}^{\alpha-1}\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{1/\beta}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\quad \beta,\alpha>0$
Weibull	$\frac{a}{\beta}(\frac{x}{\beta})^{a-1}e^{(-\frac{x}{\beta})^{a}}$	$\frac{a}{\beta}\left\{\frac{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}}{\beta}\right\}^{(a-1)}e^{\left\{-\left\{\frac{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}}{\beta}\right\}^{a}\right\}}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\quad a,c,\beta>0$
Lomax	$\frac{\alpha}{\theta}[1+\frac{x}{\theta}]^{-(\alpha+1)}$	$\frac{\alpha}{\theta}\left\{1-\frac{\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}}{\theta}\right\}^{-(\alpha+1)}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\ c,\alpha,\theta>0$
Lindley	$\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}$	$\frac{\theta^2}{1+\theta}\left\{1-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\theta}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\ c,\theta>0$
Exponentiated Exponential	$\alpha\lambda\ (1-e^{-\lambda x})^{\alpha-1}\ e^{-\lambda x}$	$\alpha\lambda\left\{1-\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\lambda}\right\}^{\alpha-1}\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\lambda}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\ c,\alpha,\lambda>0$
Akash	$\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}$	$\frac{\theta^3}{\theta^2+2}\left\{1+\left\{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}^2\right\}\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\theta}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\ c,\theta>0$
Rayleigh	$\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$	$\frac{\left\{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}}{\sigma^2}e^{-\frac{\left\{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}^2}{2\sigma^2}}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\quad c,\sigma>0$

Table 1:(Continued)

Name	The density of T	The density of the family $r(x)$
Half Normal	$\frac{1}{\sigma}(\frac{2}{\pi})^{\frac{1}{2}}e^{-\frac{t^2}{2\sigma^2}}$	$\frac{1}{\sigma}(\frac{2}{\pi})^{\frac{1}{2}}e^{-\frac{\left\{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}^2}{2\sigma^2}}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\quad \sigma>0$
Lomax	$\frac{\alpha}{\theta}[1+\frac{x}{\theta}]^{-(\alpha+1)}$	$\frac{\alpha}{\theta}\left\{1-\frac{\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}}{\theta}\right\}^{-(\alpha+1)}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\ c,\alpha,\theta>0$
Power Cauchy	$2\pi^{-1}(\alpha/\sigma)(x/\sigma)^{\alpha-1}[1+(x/\sigma)^{2\alpha}]^{-1}$	$\frac{2\pi^{-1}\alpha}{\sigma}\left\{\frac{\left\{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}}{\sigma}\right\}^{\alpha-1}\left\{1+\left\{\frac{\left\{-\log\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}\right\}}{\sigma}\right\}^{2\alpha}\right\}^{-1}$ $\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\quad c,\alpha,\sigma>0$

2. When $\lambda = 1$, the exponential-Marshall-Olkin X family reduces to Marshall-Olkin X family of distribution.

3. When $\lambda = c = 1$, the exponential-Marshall-Olkin X family reduces to a distribution with PDF $f(x)$.

3.2. Half-logistic-Marshall-Olkin X family of distributions

In the instance when the random variable T follows the half-logistic distribution with parameter λ then $j(t) = \frac{2\lambda e^{-\lambda t}}{(1+e^{-\lambda t})^2};\ t > 0,\ \lambda > 0$. Based on the Equation (4), the PDF of the half-logistic-Marshall-Olkin X (HLMO-X) family is

$$r(x) = \frac{2\lambda\{\frac{c(1-F(x))}{c+(1-c)F(x)}\}^{\lambda}}{\left\{1+\{\frac{c(1-F(x))}{c+(1-c)F(x)}\}^{\lambda}\right\}^2}\frac{f(x)}{(1-F(x))(c+(1-c)F(x))};\quad c,\lambda > 0. \tag{11}$$

When $c = 1$, the HLMO-X family reduces to half-logistic-X family of distributions. The CDF of the half-logistic distribution is $J(t) = \frac{1-e^{-\lambda t}}{1+e^{-\lambda t}}$ and hence from Equation (3) the CDF of the HLMO-X family is

$$R(x) = \frac{1-\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\lambda}}{1+\left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\lambda}}. \tag{12}$$

The corresponding HRF is given by

$$\begin{aligned} h_r(x) &= \frac{\lambda}{\left\{1+\{\frac{c(1-F(x))}{c+(1-c)F(x)}\}^{\lambda}\right\}}\frac{h_f(x)}{(c+(1-c)F(x))} \\ &= \frac{\lambda h_g(x)}{\left\{1+\{\frac{c(1-F(x))}{c+(1-c)F(x)}\}^{\lambda}\right\}}, \end{aligned} \tag{13}$$

where $h_f(x)$ and $h_g(x)$are the HRF of a distribution with PDF $f(x)$ and $g(x)$. Thus

$$\lim_{x\to-\infty} h_r(x) = \lim_{x\to-\infty}\frac{\lambda h_f(x)}{2c} = \lim_{x\to-\infty}\frac{\lambda h_g(x)}{2}$$
$$\lim_{x\to\infty} h_r(x) = \lim_{x\to\infty}\lambda h_f(x) = \lim_{x\to\infty}\lambda h_g(x).$$

It follows from Equation (13) that

$$\frac{\lambda h_f(x)}{2c} \le h_r(x) \le \lambda h_f(x) \qquad (-\infty < x < \infty, \lambda \le 2c)$$

$$\lambda h_f(x) \le h_r(x) \le \frac{\lambda h_f(x)}{2c} \qquad (-\infty < x < \infty, \lambda \ge 2c).$$

Again, Equation (13) shows that $\frac{h_r(x)}{h_f(x)}$ is increasing in x for $c \leq 1$ and decreasing for $0 < c \geq 1$.

The quantile function, $Q_r(u)$, $0 < u < 1$, is given by

$$Q_r(u) = F^{-1}\Big\{\frac{c\Big[1-[\frac{1-u}{1+u}]^{1/\lambda}\Big]}{1-[1-c]\Big[1-[\frac{1-u}{1+u}]^{1/\lambda}\Big]}\Big\}. \tag{14}$$

To generate a random variable from HLMO-X first generate a $U \sim U(0,1)$ then use

$$X = F^{-1}\Big\{\frac{c\Big[1-[\frac{1-u}{1+u}]^{1/\lambda}\Big]}{1-[1-c]\Big[1-[\frac{1-u}{1+u}]^{1/\lambda}\Big]}\Big\}.$$

Another approach to simulate the HLMO-X random variable is to simulate the half-Logistic random variable T and then calculate $X = F^{-1}\Big\{\frac{c(1-e^{-T})}{1-(1-c)(1-e^{-T})}\Big\}$
The p^{th} quantile for HLMO-X family can be obtained as

$$Q_r(p) = F^{-1}\Big\{\frac{c\Big[1-[\frac{1-p}{1+p}]^{1/\lambda}\Big]}{1-[1-c]\Big[1-[\frac{1-p}{1+p}]^{1/\lambda}\Big]}\Big\}.$$

4. Properties of HLMO-X family of distributions

This section is devoted to some important properties of HLMO-X family of distributions.

4.1. Some valuable expansions

Here we provide linear representations for the CDF and PDF of the HLMO-X family of distributions. If c $\in (0,1)$, by applying the generalized binomial expansion in Equation (12), we are getting the following result.

$$R(x) = -1 + 2\Big\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}(-1)^{j+k+l}c^{\lambda j}[1-c]^k\binom{-\lambda j}{k}\binom{\lambda j+k}{l}(F(x))^l\Big\}.$$

By swapping the indices k and l in the sum symbol,

$$R(x) = -1 + 2\Big\{\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=l}^{\infty}(-1)^{j+k+l}c^{\lambda j}[1-c]^k\binom{-\lambda j}{k}\binom{\lambda j+k}{l}(F(x))^l\Big\}.$$

and then

$$R(x) = \sum_{l=0}^{\infty} b_l[F(x)]^l, \tag{15}$$

where $a_l = 2\sum_{j=0}^{\infty}\sum_{k=l}^{\infty}(-1)^{j+k+l}c^{\lambda j}[1-c]^k\binom{-\lambda j}{k}\binom{\lambda j+k}{l}$, $b_0 = -1 + a_0$and, for $l \geq 1$, $b_l = a_l$. That is, the PDF of X can be expressed as a mixture of exponentiated-F ("exp-F" for short) densities

$$r(x) = \sum_{l=0}^{\infty} b_{l+1}h_{l+1}(x), \tag{16}$$

where $h_{l+1}(x) = (l+1)[F(x)]^l(f(x))$ represents the PDF of exp-F distribution with $(l+1)$ as the power parameter. Therefore, using Equation (16), several mathematical properties of the new distribution are able to be readily derived from those of the exp-F distribution. For instance, the ordinary and incomplete moments as well as the moment generating function of X can be derived from those quantities of the exp-F distribution.

