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Improvement Over the Bar-Lev, Bobovitch and Boukai and Tarray and Singh Randomized Response Models Through the Use of Two Variables Having Common Mean

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Abstract

Taking the clue from Odumade and Singh (2010), we have suggested a procedure to improve the randomized response model envisaged by Tarray and Singh (2014). If there exist two sensitive variables associated to the principal study sensitive variable then those variables could be used to develop ratio type adjustments to the conventional estimators of the population mean of a sensitive variable due to Tarray and Singh (2014). Conditions are obtained under which the suggested ratio-type estimators are better than estimators of Bar-Lev et al (2004) and Tarray and Singh (2014). Numerical illustrations are given in support of the present study.

Key words: Randomized response model; Study variable; Auxiliary variable; Bias; Mean Squared Error.

AMS Subject Classification: 62D05.

1. Introduction

Obtaining information pertaining to sensitive or stigmatizing characteristics has been a vexing problem that is encountered in sample surveys. The questions that make the respondent suffer embarrassment if he (or she) answers the question affirmatively prompt him (or her) to select the path that is least likely to jeopardize his (or her) reputation. This would then entail data that are mostly unreliable. Research in statistical methodology to devise schemes to elicit answers in the above context has been in the direction of finding methods that ensure anonymity to the respondent in as far as his answer is concerned. It is believed that if the interviewer does not know what the answer from the respondent to the sensitive question is, then the respondent feels safe in responding truthfully to the sensitive question. In this direction, an attempt has been made by Warner (1965) by introducing an innovative technique commonly referred to as randomized response (RR) technique for estimating the proportion of population possessing certain stigmatized character (say) by protecting the privacy of respondents and preventing the unacceptable rate of non-response.

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Since Warner's (1965) model, a rich growth of literature can be found by the researchers for collecting data on both the qualitative and the quantitative variables. For details, one can refer to Horvitz et al. (1967), Greenberg et al. (1969), Franklin (1989), Fox and Tracy (1986), Grewal et al. (2005-2006), Hong (2005-2006), Ryu et al. (2005-2006), Mahajan et al. (2007), Perri (2008), Singh and Chen (2009), Odumade and Singh (2009, 2010), Singh and Tarray (2012, 2013, 2014), Barabesi et al. (2014) and Singh and Gorey (2016), etc.

1.1 Eichhorn and Hayre’s (1983) model

Eichhorn and Hayre (1983) introduced the following RRT model that is based on multiplicative scrambling to collect information on sensitive quantitative variables like income, tax evasion, amount of drug used *etc*. If *Y* is the true response and *S* is a scrambling variable (independent of *Y*) with mean θ and standard deviation γ, then the reported response is given by

$$Z = \frac{YS}{\theta}. \tag{1.1}$$

It is assumed that the distribution of the scrambling variable *S* is known. In other words, mean (θ) and variance (γ^2) are assumed to be known and positive. Obviously $E(Z) = E(Y)$, which leads to an estimator of the population mean $\overline{Y}$ under simple random sampling with replacement (SRSWR) scheme given by

$$\bar{y}_{(EH)} = \overline{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i , \tag{1.2}$$

where $\overline{Z}$ is the sample mean of the reported responses. The variance of $\bar{y}_{(EH)}$ is given by

$$V\left(\bar{y}_{(EH)}\right) = \frac{\overline{Y}^2}{n}\left[C_y^2 + C_\gamma^2\left(1 + C_y^2\right)\right], \tag{1.3}$$

where $C_\gamma = \gamma/\theta$ and $C_y = \sigma_y/\overline{Y}$ are the coefficients of variation of scrambling variable *S* and the study variable *Y*, and σ_y is the standard deviation of the study variable *y*. We shall now discuss a randomized response model studied by Bar-Lev et al (2004), which we call BBB model hereafter.

1.2 Bar-Lev, Bobovitch and Boukai’s (2004) RR model

In the BBB model, each respondent is requested to rotate a spinner unobserved by the interviewer. If the spinner stops in the shaded area then the respondent is requested to report the real response on the sensitive variable, say Y_i. If the spinner stops in the non-shaded area then the respondent is requested to report the scrambled response, say $Y_i S$, where *S* is any scrambling variable and its distribution is assumed to be known. Let *p* be the proportion of the shaded area of the spinner and $(1-p)$ be the non-shaded area of the spinner as demonstrated in the Figure 1.1.

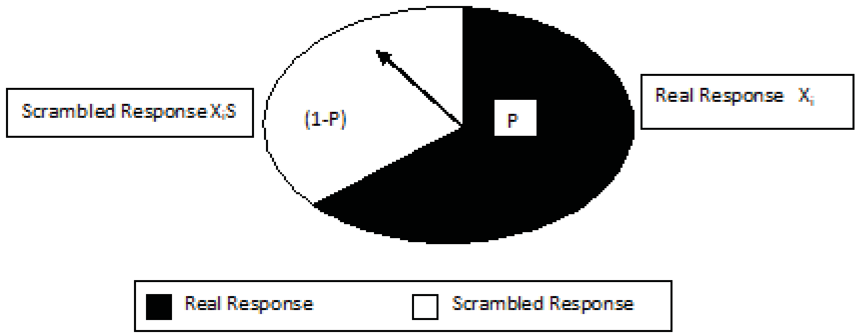

Fig. 1.1: BBB randomized response device

Let Z_i be the response from the ith respondent using BBB randomized response procedure. Then Z_i has the distribution:

$$Z_i = \begin{cases} Y_i S & with\ probability\ (1-p) \\ Y_i & with\ probability\ p \end{cases} \tag{1.4}$$

An unbiased estimator of the population mean $\bar{Y}$ is given by

$$\bar{y}_{(BBB)} = \frac{\sum_{i=1}^{n} Z_i}{n\{(1-p)\theta + p\}} . \tag{1.5}$$

The variance under SRSWR sampling is given by

$$V(\bar{y}_{(BBB)}) = \frac{\bar{Y}^2}{n}\left[C_y^2 + (1+C_y^2)C_p^2\right], \tag{1.6}$$

where

$$C_p^2 = \left\{ \frac{(1-p)\theta^2(1+C_\gamma^2) + p}{\{(1-p)\theta + p\}^2} - 1 \right\} . \tag{1.7}$$

In Section 1.3, we have revisited the Tarray and Singh (2014) RR models and in section 1.4 description of optimal model of Tarray and Singh (2014) is given.

1.3 A revisit to Tarray and Singh (2014) RR model-I

Using the knowledge of mean θ of scrambling variable S and the design parameter p, Tarray and Singh's (2014) have suggested a randomized response procedure. In the Tarray and Singh's (2014) procedure the distribution of interviewee's response to the sensitive question is:

$$Z_i = \begin{cases} \dfrac{Y_i S}{(1-p)\theta} & with\ probaility\ (1-p) \\ \dfrac{Y_i}{p} & with\ probability\ p \end{cases} \tag{1.8}$$

The expected value of Z_i is given by

$$E(Z_i) = (1-p)\frac{E(Y_i)E(S)}{(1-p)\theta} + p\frac{E(Y_i)}{p}$$
$$= \frac{E(Y_i)(1-p)\theta}{(1-p)\theta} + p\frac{E(Y_i)}{p}$$
$$= 2E(Y_i) = 2\bar{Y}.$$

Thus an unbiased estimator of the population mean μ_y is given by

$$\hat{\mu}_{y(ST1)} = \frac{\bar{Z}}{2} = \frac{1}{n}\sum_{i=1}^{n}\frac{Z_i}{2}. \tag{1.9}$$

The variance of $\bar{y}_{(ST1)}$ is given by

$$V(\bar{y}_{(ST1)}) = \frac{V(Z_i)}{4n}. \tag{1.10}$$

The variance of Z_i is obtained as follow

$$V(Z_i) = \left[E(Z_i^2) - (E(Z_i))^2\right]$$
$$= E(Z_i^2) - 4\bar{Y}^2$$
$$= (1-p)\frac{E(Y_i^2)E(S^2)}{\theta^2(1-p)^2} + \frac{pE(Y_i^2)}{p^2} - 4\bar{Y}^2$$
$$= E(Y_i^2)\left[\frac{\theta^2(1+C_\gamma^2)}{\theta^2(1-p)} + \frac{1}{p}\right] - 4\bar{Y}^2$$
$$= \frac{\bar{Y}^2(1+C_y^2)(1+pC_\gamma^2)}{p(1-p)} - 4\bar{Y}^2$$
$$= 4\bar{Y}^2\left[\frac{(1+C_y^2)(1+pC_\gamma^2)}{4p(1-p)} - 1\right]. \tag{1.11}$$

Thus, the variance of $\bar{y}_{(ST1)}$ is given by

$$V(\bar{y}_{(ST1)}) = \frac{V(Z_i)}{4n}$$
$$= \frac{\bar{Y}^2}{n}\left[\frac{(1+C_y^2)(1+pC_\gamma^2)}{4p(1-p)} - 1\right]$$
$$= \frac{\bar{Y}^2}{n}\left[C_y^2 + \frac{(1+C_y^2)(1+pC_\gamma^2)}{4p(1-p)} - 1 - C_y^2\right]$$
$$= \frac{\bar{Y}^2}{n}\left[C_y^2 + (1+C_y^2)\left\{\frac{(1+pC_\gamma^2)}{4p(1-p)} - 1\right\}\right]$$

$$= \frac{\bar{Y}^2}{n}\left[C_y^2 + \left(1 + C_y^2\right)C_{p0}^2\right], \tag{1.12}$$

where

$$C_{p0}^2 = \left[\frac{\left(1 + pC_\gamma^2\right)}{4p(1-p)} - 1\right].$$

We note that the variance of $\bar{y}_{(ST1)}$ obtained in (1.12) is correct while the variance of expression obtained by Tarray and Singh (2014, p.89, equation (2.5)) is incorrect therefore we have revisited the RR model (1.8) due to Tarray and Singh (2014).
From (1.3) and (1.12) we have

$$V\left(\bar{y}_{(EH)}\right) - V\left(\bar{y}_{(ST1)}\right) = \frac{\bar{Y}^2}{n}\left(1 + C_y^2\right)\left[1 + C_\gamma^2 - \frac{\left(1 + pC_\gamma^2\right)}{4p(1-p)}\right]$$

which is always positive if

$$\left\{1 + C_\gamma^2 - \frac{\left(1 + pC_\gamma^2\right)}{4p(1-p)}\right\} > 0. \tag{1.13}$$

Thus, the Tarray and Singh's (2014) estimator $\bar{y}_{(ST1)}$ is more efficient than Eichhorn and Hayre's (1983) estimator $\bar{y}_{(EH)}$ as long as the condition (1.13) is satisfied. Further from (1.6) and (1.12) we have

$$V\left(\bar{y}_{(BBB)}\right) - V\left(\bar{y}_{(ST1)}\right) = \frac{\bar{Y}^2\left(1 + C_y^2\right)}{n}\left(C_p^2 - C_{p0}^2\right)$$

which is positive if

$$\left(C_p^2 - C_{p0}^2\right) > 0$$

i.e. if

$$\frac{\left\{(1-p)\theta^2\left(1 + C_\gamma^2\right) + p\right\}}{\left\{(1-p)\theta + p\right\}^2} > \frac{\left(1 + pC_\gamma^2\right)}{4p(1-p)}. \tag{1.14}$$

Thus the estimator $\bar{y}_{(ST1)}$ due to Tarray and Singh (2014) is more efficient than the Bar-Lev et al (2004) estimator $\bar{y}_{(BBB)}$ if the condition (1.14) is satisfied.

To see the merits of the Tarray and Singh's (2014) unbiased estimator $\bar{y}_{(ST1)}$ we have computed the percent relative efficiency (PRE) of $\bar{y}_{(ST1)}$ with respect to $\bar{y}_{(EH)}$ and $\bar{y}_{(BBB)}$ by using the formulae:

$$PRE\left(\bar{y}_{(ST1)}, \bar{y}_{(EH)}\right) = \frac{\left[C_y^2 + \left(1 + C_y^2\right)C_{p0}^2\right]}{\left[C_y^2 + C_\gamma^2\left(1 + C_y^2\right)\right]} \times 100 \tag{1.15}$$

and

$$PRE\left(\bar{y}_{(ST1)},\bar{y}_{(BBB)}\right)=\frac{\left[C_y^2+\left(1+C_y^2\right)C_{p0}^2\right]}{\left[C_y^2+\left(1+C_y^2\right)C_p^2\right]}\times 100 \tag{1.16}$$

for different values of C_γ, C_x, θ *and* p. Findings are given in Tables 1.1 and 1.2. Tables1.1 and 1.2 show that the values of $PRE\left(\bar{y}_{(ST1)},\bar{y}_{(EH)}\right)$ and $PRE\left(\bar{y}_{(ST1)},\bar{y}_{(BBB)}\right)$ are greater than 100%. Thereby meaning is that the Tarray and Singh's (2014) estimator $\bar{y}_{(ST1)}$ is better than Eichhorn and Hayre's (1983) estimator $\bar{y}_{(EH)}$ and Bar-Lev et al's (2004) estimator $\bar{y}_{(BBB)}$ for the parametric values closed in Tables 1.1 and 1.2.

1.4 Tarray and Singh (2014) RR model- II

Tarray and Singh (2014) have suggested another RR model based on the knowledge of mean θ and square of the coefficient of variation $\left(i.e.\ C_\gamma^2\right)$ of the scrambling variable S and design parameter p. In this model, the distribution of the responses is given by

$$Z_{0i}=\begin{cases}\dfrac{Y_i S}{\theta\left(1+pC_\gamma^2\right)} & \text{with probability } (1-p)\\[2ex] \dfrac{Y_i\left(1+C_\gamma^2\right)}{\left(1+pC_\gamma^2\right)} & \text{with probability } p\end{cases} \tag{1.17}$$

An unbiased estimator of the population mean $\bar{Y}$ based on RR model (1.17) is given by

$$\bar{y}_{(ST2)}=\frac{1}{n}\sum_{i=1}^{n} Z_{0i} \tag{1.18}$$

and the variance is given by

$$V\left(\bar{y}_{(ST2)}\right)=\frac{\bar{Y}^2}{n}\left[C_y^2+\left(1+C_y^2\right)C_{p0}^{*2}\right] \tag{1.19}$$

where

$$C_{p0}^{*2}=\left[\frac{\left(1+C_\gamma^2\right)}{\left(1+pC_\gamma^2\right)}-1\right].$$

Tarray and Singh (2014) have shown that the estimator $\bar{y}_{(ST2)}$ is always better than Eichhorn and Hayre (1983) estimator $\bar{y}_{(EH)}$. They have further shown that the $\bar{y}_{(ST2)}$ is more efficient than the $\bar{y}_{(BBB)}$ due to Bar-Lev et al (2004) if the condition $C_p^2 > C_{p0}^{*2}$.

2. Proposed Ratio-Type Estimator Based on Tarray and Singh (2014) Model-I

2.1. Notations

Following Tripathi and Chaubey (1992) let $\bar{X}_{1i}=\bar{X}_{2i}=\bar{X}$ that is these two auxiliary sensitive variables have common mean. Let Y_i be the sensitive variable under study whose mean is to be estimated. A simple random sample with replacement (SRSWR) of n respondents is selected. Then each one of the respondents selected in the sample is requested to rotate three spinners.

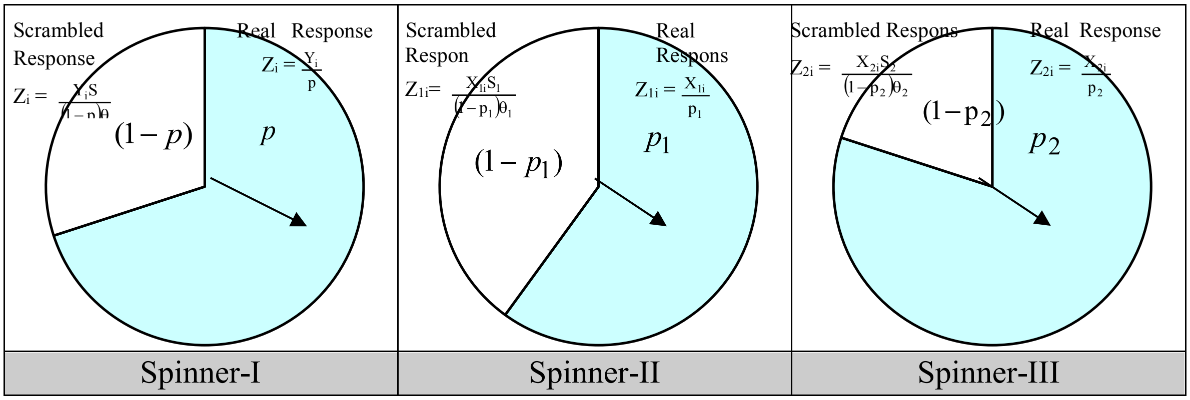

Fig. 2.1: Three spinners

The first spinner is used to gather scrambled response Z_i on the real study variable Y_i with the distribution of responses as:

$$Z_i = \begin{cases} \dfrac{Y_i}{p} & \text{with probability } p \\ \dfrac{Y_i S}{(1-p)\theta} & \text{with probaility } (1-p), \end{cases} \tag{2.1}$$

where the values of (p,θ) are known.

The second spinner is used to gather scrambled response Z_{1i} on the first auxiliary sensitive variable X_{1i} with the distribution of responses as:

$$Z_{1i} = \begin{cases} \dfrac{X_{1i}}{p_1} & \text{with probability } p_1 \\ \dfrac{X_{1i} S_1}{(1-p_1)\theta_1} & \text{with probaility } (1-p_1), \end{cases} \tag{2.2}$$

where the values of (p_1,θ) are known.

The third spinner is used to gather scrambled response Z_{2i} on the second auxiliary sensitive variable X_{2i} with the distribution of responses as:

$$Z_{2i} = \begin{cases} \dfrac{X_{2i}}{p_2} & \text{with probability } p_2 \\ \dfrac{X_{2i} S_2}{(1-p_2)\theta_2} & \text{with probaility } (1-p_2), \end{cases} \tag{2.3}$$

where the value of (p_2,θ) are known.