4.2. Moments, generating functions and mean deviation

Let $Y_{l+1}(l > 0)$ be a random variable with power parameter $l+1$ and PDF h_{l+1}. The n^{th} raw moment of X, that is n^{th} raw moment of HLMO-X family of distribution follows from Equation (16) as

$$\acute{\mu}_n = E(X^n) = \sum_{l=0}^{\infty} b_{l+1}E(Y_{l+1}^n). \tag{17}$$

Another formula for $\acute{\mu}_n$ follows from (17) as

$$\acute{\mu}_n = E(X^n) = \sum_{l=0}^{\infty}(l+1)b_{l+1}w_{n,l}, \tag{18}$$

where $w_{n,l} = \int_0^1 Q_F(u)^n u^l du$, $Q_F(u)$ is the quantile function with CDF $F(x)$.

The m^{th} central moment of X by using $\acute{\mu}_n$ in Equation (18) is given by

$$\mu_m = E(X - \acute{\mu}_1)^m = \sum_{n=0}^{m}\binom{m}{n}(-\acute{\mu}_1)^{m-n}\acute{\mu}_n. \tag{19}$$

The n^{th} incomplete moment of X is described by $m_n(y) = \int_{-\infty}^{y} x^n r(x)$. So $m_n(y)$ follows as

$$m_n(y) = \sum_{l=0}^{\infty}(l+1)b_{l+1}\int_o^{F(y)} Q_F(u)^n u^l du, \tag{20}$$

For most F distributions, the integral can be calculated at least numerically.

For the moment generating function (MGF) $M(t)$ of X, we propose two formulas. The first formula comes from Equation (16) as

$$M(t) = \sum_{l=0}^{\infty} b_{l+1}M_{l+1}(t), \tag{21}$$

where $M_{l+1}(t)$ represented as the MGF of exp-F distribution with power parameter $(l+1)$. The second formula comes from Equation (21) as

$$M(t)=\sum_{l=0}^{\infty}(l+1)b_{l+1}\tau(t,l), \tag{22}$$

Where $\tau(t,l)=\int_0^1 exp[tQ_F(u)]u^l du$.

The mean deviation about the mean($\delta_1=E(|X-\acute{\mu_1}|)$ and about the median ($\delta_2=E(|X-M|)$of X are given by

$$\delta_1=2\acute{\mu_1}R(\acute{\mu_1})-2m_1(\acute{\mu_1}) \tag{23}$$

and

$$\delta_2=\acute{\mu_1}-2m_1(M), \tag{24}$$

where $M=Q_r(0.5)$ is the median of X ,$\acute{\mu_1}=E(X)$, $R(\acute{\mu_1})$ is simply calculated from Equation (12) and $m_1(y)$ is the first incomplete moment given by Equation (20) with n = 1 that is,

$$m_1(y)=\sum_{l=0}^{\infty}(l+1)b_{l+1}\rho(y,l), \tag{25}$$

where $\rho(y,l)=\int_o^{F(y)}Q_F(u)u^l du$ can be computed numerically. Other formulae for $m_1(y)$ is

$$m_1(y)=\sum_{l=0}^{\infty}b_{l+1}j_{l+1}(y), \tag{26}$$

where $j_{l+1}(y)=\int_{-\infty}^{y}xh_{l+1}(x)dx$ is the key quantity needed to compute the first incomplete moment of the exp-F distribution. The equations Equation (25) and Equation (26) may be applied to construct Bonferroni and Lorenz curves that are useful in reliability, economics, insurance, demography, and medicine. For a given probability π the Bonferroni and Lorenz curves is defined by $B(\pi)=m_1(q)/(\pi\acute{\mu_1})$ and $L(\pi)=m_1(q)/\acute{\mu_1}$ respectively, where $q=Q(\pi)$ is the quantile function of X at π.

4.3. Order statistics

Assume that $X_1,X_2...,X_n$ is a random sample drawn from HLMO-X family of distribution and $X_{1:n},X_{2:n},...,X_{n:n}$ is the corresponding order statistic. Then the PDF $f_{i:n}(x)$ of the i^{th} order statistic, let's say $X_{i:n}$, is provided by

$$\begin{aligned} f_{i:n}(x)&=\frac{n!}{(i-1)!\,(n-i)!}r(x)R^{i-1}(x)[1-R(x)]^{n-i}\\ &=\frac{n!}{(i-1)!\,(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}r(x)[R(x)]^{i+j-1}. \end{aligned} \tag{27}$$

using Equation (16) and Equation (17) we can get

$$f_{i:n}(x) = \frac{n!}{(i-1)!} \sum_{j=0}^{n-i} \frac{(-1)^j}{(n-i-j)!\, j!} \Big[\sum_{k=0}^{\infty} b_{k+1}(k+1)[F(x)]^k f(x) \Big] \Big[\sum_{l=0}^{\infty} b_l [F(x)]^l \Big]^{i+j-1}.$$

Then we use power series expansion raised to a positive integer by Gradshteyn and Ryzhik (2014)

$$f_{i:n}(x) = \sum_{k,l=0}^{\infty} m_{k,l} h_{k+l+1}(x). \tag{28}$$

where h_{k+l+1} represents the exp-F density function with k+l+1 as its parameter, $m_{k,l} = \frac{n!(k+1)b_{k+1}}{(i-1)!(k+l+1)} \sum_{j=0}^{n-i} \frac{(-1)^j r_{j+i-1,l}}{(n-i-j)!j!}$, b_l is defined in Equation (16), the quantities $r_{j+i-1,l}$ are obtained recursively from $r_{j+i-1,0} = b_0^{j+i-1}$ and (for $l \geq 1$) $r_{j+i-1,l} = (lb_0)^{-1} \sum_{m=1}^{l} [m(i+j)-l] b_m r_{j+i-1,l-m}$. Equation (28) allows us to obtain the ordinary and incomplete moments, generating function and mean deviations of $X_{i:n}$.

4.4. Asymptotic distributions of sample extremes

A CDF R is said to belong to the domain of maximal (minimal) attraction of a non degenerate CDF $H(H^\star)$, denoted by $R \in D_{max}(H)(R \in D_{min}(H^\star))$, if there exist normalizing constants a_n and $b_n > 0$ ($a_n^\star$ and $b_n^\star > 0$) such that $R_{n:n}(a_n + b_n x) = P(X_{n:n} \leq a_n + b_n x) \to H(x)(R_{1:n}(a_n^\star + b_n^\star x) = P(X_{1:n} \leq a_n^\star + b_n^\star x) \to H(x))$ for all continuity points of $H(H^\star)$, where $H^\star(x) = 1 - H(-x)$.
As it is widely known, see (Arnold *et al.* (2008), p. 210, 213), that H belongs to any of the following types:

$$(i)\ H_1(x,\alpha) = e^{-x^{-\alpha}}, \quad x > 0, \alpha > 0.$$
$$(ii)\ H_2(x,\alpha) = e^{-(-x)^{\alpha}}, \quad x < 0, \alpha > 0.$$
$$(iii)\ H_3(x,\alpha) = e^{-e^{-x}}, \quad -\infty < x < \infty.$$

Lemma 1: (See Arnold *et al.* (2008), p. 218)
(i) $F \in D_{max}(H)$ if and only if $n\overline{F}(a_n + b_n x) \to -\log H(x)$
(ii) $F \in D_{min}(H^\star)$ if and only if $nF(a_n^\star + b_n^\star x) \to -\log[1 - H^\star(x)]$.

Theorem 1: For any CDF F, we have
(i) $R \in D_{max}(H)$ if and only if $G \in D_{max}(H)$
(ii) $R \in D_{max}(H)$ if and only if $F \in D_{max}(H)$.
More specifically, we have
(1) $G \in D_{max}(H_1(x;\alpha))$ if and only if $R \in D_{max}(H_1((2)^{-1/\alpha\lambda}x;\alpha\lambda))$
Also $F \in D_{max}(H_1(x;\alpha))$ if and only if $R \in D_{max}(H_1(2c^\lambda)^{-1/\alpha\lambda}x;\alpha\lambda))$
(2) $G \in D_{max}(H_2(x;\alpha))$ if and only if $R \in D_{max}(H_2((2)^{1/\alpha\lambda}x;\alpha\lambda))$
Also $F \in D_{max}(H_2(x;\alpha))$ if and only if $R \in D_{max}(H_2(2c^\lambda)^{1/\alpha\lambda}x;\alpha\lambda))$
(3) $G \in D_{max}(H_3(x))$ if and only if $R \in D_{max}(H_3(x\lambda - \log 2))$
Also $F \in D_{max}(H_3(x))$ if and only if $R \in D_{max}(H_3(x\lambda - \log 2c^\lambda)$
If a_n and $b_n > 0$ are the appropriate normalizing constants for the weak convergence of the upper extremes according to G(or F) in the three cases mentioned above, then $a_{\varphi(n;\lambda)}$ and $b_{\varphi(n;\lambda)} > 0$ are the appropriate normalizing constants for the weak convergence of the upper extremes according to R, where $\varphi(n;b) = [n^{1/b}]$ and $[\mu]$ indicates the integer part of μ.