Assume that the sample mean of the scrambled responses obtained from the respondents in the sample as Z_i, Z_{1i} and Z_{2i} are given by:

$$\bar{y}^* = \frac{1}{n}\sum_{i=1}^{n}\frac{Z_i}{2},\ \bar{x}_1^* = \frac{1}{n}\sum_{i=1}^{n}\frac{Z_{1i}}{2} \text{ and } \bar{x}_2^* = \frac{1}{n}\sum_{i=1}^{n}\frac{Z_{2i}}{2}.$$

Let us define:

$$\in = \frac{\bar{y}^*}{\bar{Y}} - 1,\ \delta = \frac{\bar{x}_1^*}{\bar{X}} - 1,\ \eta = \frac{\bar{x}_2^*}{\bar{X}} - 1$$

such that

$$E(\in) = E(\delta) = E(\eta) = 0$$

and it can be shown that

$$E(\in^2) = \frac{1}{\bar{Y}^2}V(\bar{y}^*) = \frac{1}{n}\left[C_y^2 + (1+C_y^2)C_{p0}^2\right],\ E(\delta^2) = \frac{1}{\bar{Y}^2}V(\bar{x}_1^*) = \frac{1}{n}\left[C_{x1}^2 + (1+C_{x1}^2)C_{p1}^2\right],$$

$$E(\eta^2) = \frac{1}{\bar{Y}^2}V(\bar{x}_2^*) = \frac{1}{n}\left[C_{x2}^2 + (1+C_{x2}^2)C_{p2}^2\right],\ E(\in\delta) = \frac{1}{n}\rho_{yx1}C_yC_{x1},$$

$$E(\in\eta) = \frac{1}{n}\rho_{yx2}C_yC_{x2}, \text{ and } E(\delta\eta) = \frac{1}{n}\rho_{x1x2}C_{.x1}C_{x2},$$

where $C_{x1} = \sigma_{x1}/\bar{X}$ and $C_{x2} = \sigma_{x2}/\bar{X}$ are the coefficients of variation of the auxiliary sensitive variables x_1 and x_2, $(\sigma_{x1}, \sigma_{x2})$ are the standard deviations of (X_1, X_2), ρ_{yx1} is the correlation coefficient between y and x_1, ρ_{yx2} is the correlation coefficient between y and x_2, ρ_{x1x2} is the correlation coefficient between x_1 and x_2,

$$C_{p0}^2 = \left[\frac{(1+pC_\gamma^2)}{4p(1-p)} - 1\right],\ C_{p1}^2 = \left[\frac{(1+p_1C_{\gamma1}^2)}{4p_1(1-p_1)} - 1\right],\ C_{p2}^2 = \left[\frac{(1+p_2C_{\gamma2}^2)}{4p_2(1-p_2)} - 1\right].$$

We note that:

$$\left.\begin{array}{l} Cov(\bar{y}^*, \bar{x}_1^*) = \dfrac{\overline{YX}_1}{n}\rho_{yx1}C_yC_{x1}, \\ Cov(\bar{y}^*, \bar{x}_2^*) = \dfrac{\overline{YX}_2}{n}\rho_{yx2}C_yC_{x2} \\ and \\ Cov(\bar{x}_1^*, \bar{x}_2^*) = \dfrac{\bar{X}_1\bar{X}_2}{n}\rho_{x1x2}C_{x1}C_{x2} \end{array}\right\} \quad (2.4)$$

Proof of the results in (2.4) is simple, so omitted.

2.2. Proposed ratio type estimator

Motivated by Odumade and Singh (2014) we define a ratio estimator for the population mean $\bar{Y}$ (based on the randomized response model-I due to Tarray and Singh (2014) as:

$$\bar{y}^*_{Ratio} = \bar{y}^*\left(\frac{\bar{x}^*_1}{\bar{x}^*_2}\right). \tag{2.5}$$

Note that

$$\bar{y}^* = \bar{Y}(1+\in), \bar{x}^*_1 = \bar{X}(1+\delta) \text{ and } \bar{x}^*_2 = \bar{X}(1+\eta).$$

Thus the ratio estimator in (2.5) can be written in terms of $\in, \delta$ *and* η as:

$$\bar{y}^*_{Ratio} = \bar{Y}(1+\in)\frac{\bar{X}(1+\delta)}{\bar{X}(1+\eta)} = \bar{Y}(1+\in)(1+\delta)(1+\eta)^{-1}$$

We assume that $|\eta|<1$ so that $(1+\eta)^{-1}$ is expandable in terms of η.

Thus

$$\bar{y}^*_{Ratio} = \bar{Y}(1+\in+\delta+\in\delta)\left[1-\eta+\eta^2+\ldots\right]$$
$$= \bar{Y}\left[1+\in+\delta-\eta+\eta^2+\in\delta-\in\eta-\delta\eta+\ldots\right]$$

or

$$\left(\bar{y}^*_{Ratio}-\bar{Y}\right) \cong \bar{Y}\left[\in+\delta-\eta+\eta^2+\in\delta-\in\eta-\delta\eta\right]. \tag{2.6}$$

Taking expectation of both sides of (2.5) we get the bias of the ratio estimator $\bar{y}^*_{Ratio}$ to the first degree of approximation as

$$B\left(\bar{y}^*_{Ratio}\right) = \left(\bar{Y}/n\right)\left[C^2_{x_2} + \left(1+C^2_{x_2}\right)C^2_{p2} + \rho_{yx_1}C_yC_{x_1} - \rho_{yx_2}C_yC_{x_2} - \rho_{x_1x_2}C_{x_1}C_{x_2}\right]$$

Thus, we obtained the following theorem.

Theorem 2.1: The bias in the proposed ratio estimator $\bar{y}^*_{Ratio}$ to the first degree of approximation is given by:

$$B\left(\bar{y}^*_{Ratio}\right) = \left(\bar{Y}/n\right)\left[C^2_{x_2} + \left(1+C^2_{x_2}\right)C^2_{p2} + \rho_{yx_1}C_yC_{x_1} - \rho_{yx_2}C_yC_{x_2} - \rho_{x_1x_2}C_{x_1}C_{x_2}\right] \tag{2.7}$$

Squaring both the sides of (2.6) and neglecting terms of $(\in, \delta, \eta)$ having power greater than two we have

$$\left(\bar{y}^*_{Ratio}-\bar{Y}\right)^2 = \bar{Y}^2\left[\in^2+\delta^2+\eta^2+2\in\delta-2\in\eta-2\eta\delta\right]. \tag{2.8}$$

Taking expectation of both sides of (2.8) we get the mean squared error (MSE) of the ratio estimator $\bar{y}^*_{Ratio}$ to the first degree of approximation, as

$$MSE\left(\bar{y}^*_{Ratio}\right) = \left(\bar{Y}^2/n\right)\Big[C^2_y + \left(1+C^2_y\right)C^2_{p0} + C^2_{x_1} + \left(1+C^2_{x_1}\right)C^2_{p1} + C^2_{x_2} + \left(1+C^2_{x_2}\right)C^2_{p2}$$
$$+2\rho_{yx_1}C_yC_{x_1} - 2\rho_{yx_2}C_yC_{x_2} - 2\rho_{x_1x_2}C_{x_1}C_{x_2}\Big]$$

Thus, we established the following theorem.

Theorem 2.2: The mean square error of the proposed ratio estimator $\bar{y}^*_{Ratio}$ to the first degree of approximation is given by:

$$MSE\left(\bar{y}^*_{Ratio}\right)=\left(\bar{Y}^2/n\right)\left[C_y^2+\left(1+C_y^2\right)C_{p0}^2+C_{x1}^2+\left(1+C_{x1}^2\right)C_{p1}^2+C_{x2}^2+\left(1+C_{x2}^2\right)C_{p2}^2\right.$$
$$\left.+2\rho_{yx1}C_yC_{x1}-2\rho_{yx2}C_yC_{x2}-2\rho_{x1x2}C_{x1}C_{x2}\right]. \quad (2.9)$$

2.3 Efficiency of the proposed ratio estimator

From (1.6), (1.12) and (2.9) it follows that the proposed ratio-type estimator $\bar{y}^*_{Ratio}$ is more efficient than:

(i) the Bar-Lev et al (2004) estimator $\bar{y}_{(BBB)}$ if

$$MSE\left(\bar{y}^*_{Ratio}\right)<MSE\left(\bar{y}_{(BBB)}\right)$$

i.e. if

$$\left[C_{x1}^2+C_{x2}^2+\left(1+C_{x1}^2\right)C_{p1}^2+\left(1+C_{x2}^2\right)C_{p2}^2\right]$$
$$<\left[2\left\{C_y\left(\rho_{yx2}C_{x2}-\rho_{yx1}C_{x1}\right)+\rho_{x1x2}C_{x1}C_{x2}\right\}+\left(1+C_y^2\right)\left(C_p^2-C_{p0}^2\right)\right]. \quad (2.10)$$

(ii) the Tarray and Singh (2014) estimator $\bar{y}_{(ST1)}$ if

$$MSE\left(\bar{y}^*_{Ratio}\right)<MSE\left(\bar{y}_{(ST1)}\right)$$

i.e. if

$$\left[C_{x1}^2+C_{x2}^2+\left(1+C_{x1}^2\right)C_{p1}^2+\left(1+C_{x2}^2\right)C_{p2}^2\right]$$
$$<2\left\{C_y\left(\rho_{yx2}C_{x2}-\rho_{yx1}C_{x1}\right)+\rho_{x1x2}C_{x1}C_{x2}\right\}. \quad (2.11)$$

Thus, the proposed ratio estimator $\bar{y}^*_{Ratio}$ will be more efficient than Bar-Lev et al's (2004) estimator $\bar{y}_{(BBB)}$ and Tarray and Singh (2014) estimator $\bar{y}_{(ST1)}$ as long as the conditions (2.10) and (2.11) are satisfied respectively. In order to see the performance of the proposed ratio-type estimator $\bar{y}^*_{Ratio}$ relative to BBB model and Tarray and Singh's (2014) model, we have computed the percent relative efficiencies (PREs) using the following formulae:

(i) Bar-Lev et al (2004) estimator $\bar{y}_{(BBB)}$

$$PRE\left(\bar{y}_{(BBB)},\bar{y}^*_{Ratio}\right)=\frac{V\left(\bar{y}_{(BBB)}\right)}{MSE\left(\bar{y}^*_{Ratio}\right)}\times 100. \quad (2.12)$$

We wrote the MATLAB code and retained those results where the percent relative efficiency (PRE) values are between 300 and 600 to discover the situations where the proposed model can perform better than the Bar-Lev et al (2004) model. In this study we have made a very reasonable choice of a few parameters such as $p, p_1, p_2, C_y, C_{x1}, C_{x2}, C_{\gamma,} C_{\gamma1}, C_{\gamma2}, \theta, \theta_1$ *and* θ_2

on which the percent relative efficiency of the ratio estimator depends. It is to be noted that the PRE is free from the sample size *n* and principal population parameter of interest $\bar{Y}$ the population mean of the study variable *y*.

We have also written the code to find the values of the parameters $C_y, C_{x_1}, C_{x_2}, C_{\gamma}, C_{\gamma_1}, C_{\gamma_2}, \rho_{yx_1}, \rho_{yx_2}, \rho_{x_1x_2}, \theta, \theta_1$ *and* θ_2 by keeping p, p_1 *and* p_2 each equal to 0.7. We changed the value of $C_y, C_{x_1}, C_{x_2}, C_{\gamma,} C_{\gamma_1}, C_{\gamma_2}$ between 0.1 to 0.5 with a step of 0.2. The values of θ, θ_1 *and* θ_2 were changed between 0 and 1 with a step of 0.5. The values $,\rho_{yx_2}$ *and* $\rho_{x_1x_2}$ were changed between 0.1 to 0.9 with a step of 0.2 and that of ρ_{yx_1} was changed between -0.9 to +0.9 with a step of 0.2. Findings are given in Table 2.1.

Table 2.3: Descriptive statistics of the percent relative efficiency

Relative Efficiency	
Mean	525.31
Standard Error	11.37
Median	553.47
Standard Deviation	90.28
Sample Variance	8150.42
Kurtosis	1.75
Skewness	–1.78
Range	285.58
Minimum	313.99
Maximum	599.57
Count	63

It is observed from Table 2.3 that the average percent relative efficiency is 525.31% with the standard deviation 90.28 with median 553.47%, minimum of 313.99% and maximum of 599.57% (see Table 2.1). We also note that there are 63 cases where the percent relative efficiency of the proposed ratio estimator remains between 300 to 600.

(ii) Tarray and Singh (2014) estimator $\bar{y}_{(ST1)}$

$$PRE\left(\bar{y}_{(ST1)}, \bar{y}^*_{Ratio}\right) = \frac{V\left(\bar{y}_{(ST1)}\right)}{MSE\left(\bar{y}^*_{Ratio}\right)} \times 100. \tag{2.13}$$

We have also written the code to find the values of the parameters $C_y, C_{x_1}, C_{x_2}, C_{\gamma,} C_{\gamma_1}, C_{\gamma_2}, \rho_{yx_1}, \rho_{yx_2}$ *and* $\rho_{x_1x_2}$ by keeping p, p_1 *and* p_2 each equal to 0.7. We changed the value of $C_y, C_{x_1}, C_{x_2}, C_{\gamma,} C_{\gamma_1}, C_{\gamma_2}$ between 0.1 to 0.5 with a step of 0.2. The values of θ, θ_1 *and* θ_2 were changed between 0 and 1 with a step of 0.5. The values $,\rho_{yx_2}$ *and* $\rho_{x_1x_2}$ were changed between 0.1 to 0.9 with a step of 0.2 and that of ρ_{yx_1} was changed between −0.9 to +0.9 with a step of 0.2. Findings are given in Table 2.2.

Table 2.4: Descriptive statistics of the percent relative efficiency

Relative Efficiency	
Mean	429.27
Standard Error	15.89
Median	387.27
Standard Deviation	111.26
Sample Variance	12379.12
Kurtosis	−1.55
Skewness	0.39
Range	285.82
Minimum	302.12
Maximum	587.94
Count	49

Table 2.4 shows that the average percent relative efficiency is 429.27% with the standard deviation 111.26 with median 387.27%, minimum of 302.12% and maximum of 587.94% (see Table 2.2). It has been observed that there are 49 cases where the percent relative efficiency of the proposed ratio estimator remains between 300 to 600.

3. Proposed Power Transformation Ratio Type Estimator Based on Tarray and Singh (2014) Model –I

Using the repeated substitution method due to Srivastava (1967) and Garcia and Cebrian (1996), we consider a new power transformation ratio type estimator $\bar{y}^*_{Power}$ for the population mean $\bar{Y}$ as:

$$\bar{y}^*_{Power} = \bar{y}^*\left(\frac{\bar{x}^*_1}{\bar{x}^*_2}\right)^{\alpha}, \tag{3.1}$$

where α is a suitably chosen real constant. For example if $\alpha = 0$ then the proposed power transformation ratio type estimator $\bar{y}^*_{Power}$ reduces to the Tarray and Singh (2014) estimator $\bar{y}_{(ST1)}$. If $\alpha = 1$ then the proposed power transformation ratio type estimator $\bar{y}^*_{Power}$ reduces to the ratio estimator $\bar{y}^*_{Ratio}$. Expressing that the proposed transformation ratio-type estimator $\bar{y}^*_{Power}$ in terms of $\in$, δ and η, we have:

$$\bar{y}^*_{Power} = \bar{Y}(1+\in)\left[\frac{\bar{X}(1+\delta)}{\bar{X}(1+\eta)}\right]^{\alpha} = Y(1+\in)(1+\delta)^{\alpha}(1+\eta)^{-\alpha}. \tag{3.2}$$

We assume that $|\delta| < 1$ and $|\eta| < 1$ so that $(1+\delta)^{\alpha}$ and $(1+\eta)^{-\alpha}$ are expandable. Now expanding the right hand side of (3.2), multiplying out and neglecting terms of $(\in, \delta, \eta)$ having power greater than two we have

$$\bar{y}^*_{power} - \bar{Y} \cong \bar{Y}\left[\in + \alpha(\delta - \eta) + \alpha(\in \delta - \in \eta) - \alpha^2 \delta\eta + \frac{\alpha(\alpha-1)}{2}\delta^2 + \frac{\alpha(\alpha+1)}{2}\eta^2\right] \quad (3.3)$$

Taking expectation of both sides of (3.3) we get the bias of $\bar{y}^*_{Power}$ to the first degree of approximation as

$$B\left(\bar{y}^*_{Power}\right) = \left(\frac{\bar{Y}}{n}\right)\left[\alpha\left(\rho_{yx1} C_y C_{x1} - \rho_{yx2} C_y C_x\right) - \alpha^2 \rho_{x1x2} C_{x1} C_{x2} + \frac{\alpha(\alpha-1)}{2}\left\{C_{x1}^2 + \left(1 + C_{x1}^2\right)C_{p1}^2\right\} + \frac{\alpha(\alpha+1)}{2}\left\{C_{x2}^2 + \left(1 + C_{x2}^2\right)C_{p2}^2\right\}\right]$$

Thus, we established the following theorem.