Proof. If $G \in D_{max}(H)$, with appropriate normalizing constants a_n and $b_n > 0$, then by applying (i) of Lemma 1, as $n \to \infty$,

$$\varphi(n;\lambda)(1 - G(a_{\varphi(n;\lambda)} + b_{\varphi(n;\lambda)}x)) \to -\log H(x),$$

which implies $n(1 - G(a_{\varphi(n;\lambda)} + b_{\varphi(n;\lambda)}x))^{\lambda} \to [-\log H(x)]^{\lambda}$. Instead, we have $1 - G(a_n + b_n x) \to 0$, for all values of x for which $-\log H(x)$ is finite. This implies that $(1 - G(a_{\varphi(n;\lambda)} + b_{\varphi(n;\lambda)}x) \to 0$, for all values of x for which $-\log H(x)$ is finite. Thus,

$$\begin{aligned} n[1 - R(a_{\varphi(n;\lambda)} + b_{\varphi(n;\lambda)}x);\lambda] &= n\Big\{\frac{2[\bar{G}(a_{\varphi(n;\lambda)} + b_{\varphi(n;\lambda)}x)]^{\lambda}}{1 + [\bar{G}(a_{\varphi(n;\lambda)} + b_{\varphi(n;\lambda)}x)]^{\lambda}}\Big\} \\ &\sim 2n[\bar{G}(a_{\varphi(n;\lambda)} + b_{\varphi(n;\lambda)}x)]^{\lambda} \to 2[-\log H(x)]^{\lambda}. \end{aligned}$$

and also noting that
$2(-\log H_1(x;\alpha))^{\lambda} = -\log(H_1((2)^{-1/\alpha\lambda}x;\alpha\lambda))$;
$2(-\log H_2(x;\alpha))^{\lambda} = -\log(H_2((2)^{1/\alpha\lambda}x;\alpha\lambda))$;
$2(-\log H_3(x))^{\lambda} = -\log(H_3(x\lambda - \log 2))$.
Moving on to the converse claim, let us assume that for a given $\lambda > 0$ we have $R \in D_{max}(H)$, with $\hat{a}_n$ and $\hat{b}_n > 0$ are the normalizing constants based on R. From (i) of Lemma 1, we then have

$$n[1 - R(\hat{a}_n + \hat{b}_n x);\lambda] \to -\log H(x),$$

as $n \to \infty$, which implies $1 - R(\hat{a}_n + \hat{b}_n x;\lambda) \to 0$, that is, $G(\hat{a}_n + \hat{b}_n x \to 1)$, as $n \to \infty$, for all values of x for which $-\log H(x)$ is finite. Thus,

$$n[1 - R(\hat{a}_n + \hat{b}_n x);\lambda] = n\Big\{\frac{2[\bar{G}(\hat{a}_n + \hat{b}_n x)]^{\lambda}}{1 + \bar{G}(\hat{a}_n + \hat{b}_n x)^{\lambda}}\Big\} \sim 2n[\bar{G}(\hat{a}_n + \hat{b}_n x)]^{\lambda}.$$

From this we get, $2n[\bar{G}(\hat{a}_n + \hat{b}_n x)]^{\lambda} \to -\log H(x)$ or equivalently, $\varphi(n;\lambda)(1 - G(\hat{a}_n + \hat{b}_n x)) \to \frac{[-\log H(x)]^{1/\lambda}}{2}$. Since the last convergence holds for all subsequence of n and specifically holds for the subsequence $\acute{n} = \varphi(n;1/\lambda) = [n^{\lambda}]$, where $\varphi(\acute{n};\lambda) = [[n^{\lambda}]^{1/\lambda}] \sim n$, we get $n(1 - G(\tilde{a}_n + \tilde{b}_n x)) \to \frac{[-\log H(x)]^{1/\lambda}}{2}$, where $\tilde{a}_n = \hat{a}_{[n^{\lambda}]}$ and $\tilde{b}_n = \hat{b}_{[n^{\lambda}]}$. Thus, we get the expected result (notice that the theorem's converse portion holds true for the normalizing constants $\tilde{a}_n$ and $\tilde{b}_n$, that is $R(\hat{a}_n + \hat{b}_n x) \in D_{max}(H)$. Implies $G(\tilde{a}_n + \tilde{b}_n x) = G(\hat{a}_{[n^{\lambda}]} + \hat{b}_{[n^{\lambda}]}x) \in D_{max}(H)$), Hence, the given theorem is proved for the part (i) scenario. The proof of theorem for the part (ii) scenario follows by similar manner by using Lemma 1, Part (i).This completes the proof.

5. Half-logistic-Marshal-Olkin Lomax distribution

Let X be a random variable following the Lomax (L) distribution with parameters α and θ then $f(x) = \frac{\alpha}{\theta}[1 + \frac{x}{\theta}]^{-(\alpha+1)};\ x > 0,\ \alpha, \theta > 0$. The PDF of half-logistic-Marshall-Olkin Lomax (HLMOL) distribution using Equation (11) is defined as

$$r(x) = \frac{2\lambda\alpha c^{\lambda}}{\theta}\frac{[(1 + \frac{x}{\theta})^{\alpha} + c - 1]^{\lambda-1}[1 + \frac{x}{\theta}]^{\alpha-1}}{\Big[[(1 + \frac{x}{\theta})^{\alpha} + c - 1]^{\lambda} + c^{\lambda}\Big]^2};\ x > 0,\ c, \lambda, \alpha, \theta > 0, \tag{29}$$

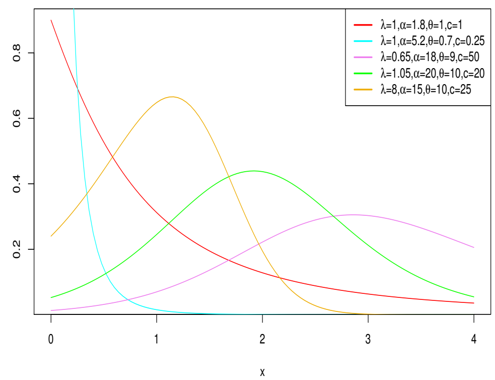

Figure 1: PDF of HLMOL for various values of α, θ, λ and c

where c, λ, α and θ are location, location scale, scale, and shape parameters, respectively. Hereafter, a random variable X with a PDF in Equation (29) will be denoted by $X \sim$ HLMOL$(c, \lambda, \alpha, \theta)$. The CDF of the Lomax distribution is F(x) $= 1 - [1 + \frac{x}{\theta}]^{-\alpha}$ and hence from Equation (12) the CDF of the HLMOL distribution is

$$R(x) = \frac{\left[(1 + \frac{x}{\theta})^{\alpha} + c - 1\right]^{\lambda} - c^{\lambda}}{\left[(1 + \frac{x}{\theta})^{\alpha} + c - 1\right]^{\lambda} + c^{\lambda}}. \tag{30}$$

In the form of graphical representations, Figure 1 displays a few plots of $r(x)$ for selected values of the parameter c, λ, α and θ. These plots demonstrate that the PDF has good shape flexibility. It can be reversed J-shape, left-skewed, right-skewed, or symmetric.

Some unique cases of HLMOL distribution:

1. When $c = 1$, the HLMOL distribution reduces to half-logistic Lomax by Anwar and Zahoor (2018) distribution.
2. When $\lambda = 1$, the HLMOL distribution reduces to Marshall-Olkin half-logistic Lomax distribution.
3. When $\lambda = 1$ and c=0.5, the HLMOL distribution reduces to Lomax distribution.

In lifetime analysis, the HRF is a useful function. Therefore, the HRF of $X \sim$ HLMOL$(c, \lambda, \alpha, \theta)$ is given by

$$h_r(x) = \frac{\frac{\lambda\alpha}{\theta}\left[(1 + \frac{x}{\theta})^{\alpha} + c - 1\right]^{\lambda - 1}[1 + \frac{x}{\theta}]^{\alpha - 1}}{\left[\left[(1 + \frac{x}{\theta})^{\alpha} + c - 1\right]^{\lambda} + c^{\lambda}\right]}. \tag{31}$$

Figure 2 displays the graphs of $h_r(x)$ for selected values of the parameters α, θ, λ and c. It can be upside down bathtub and decreasing.