Theorem 3.1. The bias in the proposed power transformation ratio type estimator $\bar{y}^*_{Power}$ is given by:

$$B\left(\bar{y}^*_{Power}\right) = \left(\frac{\alpha\bar{Y}}{n}\right)\left[\rho_{yx1} C_y C_{x1} - \rho_{yx2} C_y C_{x2} - \alpha\rho_{x1x2} C_{x1} C_{x2} + \frac{(\alpha-1)}{2}\left\{C_{x1}^2 + \left(1 + C_{x1}^2\right)C_{p1}^2\right\} + \frac{(\alpha+1)}{2}\left\{C_{x2}^2 + \left(1 + C_{x2}^2\right)C_{p2}^2\right\}\right]. \quad (3.4)$$

The mean squared error of the proposed estimator $\bar{y}^*_{Power}$ is obtained as follows. Squaring both sides of (3.3) and neglecting terms of $(\in, \delta, \eta)$ having power greater than two we have

$$\left(\bar{y}^*_{Power} - \bar{Y}\right)^2 = \bar{Y}^2\left[\in^2 + \alpha^2(\delta - \eta)^2 + 2\alpha(\delta \in - \eta \in)\right]. \quad (3.5)$$

Taking expectation of both sides of (3.5) we get the mean squared error of the estimator $\bar{y}^*_{Power}$ as

$$MSE\left(\bar{y}^*_{Power}\right) = \frac{\bar{Y}^2}{n}\left[C_y^2 + \left(1 + C_y^2\right)C_{p0}^2 + \alpha^2\left\{C_{x1}^2 + \left(1 + C_{x1}^2\right)C_{p1}^2 + C_{x2}^2 + \left(1 + C_{x2}^2\right)C_{p2}^2 - 2\rho_{x1x2} C_{x1} C_{x2}\right\} - 2\alpha\left(\rho_{yx2} C_y C_{x2} - \rho_{yx1} C_y C_{x1}\right)\right]$$

which is minimum when

$$\alpha = \frac{\left(\rho_{yx2} C_y C_{x2} - \rho_{yx1} C_y C_{x1}\right)}{\left[C_{x1}^2 + C_{x2}^2 - 2\rho_{x1x2} C_{x1} C_{x2} + \left(1 + C_{x1}^2\right)C_{p1}^2 + \left(1 + C_{x2}^2\right)C_{p2}^2\right]}$$
$$= \alpha_0 \text{(say)}.$$

Thus the resulting minimum MSE of $\bar{y}^*_{Power}$ is given by

$$Min.MSE\left(\bar{y}^*_{Power}\right)=\frac{\bar{Y}^2}{n}\left[C_y^2+\left(1+C_y^2\right)C_{p0}^2-\frac{C_y^2\left(\rho_{yx2}C_{x2}-\rho_{yx1}C_{x1}\right)^2}{\left\{C_{x1}^2+C_{x2}^2-2\rho_{x1x2}C_{x1}C_{x2}+\left(1+C_{x1}^2\right)C_{p1}^2+\left(1+C_{x2}^2\right)C_{p2}^2\right\}}\right]$$

Thus, we arrived at the following theorems.

Theorem 3.2. The mean squared error of the estimator $\bar{y}^*_{Power}$ to the first degree of approximation is given by

$$MSE\left(\bar{y}^*_{Power}\right)=\left(\frac{\bar{Y}^2}{n}\right)\left[C_y^2+\left(1+C_y^2\right)C_{p0}^2+\alpha^2\left\{C_{x1}^2+C_{x2}^2-2\rho_{x1x2}C_{x1}C_{x2}+\left(1+C_{x1}^2\right)C_{p1}^2+\left(1+C_{x2}^2\right)C_{p2}^2\right\}-2\alpha\left(\rho_{yx2}C_yC_{x2}-\rho_{yx1}C_yC_{x1}\right)\right] \quad (3.6)$$

Theorem 3.3. The optimum value of α (for which the MSE $\left(\bar{y}^*_{Power}\right)$ in (3.6) is minimum) and the minimum MSE of the estimator $\bar{y}^*_{Poewr}$ are respectively given by

$$\alpha=\frac{C_y\left(\rho_{yx2}C_{x2}-\rho_{yx1}C_{x1}\right)}{\left[C_{x1}^2+C_{x2}^2-2\rho_{x1x2}C_{x1}C_{x2}+\left(1+C_{x1}^2\right)C_{p1}^2+\left(1+C_{x2}^2\right)C_{p2}^2\right]}$$
$$=\alpha_0 \text{ (say)} \quad (3.7)$$

and

$$Min.MSE\left(\bar{y}^*_{Power}\right)=\frac{\bar{Y}^2}{n}\left[C_y^2+\left(1+C_y^2\right)C_{p0}^2-\frac{C_y^2\left(\rho_{yx2}C_{x2}-\rho_{yx1}C_{x1}\right)^2}{\left\{C_{x1}^2+C_{x2}^2-2\rho_{x1x2}C_{x1}C_{x2}+\left(1+C_{x1}^2\right)C_{p1}^2+\left(1+C_{x2}^2\right)C_{p2}^2\right\}}\right] \quad (3.8)$$

3.1. Efficiency comparison

3.1.1. When the scalar α does not coincide exactly with its optimum value α_0

From (1.6) and (3.6)

$$MSE\left(\bar{y}^*_{Power}\right)-V\left(\bar{y}_{BBB}\right)=\left(\frac{\bar{Y}^2}{n}\right)\left[C_y^2+\left(1+C_y^2\right)\left(C_{p0}^2-C_p^2\right)+\alpha^2\left\{C_{x1}^2+C_{x2}^2-2\rho_{x1x2}C_{x1}C_{x2}+\left(1+C_{x1}^2\right)C_{p1}^2+\left(1+C_{x2}^2\right)C_{p2}^2\right\}-2\alpha\left(\rho_{yx2}C_yC_{x2}-\rho_{yx1}C_yC_{x1}\right)\right]$$

which is less than zero if

$$\alpha^2A-2\alpha B+C<0$$

i.e. if

$$\frac{B-\sqrt{\left(B^2-AC\right)}}{A}<\alpha<\frac{B+\sqrt{\left(B^2-AC\right)}}{A} \tag{3.9}$$

where

$$A=\left[C_{x_1}^2+C_{x_2}^2-2\rho_{x_1x_2}C_{x_1}C_{x_2}+\left(1+C_{x_1}^2\right)C_{p_1}^2+\left(1+C_{x_2}^2\right)C_{p_2}^2\right],$$
$$B=\left[\rho_{yx_2}C_yC_{x_2}-\rho_{yx_1}C_yC_{x_1}\right] \text{ and}$$
$$C=\left[C_y^2+\left(1+C_y^2\right)\left(C_{p_0}^2-C_p^2\right)\right].$$

Thus, we state the following theorem.

Theorem 3.4. The proposed power transformation ratio-type estimator $\bar{y}_{Power}^*$ is more efficient than the Bar-Lev et al's (2004) estimator $\bar{y}_{BBB}$ as long as the condition (3.9) is satisfied.

Further from (1.12) and (3.6)

$$V\left(\bar{y}_{(ST1)}\right)-MSE\left(\bar{y}_{Power}^*\right)=\frac{\bar{Y}^2}{n}\left[2\alpha C_y\left(\rho_{yx_2}C_{x_2}-\rho_{yx_1}C_{x_1}\right)\right.$$
$$\left.-\alpha^2\left\{C_{x_1}^2+C_{x_2}^2-2\rho_{x_1x_2}C_{x_1}C_{x_2}+\left(1+C_{x_1}^2\right)C_{p_1}^2+\left(1+C_{x_2}^2\right)C_{p_2}^2\right\}\right]$$

which is non-negative if

$$\alpha\left(2\alpha_0-\alpha\right)>0$$

i.e. if $\alpha\left(\alpha-2\alpha_0\right)<0$

i.e. if $\left|\left(\alpha-\alpha_0\right)\right|<\left|\alpha_0\right|,$ (3.10)

where α_0 is given by (3.7) .

Thus. we state the following theorem.

Theorem 3.5. The proposed power transformation ratio-type estimator $\bar{y}_{Poewr}^*$ is more efficient than the Tarray and Singh's (2014) estimator $\bar{y}_{(ST1)}$ as long as the condition (3.10) is satisfied.

Further from (2.8) and (3.6)

$$MSE\left(\bar{y}_{Ratio}^*\right)-MSE\left(\bar{y}_{Power}^*\right)=\left(\frac{\bar{Y}^2}{n}\right)\left[\left(1-\alpha^2\right)A-2(1-\alpha)B\right]$$
$$=\left(\frac{\bar{Y}^2A}{n}\right)\left[\left(1-\alpha^2\right)-2(1-\alpha)\frac{B}{A}\right]$$
$$=\left(\frac{\bar{Y}^2A}{n}\right)\left[\left(1-\alpha^2\right)-2(1-\alpha)\alpha_0\right]$$

which is positive if

$$\left(1-\alpha^2-2\alpha_0+2\alpha\alpha_0\right)>0$$

i.e. if $\left(\alpha^2 - 2\alpha\alpha_0 - 1 + 2\alpha_0\right) < 0$

i.e. if $\left| \left(\alpha - \alpha_0\right) \right| < \left| \alpha_0 \right|$ (3.11)

where $A = \left[C_{x_1}^2 + C_{x_2}^2 - 2\rho_{x_1x_2} C_{x_1} C_{x_2} + \left(1 + C_{x_1}^2\right)C_{p_1}^2 + \left(1 + C_{x_2}^2\right)C_{p_2}^2\right]$

$B = \left[\rho_{yx_2} C_y C_{x_2} - \rho_{yx_1} C_y C_{x_2}\right]$

and $\alpha_0 = \dfrac{B}{A}$ is same as given by (3.7).

Thus, we established the following theorem.