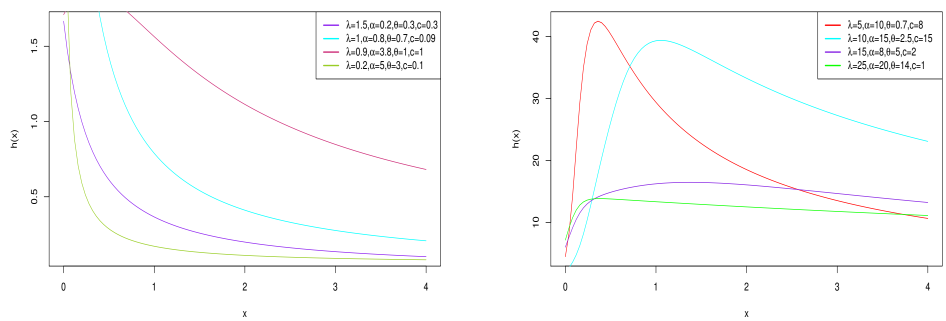

Figure 2: HRF of HLMOL for various values of α, θ, λ and c

By using Equation (14) the quantile function, $Q_r(u)$, $0 < u < 1$, is given by

$$Q_r(u) = \theta\Big\{\Big[\frac{[\frac{1-u}{1+u}]^{1/\lambda}}{1-[1-c][1-[\frac{1-u}{1+u}]^{1/\lambda}]}\Big]^{-1/\alpha} - 1\Big\}.$$

To generate a random variable from HLMOL, first generate a $U \sim U(0,1)$ then use

$$X = \theta\Big\{\Big[\frac{[\frac{1-u}{1+u}]^{1/\lambda}}{1-[1-c][1-[\frac{1-u}{1+u}]^{1/\lambda}]}\Big]^{-1/\alpha} - 1\Big\}.$$

Another approach to simulate the HLMOL random variable is by simulating the half-logistic random variable T and then calculate

$$X = \theta\Big\{\Big[\frac{[e^{-T}]^{1/\lambda}}{1-[1-c][1-[e^{-T}]^{1/\lambda}]}\Big]^{-1/\alpha} - 1\Big\}.$$

The p^{th} quantile for HLMOL distribution can be obtained as

$$Q_r(p) = \theta\Big\{\Big[\frac{[\frac{1-p}{1+p}]^{1/\lambda}}{1-[1-c][1-[\frac{1-p}{1+p}]^{1/\lambda}]}\Big]^{-1/\alpha} - 1\Big\}.$$

If p =1/2, that is median of HLMOL is given by

$$M = \theta\Big\{\Big[\frac{[\frac{1}{3}]^{1/\lambda}}{1-[1-c][1-[\frac{1}{3}]^{1/\lambda}]}\Big]^{-1/\alpha} - 1\Big\}.$$

5.1. Linear representation

By using Equations (16) and (17) the linear representation of CDF and PDF of HLMOL distribution is given by

$$R(x) = \sum_{l=0}^{\infty} b_l[1-[1+\frac{x}{\theta}]^{-\alpha}]^l, \tag{32}$$

Where $a_l = 2\sum_{j=0}^{\infty}\sum_{k=l}^{\infty}(-1)^{j+k+l}c^{\lambda j}[1-c]^k\binom{-\lambda j}{k}\binom{\lambda j+k}{l}$, $b_0 = -1 + a_0$and, for $l \geq 1$,$b_l = a_l$

$$\begin{aligned} r(x) &= \sum_{l=0}^{\infty}\sum_{j=0}^{l}(l+1)b_{l+1}(-1)^j\binom{l}{j}\frac{\alpha}{\theta}[1+\frac{x}{\theta}]^{-(\alpha+\alpha j+1)} \\ &= \sum_{l=0}^{\infty}\sum_{j=0}^{l}(l+1)b_{l+1}(-1)^j\binom{l}{j}\frac{\alpha}{\alpha+\alpha j}L(x;\alpha+\alpha j,\theta), \end{aligned} \tag{33}$$

Where $L(x;\alpha+\alpha j,\theta)$ denoted the Lomax PDF with parameter θ and $\alpha+\alpha j$. So the PDF of HLMOL is simply an infinite linear combination of Lomax distribution. Thus, some mathematical properties of the new distribution can be obtained straightly from those Lomax distribution properties based on Equation (33).

5.2. Moments and generating functions

The n^{th} raw moment of X is obtained from Equation (20)

$$\acute{\mu}_n = E(X^n) = \sum_{l=0}^{\infty}\sum_{i=0}^{n}(l+1)b_{l+1}\theta^n(-1)^i\binom{n}{i}\beta(l+1, 1+\frac{i-n}{\alpha}), \qquad n < \alpha.$$

If n=1, That is the mean of HLMOL distribution is given by

$$\acute{\mu}_1 = E(X) = \sum_{l=0}^{\infty}(l+1)b_{l+1}\theta[\beta(l+1, 1-1/\alpha) - \beta(l+1, 1)], \qquad \alpha > 1.$$

If n=2

$$\begin{aligned} \acute{\mu}_2 &= E(X^2) \\ &= \sum_{l=0}^{\infty}(l+1)b_{l+1}\theta^2[\beta(l+1, 1-2/\alpha) - 2\beta(l+1, 1-1/\alpha) + \beta(l+1, 1)], \qquad \alpha > 2. \end{aligned}$$

The m^{th} central moment of X by using $\acute{\mu}_n$ in Equation (19) is given by

$$\mu_m = E(X - \acute{\mu}_1)^m = \sum_{n=0}^{m}\binom{m}{n}(-\acute{\mu}_1)^{m-n}\acute{\mu}_n.$$

If m=2, That is the variance of HLMOL distribution is given by

$$\begin{aligned} \mu_2 = &\sum_{l=0}^{\infty}(l+1)b_{l+1}\theta^2[\beta(l+1, 1-2/\alpha) - 2\beta(l+1, 1-1/\alpha) + \beta(l+1, 1)] \\ &- \Big[\sum_{l=0}^{\infty}(l+1)b_{l+1}\theta[\beta(l+1, 1-1/\alpha) - \beta(l+1, 1)]\Big]^2, \qquad \alpha > 2. \end{aligned}$$

Then, the moment measure of skewness $S = \frac{\mu_3^2}{\mu_2^3}$ and moment measure of kurtosis $K = \frac{\mu_4}{\mu_2^2}$ can be calculated from the second, third and fourth central moments.

The n^{th} incomplete moment of X is defined by using Equation (20) is given by

$$m_n(y) = \sum_{l=0}^{\infty}\sum_{i=0}^{n}(l+1)b_{l+1}\theta^n(-1)^i\binom{n}{i}\beta_{F(y)}(l+1, 1+\frac{i-n}{\alpha}), \qquad n < \alpha.$$

By using Equation (22) the MGF of X is given by

$$\begin{aligned} M(t) &= \sum_{l=0}^{\infty}(l+1)b_{l+1}\int_0^1\sum_{n=0}^{\infty}\frac{(t\theta)^n}{n!}[(1-u)^{-1/\alpha}-1]^n u^l du \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\acute{\mu_n}, \end{aligned}$$

where $\acute{\mu_n}$ is the n^{th} raw moment of the HLMOL distribution. The Bonferroni and the Lorenz curve are given by

$$B(\pi) = \frac{\sum_{l=0}^{\infty}(l+1)b_{l+1}[\beta_{F(q)}(l+1,1-1/\alpha)-\beta_{F(q)}(l+1,1)]}{\pi\sum_{l=0}^{\infty}(l+1)b_{l+1}[\beta(l+1,1-1/\alpha)-\beta(l+1,1)]}, \qquad \alpha > 1.$$

$$L(\pi)) = \frac{\sum_{l=0}^{\infty}(l+1)b_{l+1}[\beta_{F(q)}(l+1,1-1/\alpha)-\beta_{F(q)}(l+1,1)]}{\sum_{l=0}^{\infty}(l+1)b_{l+1}[\beta(l+1,1-1/\alpha)-\beta(l+1,1)]}, \qquad \alpha > 1, \tag{34}$$

where $q = Q(\pi)$ is the quantile function of X at π.