Theorem 3.6. The proposed power transformation ratio-type estimator $\bar{y}^*_{Power}$ is more efficient than the proposed ratio type estimator $\bar{y}^*_{Ratio}$ as long as the condition (3.11) is satisfied.

3.1.2. When the Optimum Value α_0 of the Scalar α is Exactly Known

$$V\left(\bar{y}_{(BBB)}\right) - Min.MSE\left(\bar{y}^*_{Power}\right) = \bar{Y}^2 \left[\left(1 + C_y^2\right)\left(C_p^2 - C_{p0}^2\right) + \frac{C_y^2\left[\rho_{yx_2} C_{x_2} - \rho_{yx_1} C_{x_2}\right]^2}{\left[C_{x_1}^2 + C_{x_2}^2 - 2\rho_{x_1x_2} C_{x_1} C_{x_2} + \left(1 + C_{x_1}^2\right)C_{p_1}^2 + \left(1 + C_{x_2}^2\right)C_{p_2}^2\right]}\right]$$

which is non-negative if

$$C_p^2 > C_{p0}^2 \tag{3.12}$$

Thus, we state the following theorem.

Theorem 3.7. The proposed power transformation ratio-type estimator $\bar{y}^*_{power}$ (at its optimum condition i.e. when $\alpha = \alpha_0$) is better than Bar-Lev et al (2004) estimator $\bar{y}_{(BBB)}$ if $C_p^2 > C_{p0}^2$.

Further from (1.12) and (3.8) we have

$$V\left(\bar{y}_{(ST1)}\right) - Min.MSE\left(\bar{y}^*_{Power}\right) = \frac{S_Y^2}{n} \frac{\left(\rho_{yx_2} C_{x_2} - \rho_{yx_1} C_{x_1}\right)}{\left[C_{x_1}^2 + C_{x_2}^2 - 2\rho_{x_1x_2} C_{x_1} C_{x_2} + \left(1 + C_{x_1}^2\right)C_{p_1}^2 + \left(1 + C_{x_2}^2\right)C_{p_2}^2\right]}$$

$$> 0 \text{ provided } \rho_{yx_1} C_{x_1} \neq \rho_{yx_2} C_{x_2} \tag{3.13}$$

Thus, we state the following theorem.

Theorem 3.8. The proposed power transformation ratio-type estimator $\bar{y}^*_{power}$ (at its optimum condition i.e. when $\alpha = \alpha_0$) is better than Tarray and Singh's (2014) estimator $\bar{y}_{(ST1)}$ unless $\rho_{yx_1} C_{x_1} \neq \rho_{yx_2} C_{x_2}$, the case where both the estimator $\bar{y}_{(ST1)}$ and $\bar{y}^*_{power}$ are equally efficient.

Next from (2.9) and (3.8) we have

$$MSE\left(\bar{y}^*_{Ratio}\right) - Min.MSE\left(\bar{y}^*_{Power}\right) = \frac{\bar{Y}^2}{n}\frac{(A-B)^2}{A}$$
$$> 0 \text{ provided } A \neq B. \tag{3.14}$$

Thus, we state the following theorem

Theorem 3.9. The proposed power transformation ratio-type estimator $\bar{y}^*_{power}$ (at its optimum condition i.e. when $\alpha = \alpha_0$) is more efficient than the proposed ratio type estimator $\bar{y}^*_{Ratio}$ unless $A{=}B$, the case where both estimators $\bar{y}^*_{Ratio}$ and $\bar{y}^*_{power}$ are equally efficient.

3.2. Relative efficiency of the power transformation ratio type estimator

In order to see the magnitude, we computed the percent relative efficiency of the proposed power transformation ratio-type estimator $\bar{y}^*_{power}$ with respect to:

(i) Bar-Lev et al (2004) estimator $\bar{y}_{BBB}$

$$PRE\left(\bar{y}_{(BBB)}, \bar{y}^*_{Power}\right) = \frac{V\left(\bar{y}_{(BBB)}\right)}{MSE\left(\bar{y}^*_{Power}\right)} \times 100 \tag{3.15}$$

We have also written the code to find the values of the parameter $C_y, C_{x1}, C_{x_2}, C_{\gamma}, C_{\gamma 1}, C_{\gamma 2}, \rho_{yx1}, \rho_{yx_2}, \rho_{x1x_2}, \theta, \theta_1$ *and* θ_2 by keeping p, p_1 *and* p_2 each equal to 0.7. We changed the value of $C_y, C_{x1}, C_{x_2}, C_{\gamma}, C_{\gamma 1}, C_{\gamma 2}$ between 0.1 to 0.5 with a step of 0.2. The values of θ, θ_1 *and* θ_2 were changed between 0 and 1 with a step of 0.5. The values ρ_{yx_1}, ρ_{yx_2} and $\rho_{x_1x_2}$ were changed between 0.1 to 0.9 with a step of 0.2 and that of ρ_{yx1} was changed between –0.9 to +0.9 with a step of 0.2.

Findings are shown in Table 3.1.

Table 3.3: Descriptive statistics of the percent relative efficiency

Relative Efficiency	
Mean	507.30
Standard Error	13.64
Median	551.71
Standard Deviation	95.50
Sample Variance	9121.35
Kurtosis	0.38
Skewness	–1.47
Range	292.18
Minimum	300.17
Maximum	592.35
Count	49

Table 3.3 depicts that the average percent relative efficiency is 507.30% with the standard deviation 95.50 with median 551.71%, minimum of 300.17% and maximum of 592.35% (see Table 3.1). It has been observed that there are 49 cases where the percent relative efficiency of the proposed ratio estimator remains between 300 to 600. It has been observed that a choice of larger values of θ, θ_1 and θ_2 may lead to inefficient results, thus the choice of these values is must while using the proposed ratio method in actual practice.

(ii) Tarray and Singh (2014) estimator $\bar{y}_{(ST1)}$

$$PRE\left(\bar{y}_{(ST1)}, \bar{y}^*_{Power}\right) = \frac{V\left(\bar{y}_{(ST1)}\right)}{MSE\left(\bar{y}^*_{Power}\right)} \times 100 \tag{3.16}$$

We have also written the code to find the values of the parameters $C_y, C_{x_1}, C_{x_2}, C_{\gamma}, C_{\gamma_1}, C_{\gamma_2}, \rho_{yx_1}, \rho_{yx_2}$ *and* $\rho_{x_1x_2}$ by keeping p, p_1 *and* p_2 each equal to 0.7. We changed the value of $C_y, C_{x_1}, C_{x_2}, C_{\gamma}, C_{\gamma_1}, C_{\gamma_2}$ between 0.1 to 0.5 with a step of 0.2. The values of θ, θ_1 *and* θ_2 were changed between 0 and 1 with a step of 0.5. The values $,\rho_{yx_2}$ *and* $\rho_{x_1x_2}$ were changed between 0.1 to 0.9 with a step of 0.2 and that of ρ_{yx_1} was changed between -0.9 to +0.9 with a step of 0.2. Findings are shown in Table 3.2.

Table 3.4: Descriptive statistics of the percent relative efficiency

Relative Efficiency	
Mean	441.89
Standard Error	13.04
Median	379.03
Standard Deviation	92.18
Sample Variance	8497.77
Kurtosis	–1.83
Skewness	0.09
Range	253.31
Minimum	300.90
Maximum	554.21
Count	50

It is observed from Table 3.4 that the average percent relative efficiency is 441.89% with the standard deviation 92.18 with median 379.03%, minimum of 300.90% and maximum of 554.21% (see Table 3.2). It has been observed that there are 50 cases where the percent relative efficiency of the proposed ratio estimator remains between 300 to 600.

4. Proposed Ratio-Type Estimator Based on Tarray And Singh (2014) Model-II

4.1. Notations

By Tripathi and Chaubey (1992) let $\bar{X}_{1i} = \bar{X}_{2i} = \bar{X}$ that is these two auxiliary sensitive variables have common mean. Let Y_i be the sensitive variable under study whose mean is to be estimated. Consider we selected a simple random sample with replacement (SRSWR) of n respondents. Then each one of the respondents selected in the sample is requested to rotate three spinners.

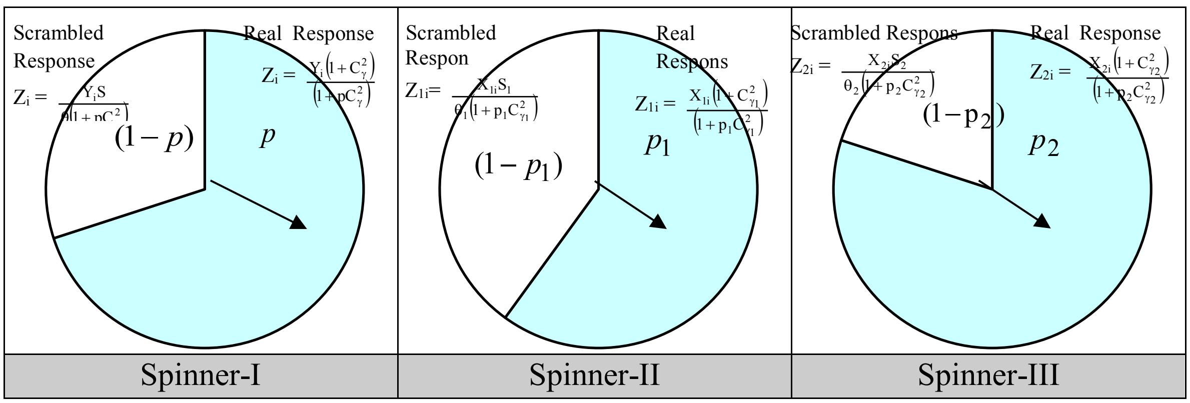

Fig. 4.1: Three spinners

The first spinner is used to collect scrambled response Z_i on the real study variable Y_i with the distribution of responses as:

$$Z_{0i} = \begin{cases} \dfrac{Y_i\left(1 + C_\gamma^2\right)}{\left(1 + pC_\gamma^2\right)} & \text{with probability } p \\ \dfrac{Y_i S}{\theta\left(1 + pC_\gamma^2\right)} & \text{with probaility } (1-p), \end{cases} \tag{4.1}$$

where the value of p is assumed to be known.

The second spinner is used to collect scrambled response Z_{01i} on the first auxiliary sensitive variable X_{1i} with the distribution of responses as:

$$Z_{01i} = \begin{cases} \dfrac{X_{1i}\left(1 + C_{\gamma 1}^2\right)}{\left(1 + p_1 C_{\gamma 1}^2\right)} & \text{with probability } p_1 \\ \dfrac{X_{1i} S_1}{\theta_1\left(1 + p_1 C_{\gamma 1}^2\right)} & \text{with probaility } (1-p_1), \end{cases} \tag{4.2}$$

where the value of p_1 is assumed to be known.

The third spinner is used to collect scrambled response Z_{02i} on the second auxiliary sensitive variable X_{2i} with the distribution of responses as:

$$Z_{2i} = \begin{cases} \dfrac{X_{2i}\left(1+C_{\gamma 2}^2\right)}{\left(1+p_2 C_{\gamma 2}^2\right)} & \textit{with probability } p_2 \\ \dfrac{X_{2i} S_2}{\theta_2\left(1+p_2 C_{\gamma 2}^2\right)} & \textit{with probaility } (1-p_2), \end{cases} \tag{4.3}$$

where the value of p_2 is assumed to be known.

Assume that the sample mean of the scrambled responses obtained from the respondents in the sample as Z_{0i}, Z_{01i} and Z_{02i} are given by:

$$\bar{y}^{**} = \frac{1}{n}\sum_{i=1}^{n} Z_{0i}\,,\ \bar{x}_1^{**} = \frac{1}{n}\sum_{i=1}^{n} Z_{01i} \text{ and } \bar{x}_2^{**} = \frac{1}{n}\sum_{i=1}^{n} Z_{02i}\,.$$

Let us define:

$$\in_1 = \frac{\bar{y}^{**}}{\bar{Y}} - 1,\ \delta_1 = \frac{\bar{x}_1^{**}}{\bar{X}} - 1,\ \eta_1 = \frac{\bar{x}_2^{**}}{\bar{X}} - 1$$

such that

$$E(\in_1) = E(\delta_1) = E(\eta_1) = 0$$

and it can be shown that

$$E\left(\in_1^2\right) = \frac{1}{\bar{Y}^2} V\left(\bar{y}^{**}\right) = \frac{1}{n}\left[C_y^2 + \left(1+C_y^2\right)C_{p0}^{*2}\right],$$

$$E\left(\delta_1^2\right) = \frac{1}{\bar{Y}^2} V\left(\bar{x}_1^{**}\right) = \frac{1}{n}\left[C_{x1}^2 + \left(1+C_{x1}^2\right)C_{p1}^{*2}\right],$$

$$E\left(\eta_1^2\right) = \frac{1}{\bar{Y}^2} V\left(\bar{x}_2^{**}\right) = \frac{1}{n}\left[C_{x2}^2 + \left(1+C_{x2}^2\right)C_{p2}^{*2}\right],$$

$$E(\in_1\ \delta_1) = \frac{1}{n}\rho_{yx_1} C_y C_{x_1}\,,$$

$$E(\in_1\ \eta_1) = \frac{1}{n}\rho_{yx_2} C_y C_{x2}\,,$$

and

$$E(\delta_1 \eta_1) = \frac{1}{n}\rho_{x_1 x_2} C_{.x1} C_{x2}\,,$$

where $C_{x_1} = \sigma_{x_1}/\bar{X}$ and $C_{x_2} = \sigma_{x_2}/\bar{X}$ are the coefficients of variation of the auxiliary sensitive variables x_1 and x_2 respectively,

ρ_{yx_1} : is the correlation coefficient between y *and* x_1,

ρ_{yx_2} : is the correlation coefficient between y *and* x_2,

$\rho_{x_1x_2}$: is the correlation coefficient between x_1 *and* x_2,

$$C_{p0}^{*2} = \left[\frac{\left(1 + C_{\gamma}^2\right)}{\left(1 + pC_{\gamma}^2\right)} - 1 \right],\ C_{p1}^{*2} = \left[\frac{\left(1 + C_{\gamma 1}^2\right)}{\left(1 + p_1 C_{\gamma 1}^2\right)} - 1 \right] \text{ and } C_{p2}^2 = \left[\frac{\left(1 + C_{\gamma 2}^2\right)}{\left(1 + p_2 C_{\gamma 2}^2\right)} - 1 \right].$$

4.2. Proposed ratio type estimator

We define a ratio estimator for the population mean $\bar{Y}$ (based on the randomized response model-II due to Tarray and Singh (2014) as:

$$\bar{y}_{Ratio}^{**} = \bar{y}^{**} \left(\frac{\bar{x}_1^{**}}{\bar{x}_2^{**}} \right) \tag{4.4}$$

Note that

$$\bar{y}^{**} = \bar{Y}\left(1 + \in_1\right), \bar{x}_1^{**} = \bar{X}\left(1 + \delta_1\right) \, and \, \bar{x}_2^{**} = \bar{X}\left(1 + \eta_1\right).$$

Thus the ratio estimator in (4.4) can be written in terms of $\in_1, \delta_1$ *and* η_1 as:

$$\bar{y}_{Ratio}^{**} = \bar{Y}\left(1 + \in_1\right) \frac{\bar{X}\left(1 + \delta_1\right)}{\bar{X}\left(1 + \eta_1\right)} = \bar{Y}\left(1 + \in_1\right)\left(1 + \delta_1\right)\left(1 + \eta_1\right)^{-1}$$

We assume that $|\eta_1| < 1$ so that $\left(1 + \eta_1\right)^{-1}$ is expandable in terms of η_1.

Thus

$$\begin{aligned} \bar{y}_{Ratio}^{**} &= \bar{Y}\left(1 + \in_1 + \delta_1 + \in_1 \delta_1\right)\left[1 - \eta_1 + \eta_1^2 + \ldots\right] \\ &= \bar{Y}\left[1 + \in_1 + \delta_1 - \eta_1 + \eta_1^2 + \in_1 \delta_1 - \in_1 \eta_1 - \delta_1 \eta_1 + \ldots\right] \end{aligned}$$

or

$$\left(\bar{y}_{Ratio}^{**} - \bar{Y}\right) \cong \bar{Y}\left[\in_1 + \delta_1 - \eta_1 + \eta_1^2 + \in_1 \delta_1 - \in_1 \eta_1 - \delta_1 \eta_1\right] \tag{4.5}$$

Theorem 4.1: The bias in the proposed ratio estimator $\bar{y}_{\text{Ratio}}^{**}$ to the first degree of approximation is given by:

$$B\left(\bar{y}_{Ratio}^{**}\right) = \left(\bar{Y}/n\right)\left[C_{x2}^2 + \left(1 + C_{x2}^2\right)C_{p2}^{*2} + \rho_{yx1} C_y C_{x1} - \rho_{yx2} C_y C_{x2} - \rho_{x1x2} C_{x1} C_{x2}\right] \tag{4.6}$$

Proof- Taking expectation of both sides of (4.5) we get the bias of the ratio estimator $\bar{y}_{Ratio}^{**}$ to the first degree of approximation as

$$\begin{aligned} B\left(\bar{y}_{Ratio}^{**}\right) &= E\left(\bar{y}_{Ratio}^{**}\right) - \bar{Y} \\ &= \bar{Y}\left[E\left(\eta_1^2\right) + E\left(\in_1 \delta_1\right) - E\left(\in_1 \eta_1\right) - E\left(\delta_1 \eta_1\right)\right] \\ &= \left(\bar{Y}/n\right)\left[C_{x2}^2 + \left(1 + C_{x2}^2\right)C_{p2}^2 + \rho_{yx1} C_y C_{x1} - \rho_{yx2} C_y C_{x2} - \rho_{x1x2} C_{x1} C_{x2}\right] \end{aligned}$$

which proves the theorem.