Table 2 gives the mean, variance, third raw moment, skewness and kurtosis of HLMOL distribution for different choices of parameter values. For fixed λ and c, the mean and variance of the HLMOL distribution are increasing functions of θ and α. Also the distribution of the HLMOL distribution tends to be skewed more to the right as θ and α decreases. For fixed λ, θ and α, the HLMOL distribution can be platykurtic, mesokurtic and leptokurtic as c increases. Also the distribution of the HLMOL distribution tends to be skewed more to the left as c increases. That is, the HLMOL is positively and negatively skewed, platykurtic, mesokurtic and leptokurtic distribution.

5.3. Order statistics

Assume that $X_1, X_2..., X_n$ is a random sample drawn from HLMOL distribution and $X_{1:n}, X_{2:n}, ..., X_{n:n}$ is the corresponding order statistic. Then the PDF $f_{i:n}(x)$ of the i^{th} order statistic, let's say $X_{i:n}$, is provided by

$$f_{i:n}(x) = \sum_{k,l=0}^{\infty}\sum_{j=0}^{k+l}m_{k,l}(-1)^j(k+l+1)\binom{k+l}{j}\frac{\alpha}{\alpha+\alpha j}L(x;\alpha+\alpha j,\theta), \tag{35}$$

where $m_{k,l} = \frac{n!(k+1)b_{k+1}}{(i-1)!(k+l+1)}\sum_{j=0}^{n-i}\frac{(-1)^j r_{j+i-1,l}}{(n-i-j)!j!}$, the quantities $r_{j+i-1,l}$ are obtained recursively from $r_{j+i-1,0} = b_0^{j+i-1}$ and (for $l \geq 1$) $r_{j+i-1,l} = (lb_0)^{-1}\sum_{m=1}^{l}[m(i+j)-l]b_m r_{j+i-1,l-m}$ and $L(x;\alpha+\alpha j,\theta)$ denoted the Lomax PDF with parameter θ and $\alpha+\alpha j$. So the PDF of i^{th} order statistic of HLMOL distribution is simply an infinite linear combination of Lomax distribution.

Table 2: $\acute{\mu_1}$, μ_2, μ_3, S and K for various choices of parameters

Parameter	$\acute{\mu_1}$	μ_2	μ_3	S	K
$\lambda = 0.95$ c=0.25 θ=0.7 α=5.2	0.1218	0.0343	0.03572	31.6755	107.1577
$\lambda = 0.95$ c=0.25 θ=0.9 α=6	0.1319	0.0370	0.0342	23.0538	61.2956
$\lambda = 0.95$ c=0.25 θ=10 α=20	0.3904	0.2398	0.3475	8.7539	17.3673
λ =3 c=20 θ=50 α=50	2.3191	1.1223	0.1005	0.0071	2.7521
λ =3 c=74.4263 θ=50 α=50	3.5554	1.5711	-0.5373	0.0744	3
λ =3 c=80 θ=50 α=50	3.6277	1.5920	-0.5798	0.0833	3.0237
$\lambda = 0.6$ c=0.2 θ=0.7 α=7	0.1686	0.0757	0.1411	45.8365	433.6134
$\lambda = 10$ c=1.1 θ=1.1 α=1.1	0.1706	0.0257	0.0091	4.8254	12.1528
$\lambda = 4.9$ c=1 θ=1 α=1	0.3709	0.1731	0.2933	16.5825	62.3333

5.4. Asymptotic distributions of sample extremes

Consider the asymptotic distributions of first order statistic $X_{1:n}$ and n^{th} order statistic $X_{n:n}$. We using the asymptotic results for $X_{1:n}$ and $X_{n:n}$ by Arnold *et al.* (2008) and Theorem 1, we can find the limiting distribution of extreme order statistic.

For HLMOL distribution $R^{-1}(O) = 0$ which is finite and by using L'Hospital's

$$\lim_{\epsilon \to 0+} \frac{R[R^{-1}(0) + \epsilon x]}{R[R^{-1}(0) + \epsilon]} = \lim_{\epsilon \to 0+} x \frac{r[\epsilon x]}{r[x]} = x.$$

Therefore the asymptotic distribution $X_{1:n}$ is of Weibull type with $\alpha = 1$, that is $R \in D_{min}(H_2^{\star}(x;1)) = 1 - e^{-x}, x > 0$. Here the normalizing constants based on R are given by $a_n^{\star} = R^{-1}(O) = 0$, $b_n^{\star} = R^{-1}(1/n) - R^{-1}(O) = \theta\Big\{\Big[\frac{\left[\frac{n-1}{n+1}\right]^{1/\lambda}}{1-[1-c]\left[1-\left[\frac{n-1}{n+1}\right]^{1/\lambda}\right]}\Big]^{-1/\alpha} - 1\Big\}$.

For Lomax distribution $F^{-1}(1) = \infty$, by using L'Hospital's

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t \to \infty} x \frac{f(tx)}{f(t)} = x^{-\alpha}.$$

Therefore the asymptotic distribution of $X_{n:n}$ based on F is Fréchet type. From Theorem 1 the asymptotic distribution of $X_{n:n}$ based on R is Fréchet type, that is $R \in D_{max}(H_1(x;\alpha)) = e^{-x^{-\alpha}}, x > 0, \alpha > 0$. Here the normalizing constants based on R are given by $a_n = 0$, $b_n = R^{-1}(1 - 1/n) = \theta\Big\{\Big[\frac{\left[\frac{1}{2n-1}\right]^{1/\lambda}}{1-[1-c]\left[1-\left[\frac{1}{2n-1}\right]^{1/\lambda}\right]}\Big]^{-1/\alpha} - 1\Big\}$.

6. Estimation of parameters by maximum likelihood method

Here, we discuss maximum likelihood estimation of HLMO-X family of distribution along with a simulation study of HLMOL. Let $x_1, ..., x_n$ be a sample from $X \sim$ HLMO-X(λ, c, ξ). Let $\Theta = (\lambda, c, \xi)^T$ be the parameter vector and ξ corresponds to the parameter vector of the baseline distribution F, $F(x) = F(x_i;\xi)$, $f(x) = f(x_i;\xi)$. The total log-likelihood function for Θ is given by

$$\begin{aligned}\ell_n = \ell_n(\Theta|x_1, ..., x_n) = {} & n\log(2\lambda) + \lambda\sum_{i=1}^{n}\log\left\{\frac{c[1-F(x_i;\xi)]}{c+(1-c)F(x_i;\xi)}\right\} + \sum_{i=1}^{n}\log[f(x_i;\xi)] \\ & - \sum_{i=1}^{n}\log[1-F(x_i;\xi)] - 2\sum_{i=1}^{n}\log\left\{1 + \left[\frac{c[1-F(x_i;\xi)]}{c+(1-c)F(x_i;\xi)}\right]^{\lambda}\right\} \\ & - \sum_{i=1}^{n}\log[c+(1-c)F(x_i;\xi)].\end{aligned}$$

The score function $U_n(\Theta) = (\frac{\partial \ell_n}{\partial \lambda}, \frac{\partial \ell_n}{\partial c}, \frac{\partial \ell_n}{\partial \xi})^T$ has components given by

$$\begin{aligned}\frac{\partial \ell_n}{\partial \lambda} = {} & \frac{n}{\lambda} - 2\sum_{i=1}^{n}\log\left\{\frac{c[1-F(x_i;\xi)]}{c+(1-c)F(x_i;\xi)}\right\}\frac{[c[1-F(x_i;\xi)]]^{\lambda}}{[c+(1-c)F(x_i;\xi)]^{\lambda} + [c[1-F(x_i;\xi)]]^{\lambda}} \\ & + \sum_{i=1}^{n}\log\left\{\frac{c[1-F(x_i;\xi)]}{c+(1-c)F(x_i;\xi)}\right\},\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell_n}{\partial c} &= \lambda \sum_{i=1}^{n} \frac{F(x_i;\xi)}{c[c+(1-c)F(x_i;\xi)]} - \frac{[1-F(x_i;\xi)]}{[c+(1-c)F(x_i;\xi)]} \\ &- 2\lambda \sum_{i=1}^{n} \frac{c^{(\lambda-1)}[1-F(x_i;\xi)]^{\lambda}F(x_i;\xi)}{[c+(1-c)F(x_i;\xi)]\{[c+(1-c)F(x_i;\xi)]^{\lambda}+[c[1-F(x_i;\xi)]]^{\lambda}\}},\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell_n}{\partial \xi} &= -\lambda \sum_{i=1}^{n} \frac{F^{(\xi)}(x_i;\xi)}{[1-F(x_i;\xi)][c+(1-c)F(x_i;\xi)]} + \sum_{i=1}^{n} \frac{f^{(\xi)}(x_i;\xi)}{f(x_i;\xi)} \\ &- (1-c)\sum_{i=1}^{n} \frac{F^{(\xi)}(x_i;\xi)}{c+(1-c)F(x_i;\xi)} + \sum_{i=1}^{n} \frac{F^{(\xi)}(x_i;\xi)}{1-F(x_i;\xi)} \\ &- 2c \sum_{i=1}^{n} \frac{[1-F(x_i;\xi)]^{\lambda-1}F^{(\xi)}(x_i;\xi)}{[c+(1-c)F(x_i;\xi)]\{[c+(1-c)F(x_i;\xi)]^{\lambda}+[c[1-F(x_i;\xi)]]^{\lambda}\}},\end{aligned}$$

where $f^{(\xi)}(x_i;\xi) = \frac{\partial f(x_i;\xi)}{\partial \xi}$ and $F^{(\xi)}(x_i;\xi) = \frac{\partial F(x_i;\xi)}{\partial \xi}$. The maximum likelihood estimates (MLEs) of Θ, say $\hat{\Theta} = (\hat{\lambda}, \hat{c}, \hat{\xi})$, are the simultaneous solutions of the following equations: $\frac{\partial \ell_n}{\partial \lambda} = 0$, $\frac{\partial \ell_n}{\partial c} = 0$ and $\frac{\partial \ell_n}{\partial \xi} = 0$. These equations cannot be solved analytically and statistical software can be used to solve them numerically.

6.1. Simulation study

Here we perform a simulation study evaluating the performance of the MLEs presented above for the HLMOL distribution for selected values of the parameters θ, α, λ and c. The simulation experiment was repeated 1000 times each with sample sizes 50, 100, 150, 200 and parameter combinations are

1. λ=1.5, α=4 fixed c=1 and θ=1.
2. α=0.5, θ=0.6 fixed λ=1 and c=0.5.
3. θ= 0.2, c=0.5 fixed λ=1 and α=1.
4. λ=1, θ= 1 fixed c=1 and α=1.
5. λ=0.5, c= 0.2, θ=0.4 and fixed α=1.
6. λ= 0.75, α=0.15, c=0.1, θ= 0.05.

Table 3 presents the average estimates (*AEs*), average bias (*Bias*) and mean square error (*MSE*) values of parameters for different sample sizes. It can be noted that as sample size increases, the *Bias* decay towards zero and *MSE* decreases. That is, the estimators are asymptotically unbiased and consistent. Therefore the maximum likelihood estimation method works quite well to estimate the parameters of the HLMOL distribution.

Table 3: *AE*s, *Bias* and *MSE* of parameters based on 1000 simulations of the HLMOL distribution

	n	Parameter	*AE*s	*Bias*	*MSE*
I	50	λ	1.1136	-0.3864	5.6075
		α	3.8552	-0.1448	0.7885
	100	λ	1.3263	-0.1736	4.0699
		α	3.9348	-0.0652	0.5723
	150	λ	1.4484	-0.0516	1.8639
		α	3.9805	-0.0195	0.2621
	200	λ	1.5022	0.0022	0.0059
		α	4.0006	0.0006	0.0008
II	50	α	0.5389	0.0388	0.0182
		θ	0.7409	0.1409	0.0205
	100	α	0.5196	0.0196	0.0069
		θ	0.6635	0.0635	0.0131
	150	α	0.5113	0.0113	0.0043
		θ	0.6367	0.0367	0.0120
	200	α	0.5080	0.0079	0.0033
		θ	0.6256	0.0256	0.0117
III	50	c	0.5013	0.0013	0.0002
		θ	0.2051	0.0051	0.0019
	100	c	0.5002	0.0002	0.0001
		θ	0.2017	0.0017	0.0010
	150	c	0.5001	6.5753e-05	0.0001
		θ	0.2016	0.0016	0.0007
	200	c	0.4999	-1.2853e-05	1.2186e-05
		θ	0.2001	5.4980e-05	7.6654e-05
IV	50	λ	1.1002	0.1002	0.1468
		θ	1.2458	0.2458	0.8529
	100	λ	1.0464	0.0464	0.0350
		θ	1.1082	0.1082	0.1621
	150	λ	1.0325	0.0325	0.0224
		θ	1.0732	0.0732	0.1109
	200	λ	1.0233	0.0233	0.0149
		θ	1.0489	0.0489	0.0635

Continued on the next page

Table 3:(Continued)

	n	Parameter	AEs	$Bias$	MSE
V	50	λ	0.5263	0.0263	0.0105
		c	0.2272	0.0272	0.0096
		θ	0.4128	0.0128	0.0023
	100	λ	0.5132	0.0132	0.0043
		c	0.2136	0.0136	0.0040
		θ	0.4067	0.0067	0.0010
	150	λ	0.5085	0.0085	0.0028
		c	0.2076	0.0076	0.0024
		θ	0.4035	0.0035	0.0006
	200	λ	0.5032	0.0032	0.0020
		c	0.2049	0.0049	0.0017
		θ	0.4022	0.0022	0.0004
VI	50	λ	0.4659	-0.2840	0.8821
		α	0.1225	-0.0274	0.0375
		c	0.0669	-0.0331	0.0938
		θ	0.2106	0.1606	1.5323
	100	λ	0.8875	0.1375	0.5642
		α	0.1976	0.04764	0.0187
		c	0.0882	-0.0118	0.0074
		θ	0.1965	0.1465	0.2604
	150	λ	0.8732	0.1232	0.4111
		α	0.1873	0.0373	0.0147
		c	0.0885	-0.0114	0.0061
		θ	0.1566	0.1066	0.0987
	200	λ	0.8688	0.1188	0.2939
		α	0.1763	0.0263	0.0119
		c	0.0901	-0.0099	0.0048
		θ	0.1357	0.0857	0.0739

7. Test to compare HLMOL with Lomax and Half-logistic-Lomax distributions

Since Lomax (L), half-logistic-Lomax (HLL) by Anwar and Zahoor (2018) and HLMOL distributions are nested models, To distinguish between them, the likelihood ratio (LR) test is employed. For the nested models, the LR statistic is

$$LR = -2\left\{\frac{likelihood\ under\ the\ null\ hypothesis}{likelihood\ under\ the\ alternative\ hypothesis}\right\}.$$

This statistic is asymptotically (as $n \to \infty$) distributed as chi-square distribution with m degrees of freedom (df), where m is the number of additional parameters .

When c is equal to 1, the HLMOL distribution becomes the HLL distribution. So, in order to compare the HLMOL with the HLL distribution, we test the null hypothesis that $H_0 : c = 1$ *against* $H_1 : c \neq 1$, and the corresponding LR statistic asymptotically (as $n \to$

∞) distributed as chi-square distribution with 1 df. To investigate how well the test statistic performed for the above hypothesis , we conducted a simulation study.The simulation experiment was performed 1000 times, with sample sizes of 100, 250, and 500 with different parameter combinations. From the HLMOL distribution, a random sample is created, and the test is then run with a 5% level of significance . Calculating the proportion of times the null hypothesis H_0 is rejected requires running the simulation 1000 times for each set of parameter combinations. In order to estimate the test's power, we look at the proportion of times that H_0 is rejected. Table 4 provides the proportions for the 5% level of significance.

The findings in Table 4 show that, for fixed c, θ and α the power of the tests increases as a function of λ. Additionally, given a fixed value of θ, α and λ the tests' power is a diminishing function of c. In general, as sample sizes grow, power grows as well

Table 4: The proportion of times (out of 1000) that the H_0 is rejected at 5% level of significance.

Parameter value				n=100	n=250	n=500
c	θ	α	λ			
			0.15	0.979	0.989	0.993
		0.9	1.25	0.984	0.991	0.999
	0.05		2	0.987	0.993	0.999
			0.15	0.969	0.971	0.999
		1.25	1.25	0.972	0.993	1
			2	0.986	0.988	1
0.1			0.15	0.96	0.976	0.998
		0.9	1.25	0.971	0.991	1
	0.5		2	0.987	0.993	1
			0.15	0.973	0.981	0.986
		1.25	1.25	0.984	0.985	1
			2	0.989	0.994	1
			0.15	0.848	0.886	0.904
		0.9	1.25	0.924	0.935	0.946
	0.05		2	0.902	0.945	0.985
			0.15	0.907	0.952	0.954
		1.25	1.25	0.924	0.956	0.983
			2	0.972	0.980	0.983
0.25			0.15	0.893	0.899	0.95
		0.9	1.25	0.9	0.912	0.954
	0.5		2	0.911	0.921	0.962
			0.15	0.87	0.901	0.915
		1.25	1.25	0.907	0.927	0.939
			2	0.916	0.949	0.966

Similarly, When λ is equal to 1 and c=0.5, the HLMOL distribution becomes the L distribution. So, in order to compare the HLMOL with the L distribution, we test the null hypothesis that

$H_0 : \lambda = 1,\ c = 0.5$ against $H_1 : \lambda \neq 1,\ c \neq 0.5$, and the corresponding LR statistic asymptotically (as n $\rightarrow \infty$) distributed as chi-square distribution with 2 DF. To investigate how well the test statistic performed for the above hypothesis, we conducted a simulation study. The simulation experiment was performed 1000 times, with sample sizes of 100, 250, and 500 with different combination of parameters. From the HLMOL distribution, a random sample is created, and the test is then run with a 5% level of significance . Calculating the proportion of times the null hypothesis H_0 is rejected requires running the simulation 1000 times for each set of combination of parameters. In order to estimate the test's power, we look at the proportion of times that H_0 is rejected. Table 5 provides the proportions for the 5% level of significance.

Table 5: The proportion of times (out of 1000) that the H_0 is rejected at 5% level of significance.

Parameter value				n=100	n=250	n=500
c	θ	α	λ			
0.1			0.2	0.986	0.998	1
		0.15	1	0.987	0.999	1
	0.05		2	0.991	1	1
			0.2	0.998	0.999	1
		1.5	1	0.972	0.973	0.986
			2	0.975	0.982	0.994
0.1			0.2	0.989	0.998	1
		0.15	1	0.993	0.999	1
	0.5		2	0.998	1	1
			0.2	0.996	0.997	1
		1.5	1	0.997	0.997	1
			2	0.997	1	1
			0.2	0.961	0.986	0.993
		0.15	1	0.982	0.988	0.994
	0.05		2	0.982	0.985	0.997
			0.2	0.997	0.999	1
		1.5	1	0.914	0.95	0.998
			2	0.95	0.981	1
0.25			0.2	0.97	0.984	0.99
		0.15	1	0.979	0.985	0.991
	0.5		2	0.98	0.985	0.991
			0.2	0.987	0.992	0.999
		1.5	1	0.988	0.994	1
			2	0.993	0.994	1

The results in Table 5 demonstrate that, for fixed c, θ and α, the power of the tests generally increases as a function of λ. Additionally, the power of the tests is a decreasing function of c for a certain value of λ, θ and α. In general, power increases as sample size increase.

8. Applications

Under this head, we exhibit the importance of the proposed family. We fit the HLMOL distribution to two data sets and compare this distribution with four other models, namely: Kumaraswamy-generalized Lomax (Kw-GL) distribution by Shams (2013), Weibull Lomax (WL) distribution by Tahir *et al.* (2015), HLL and L distribution. The MLEs of the parameters of the models are calculated and goodness-of-fit statistics for the models are compared. The measures including the Akaike information criterion (AIC), Bayesian information criterion (BIC) and Kolmogorov-Smirnov (K-S) statistic with p-value (p-V). Additionally, we employ the LR test to compare the HLMOL distribution with the L and HLL distributions.

8.1. The secondary reactor pumps data set

This data represents the time period between secondary reactor pump failures. The data was originally discussed in Suprawhardana and Prayoto (1999). and was previously used by Bebbington *et al.* (2007). Following are the time between failures for 23 secondary reactor pumps.
{2.160, 0.150, 4.082, 0.746, 0.358, 0.199, 0.402, 0.101, 0.605, 0.954, 1.359, 0.273, 0.491, 3.465, 0.070, 6.560, 1.060, 0.062, 4.992, 0.614, 5.320, 0.347, 1.921}

The necessary numerical summaries for the five fits using the secondary reactor pumps data set includes the estimated log-likelihood function ($\hat{\ell}$), AIC, BIC and K-S with p-V are provided in Tables 6 and 7. Additionally, Table 8 provides two LR statistics based on data set from secondary reactor pumps along with (p-V).

Table 6: Estimated values, log-likelihood, AIC and BIC for the secondary reactor pumps data set

Distribution	Estimates	$-ln(L)$	AIC	BIC
HLMOL	$\boldsymbol{\hat{\lambda} = 0.5250}$ $\boldsymbol{\hat{\alpha} = 8442.2096}$ $\boldsymbol{\hat{c} = 0.1662}$ $\boldsymbol{\hat{\theta} = 9025.7431}$	**31.862**	**67.7242**	**69.9952**
Kw-GL	$\hat{a} = 0.8085$ $\hat{b} = 185.7834$ $\hat{\lambda} = 297.5083$ $\hat{\alpha} = 0.3337$	32.51709	73.03418	77.57616
WL	$\hat{a} = 7.2122$ $\hat{b} = 0.8163$ $\hat{\beta} = 12.6936$ $\hat{\alpha} = 0.8239$	32.51238	73.02476	77.56674
HLL	$\hat{\lambda}= 0.6797$ $\hat{\alpha} = 2.3802$ $\hat{\theta} = 0.7796$	32.64682	71.29364	74.70013
L	$\hat{\alpha} = 2.2425$ $\hat{\theta}= 2.1699$	32.4952	68.9903	71.2613

Table 7: K-S with p-V for the secondary reactor pumps data set

Distributions	K-S	p-V
HLMOL	**0.0954**	**0.9718**
Kw-GL	0.1186	0.8654
WL	0.1176	0.8717
HLL	0.096283	0.9695
L	0.099734	0.9589

Table 8: The values of LR statistic for different hypothesis and data sets

Models	Hypothesis	Secondary reactor pumps data set		
		LR	df	p-V
HLMOL vs. L	$H_0: \lambda = 1, c=0.5$ vs. $H_1: H_0$ is false	7.2334	2	0.0269
HLMOL vs. HLL	$H_0: c=1$ vs. $H_1: H_0$ is false	14.6222	1	< 0.001

Figure 3 display the total time test (TTT) plot for the secondary reactor pumps data set, and Figure 4 display the graphs of estimated PDF and CDF of the considered distributions for secondary reactor pumps data set.

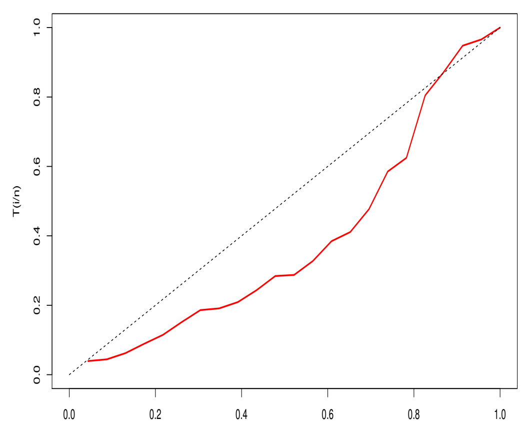

Figure 3: TTT-plot for the secondary reactor pumps data set

8.2. Bladder cancer patients data set

The data set was given by Almheidat *et al.* (2015). It is corresponding to remission times (months) of a random sample of 128 bladder cancer patients. The data are as given below
{0.080, 0.200, 0.400, 0.500, 0.510, 0.810, 0.900, 1.050, 1.190, 1.260, 1.350, 1.400,1.460, 1.760, 2.020, 2.020, 2.070, 2.090, 2.230, 2.260, 2.460, 2.540, 2.620, 2.640, 2.690, 2.690, 2.750,

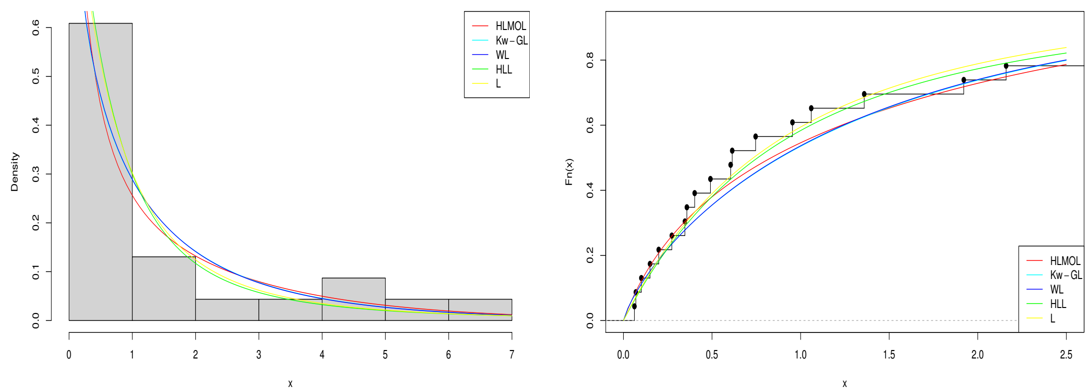

Figure 4: Estimated PDF and CDF for the HLMOL, Kw-GL, WL, HLL and L distributions for secondary reactor pumps data set

2.830, 2.870, 3.020, 3.250, 3.310, 3.360, 3.360, 3.480, 3.520, 3.570, 3.640, 3.700, 3.820, 3.880, 4.180, 4.230, 4.260, 4.330, 4.340, 4.400, 4.500, 4.510, 4.870, 4.980, 5.060, 5.090, 5.170, 5.320, 5.320, 5.340, 5.410, 5.410, 5.490,5.620, 5.710, 5.850, 6.250, 6.540, 6.760, 6.930, 6.940, 6.970, 7.090, 7.260, 7.280, 7.320, 7.390, 7.590, 7.620, 7.630, 7.660, 7.870, 7.930, 8.260, 8.370, 8.530, 8.650,8.660, 9.020, 9.220, 9.470, 9.740, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 3.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05}

The necessary numerical summaries for the five fits using the bladder cancer patients data set includes $\hat{\ell}$, AIC, BIC and K-S with p-V are provided in Tables 9 and 10. Additionally, Table 11 provides two LR statistics based on data set bladder cancer patients along with p-V.

Figure 5 display the TTT-plot for the bladder cancer patients data set, and Figure 6 displays the graphs of estimated PDF and CDF of the considered distributions for bladder cancer patients data sets.

In Tables 6, 7, 9 and 10, the MLEs of the parameters for the fitted distributions along with -log-likelihood, AIC, BIC, K-S with p-V values are given for two distinct data sets. The HLMOL distribution proves to be a superior model than the Kw-GL, WLo, HLL, and L models because it has the lowest values of AIC, BIC, K-S, and the highest p-V of the K-S statistic. Tables 8 and 11 also show the LR statistic values and p-V. In light of these results, we reject the null hypothesis for the aforementioned data sets and come to the conclusion that the HLMOL distribution offers a much more accurate depiction than the L and HLL distributions.

Figures 3 and 5 indicates decreasing HRF for he secondary reactor pumps data set and upside-down bathtub shaped HRF for the bladder cancer patients data set. Therefore, the HLMOL distribution can fit these data sets.

Figures 4 and 6 present a diagrammatic comparison of the closeness of the fitted

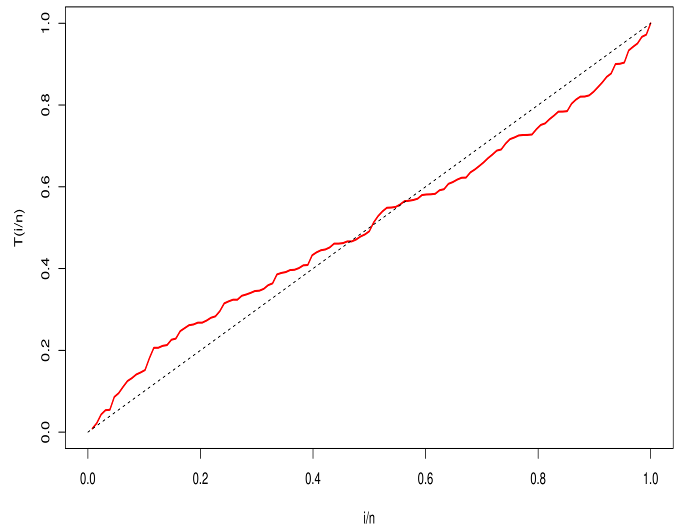

Figure 5: TTT-plot for the bladder cancer patients data set

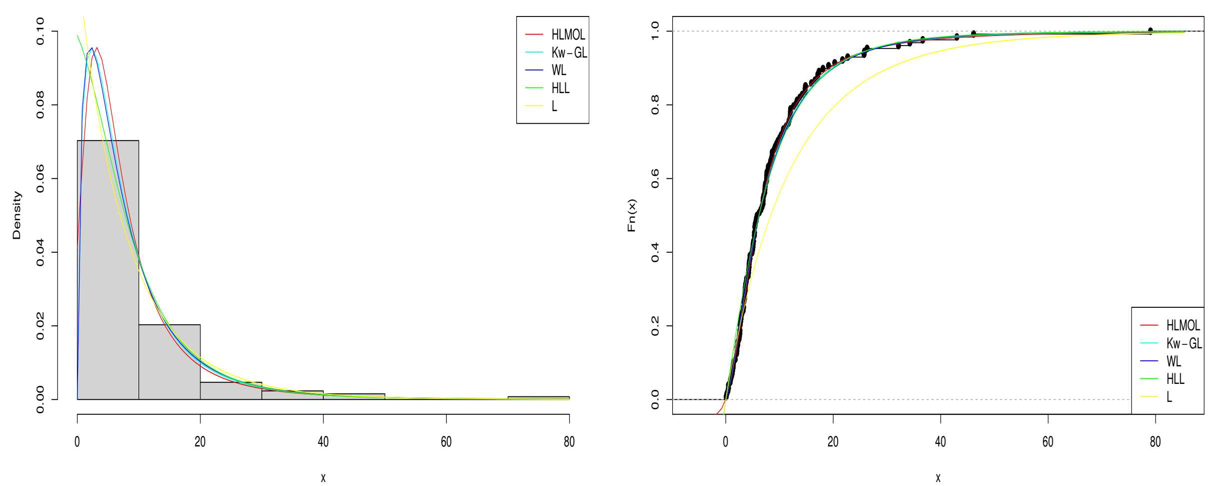

Figure 6: Estimated PDF and CDF for the HLMOL, Kw-GL, WL, HLL and L distributions for bladder cancer patients data sets

Table 9: Estimated values, log-likelihood, AIC and BIC bladder cancer patients data set

Distribution	Estimates	$-ln(L)$	AIC	BIC
HLMOL	$\boldsymbol{\hat{\lambda} = 0.3401}$ $\boldsymbol{\hat{\alpha} = 8.6402}$ $\boldsymbol{\hat{c} = 4.0693}$ $\boldsymbol{\hat{\theta} = 8.7546}$	**406.579**	**821.158**	**832.5661**
Kw-GL	$\hat{a} = 1.5493$ $\hat{b} = 10.3464$ $\hat{\lambda} = 11.5419$ $\hat{\alpha} = 0.4372$	407.3357	822.6713	834.0794
WL	$\hat{a} = 16.3314$ $\hat{b} = 1.5541$ $\hat{\beta} = 5.3873$ $\hat{\alpha} = 0.1607$	407.611	823.222	834.6301
HLL	$\hat{\lambda}= 0.5540$ $\hat{\alpha} = 0.4941$ $\hat{\theta} = 26.6014$	409.4457	824.8915	833.4476
L	$\hat{\alpha} = 13.0380$ $\hat{\theta}=110.7043$	411.5897	827.1794	832.8835

Table 10: K-S with p-V for the secondary reactor pumps data set

Distributions	K-S	p-V
HLMOL	**0.0286**	**0.9999**
Kw-GL	0.0404,	0.985
WL	0.0449	0.9587
HLL	0.0808	0.3738
L	0.1006	0.1498

densities with the observed histogram and CDFs with the empirical CDFs of the data sets. These diagrams demonstrate that the proposed distribution renders a closer fit the above two data sets.

9. Conclusion

In this article, the T-X method was utilized to introduce the T-Marshall Olkin X family of distribution, a novel family of distributions. HLMO-X and one of its members, HLMOL, are investigated in depth as a particular case. The quantile function, moments, incomplete moments, moment generating function, Lorenz curve, Bonferroni curve, skewness, kurtosis, order statistics, and asymptotic distributions of order statistics are some of the structural characteristics are investigated. The maximum likelihood approach, together with simulation analysis, is the technique utilized to estimate the model parameters. The distribution fit between HLL and HLMOL and also between L and HLMOL is tested using the LR test with simulation research. The outcome demonstrates that the HLMOL dis-

Table 11: The values of LR statistic for different hypothesis and data sets

Models	Hypothesis	Bladder cancer patients data set		
		LR	df	p-V
HLMOL vs. L	$H_0 : \lambda = 1, c{=}0.5$ vs. $H_1 : H_0$ is false	709.4762	2	< 0.001
HLMOL vs. HLL	$H_0 : c{=}1$ vs. $H_1 : H_0$ is false	24.6404	1	< 0.001

tribution is superior to the other two. When compared to the Kw-GL, WL, GL, and EL distributions, fitting to two real-world data produce good results in favour of the suggested distribution. As a result, the proposed distribution can be viewed as making a worthwhile contribution to the existing knowledge. Future research will include more generalizations that can be made for both continuous and discrete cases. One such generalization is the exponential-Marshall-Olkin X family of distributions. For evaluating the accuracy of the new models, different inferential investigations will be taken into consideration.

Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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