Theorem 4.2: The mean square error of the proposed ratio estimator $\bar{y}^{**}_{Ratio}$ to the first degree of approximation is given by:

$$MSE\left(\bar{y}^{**}_{Ratio}\right)=\left(\bar{Y}^2/n\right)\left[C_y^2+\left(1+C_y^2\right)C_{p0}^{*2}+C_{x1}^2+\left(1+C_{x1}^2\right)C_{p1}^{*2}+C_{x2}^2+\left(1+C_{x2}^2\right)C_{p2}^{*2}\right.$$
$$\left.+2\rho_{yx1}C_yC_{x1}-2\rho_{yx2}C_yC_{x2}-2\rho_{x1x2}C_{x1}C_{x2}\right] \quad (4.7)$$

Proof. Squaring both the sides of (4.5) and neglecting terms of $\left(\in_1,\delta_1,\eta_1\right)$ having power greater than two we have

$$\left(\bar{y}^{**}_{Ratio}-\bar{Y}\right)^2=\bar{Y}^2\left[\in_1^2+\delta_1^2+\eta_1^2+2\in_1\delta_1-2\in_1\eta_1-2\eta_1\delta_1\right] \quad (4.8)$$

Taking expectation of both sides of (4.8) we get the mean squared error (MSE) of the ratio estimator $\bar{y}^{**}_{Ratio}$ as

$$MSE\left(\bar{y}^{**}_{Ratio}\right)=\left(\bar{Y}^2/n\right)\left[C_y^2+\left(1+C_y^2\right)C_{p0}^{2}+C_{x1}^2+\left(1+C_{x1}^2\right)C_{p1}^{2}+C_{x2}^2+\left(1+C_{x2}^2\right)C_{p2}^{2}\right.$$
$$\left.+2\rho_{yx1}C_yC_{x1}-2\rho_{yx2}C_yC_{x2}-2\rho_{x1x2}C_{x1}C_{x2}\right]$$

which proves the theorem.

4.3. Efficiency of the proposed ratio estimator

From (1.6), (1.19) and (4.7) it follows that the proposed ratio-type estimator $\bar{y}^{**}_{Ratio}$ is more efficient than:

(i) the Bar-Lev et al (2004) estimator $\bar{y}_{(BBB)}$ if

$$MSE\left(\bar{y}^{**}_{Ratio}\right)<MSE\left(\bar{y}_{(BBB)}\right)$$

i.e. if

$$\left[C_{x1}^2+C_{x2}^2+\left(1+C_{x1}^2\right)C_{p1}^{*2}+\left(1+C_{x2}^2\right)C_{p2}^{*2}\right]<$$
$$<\left[2\left\{C_y\left(\rho_{yx2}C_{x2}-\rho_{yx1}C_{x1}\right)+\rho_{x1x2}C_{x1}C_{x2}\right\}+\left(1+C_y^2\right)\left(C_p^2-C_{p0}^{*2}\right)\right] \quad (4.9)$$

(ii) the Tarray and Singh (2014) estimator $\bar{y}_{(ST2)}$ if

$$MSE\left(\bar{y}^{**}_{Ratio}\right)<MSE\left(\bar{y}_{(ST2)}\right)$$

i.e. if

$$\left[C_{x1}^2+C_{x2}^2+\left(1+C_{x1}^2\right)C_{p1}^{*2}+\left(1+C_{x2}^2\right)C_{p2}^{*2}\right]$$
$$<2\left\{C_y\left(\rho_{yx2}C_{x2}-\rho_{yx1}C_{x1}\right)+\rho_{x1x2}C_{x1}C_{x2}\right\} \quad (4.10)$$

Thus, the proposed ratio estimator $\bar{y}^{**}_{Ratio}$ will be more efficient than Bar-Lev et al's (2004) estimator $\bar{y}_{(BBB)}$ and Tarray and Singh (2014) estimator $\bar{y}_{(ST2)}$ as long as the conditions (4.9) and (4.10) are satisfied respectively.

We have computed the percent relative efficiencies (PREs) in order to see the performance of the proposed ratio-type estimator $\bar{y}^{**}_{Ratio}$ with respect to Bar-Lev et al (2004) estimator $\bar{y}_{(BBB)}$ by using the formula:

$$PRE\left(\bar{y}_{(BBB)}, \bar{y}^{**}_{Ratio}\right) = \frac{V\left(\bar{y}_{(BBB)}\right)}{MSE\left(\bar{y}^{*}_{Ratio}\right)} \times 100 \tag{4.11}$$

We have also written the code to find the values of the parameter $C_y, C_{x1}, C_{x2}, C_{\gamma,} C_{\gamma1}, C_{\gamma2}, \rho_{yx1}, \rho_{yx2}, \rho_{x1x2}, \theta, \theta_1$ *and* θ_2 by keeping p, p_1 *and* p_2 each equal to 0.7. We changed the value of $C_y, C_{x1}, C_{x2}, C_{\gamma,} C_{\gamma1}, C_{\gamma2}$ between 0.1 to 0.5 with a step of 0.2. The values of θ, θ_1 *and* θ_2 were changed between 0 and 1 with a step of 0.5. The values ,ρ_{yx2} *and* ρ_{x1x2} were changed between 0.1 to 0.9 with a step of 0.2 and that of ρ_{yx1} was changed between -0.9 to +0.9 with a step of 0.2. Findings are given in Table 4.1.

Table 4.3: Descriptive statistics of the percent relative efficiency

Relative Efficiency	
Mean	542.31
Standard Error	1.41
Median	539.91
Standard Deviation	10.00
Sample Variance	99.92
Kurtosis	–1.47
Skewness	0.44
Range	24.00
Minimum	531.91
Maximum	555.91
Count	50

It is observed from Table 4.3 that the average percent relative efficiency is 542.31% with the standard deviation 10.00with median 539.91%, minimum of 531.91% and maximum of 555.91% (see Table 4.1). It has been observed that there are 50 cases where the percent relative efficiency of the proposed ratio estimator remains between 300 to 600. It has been observed that a choice of larger values of θ, θ_1 and θ_2 may lead to inefficient results, thus the choice of these values is must while using the proposed ratio method in actual practice.

We have also computed the percent relative efficiencies (PREs) of the proposed ratio-type estimator $\bar{y}^{**}_{Ratio}$ with respect to Tarray and Singh (2014) estimator $\bar{y}_{ST2}$ by using the formula:

$$PRE\left(\bar{y}_{(ST2)}, \bar{y}^{**}_{Ratio}\right) = \frac{V\left(\bar{y}_{(ST2)}\right)}{MSE\left(\bar{y}^{**}_{Ratio}\right)} \times 100 \tag{4.12}$$

We have also written the code to find the values of the parameter $C_y, C_{x_1}, C_{x_2}, C_{\gamma,} C_{\gamma 1}, C_{\gamma 2}, \rho_{yx_1}, \rho_{yx_2}$ *and* $\rho_{x_1 x_2}$ by keeping p, p_1 *and* p_2 each equal to 0.7. We changed the value of $C_y, C_{x_1}, C_{x_2}, C_{\gamma,} C_{\gamma_1}, C_{\gamma 2}$ between 0.1 to 0.5 with a step of 0.2. The values $,\rho_{yx_2}$ *and* $\rho_{x_1 x_2}$ were changed between 0.1 to 0.9 with a step of 0.2 and that of ρ_{yx_1} was changed between –0.9 to +0.9 with a step of 0.2. Findings are given in Table 4.2.

Table 4.4: Descriptive statistics of the percent relative efficiency

Relative Efficiency	
Mean	542.31
Standard Error	1.41
Median	539.91
Standard Deviation	10.00
Sample Variance	99.92
Kurtosis	–1.47
Skewness	0.44
Range	24.00
Minimum	531.91
Maximum	555.91
Count	50

Table 4.4 demonstrates that the average percent relative efficiency is 369.77% with the standard deviation 17.03 with median 368.97%, minimum of 348.97% and maximum of 396.97% (see Table 8). It has been observed that there are 50 cases where the percent relative efficiency of the proposed ratio estimator remains between 300 to 600.

5. Proposed Power Transformation Ratio Type Estimator Based on Tarray and Singh (2014) Model –II

A generalized version of the ratio-type estimator $\bar{y}^{**}_{Ratio}$ is given by:

$$\bar{y}^{**}_{Power} = \bar{y}^{**}\left(\frac{\bar{x}^{**}_1}{\bar{x}^{**}_2}\right)^{\alpha_1}, \tag{5.1}$$

where α_1 is a suitably chosen real constant. For example if $\alpha_1 = 0$ then the proposed power transformation ratio type estimator $\bar{y}^{**}_{Power}$ reduces to the Tarray and Singh (2014) estimator $\bar{y}_{ST2}$. If $\alpha_1 = 1$ then the proposed estimator $\bar{y}^{**}_{Power}$ reduces to the ratio estimator $\bar{y}^{**}_{Ratio}$.

Proceeding as earlier the bias and MSE of the estimator $\bar{y}^{**}_{Power}$ can be easily obtained and given in the following theorems.

Theorem 5.1. The bias in the proposed estimator $\bar{y}^{**}_{Power}$ is given by:

$$B\left(\bar{y}^{**}_{Power}\right)=\left(\frac{\alpha_1\bar{Y}}{n}\right)\Big[\rho_{yx1}C_yC_{x1}-\rho_{yx2}C_yC_{x2}-\alpha_1\rho_{x1x2}C_{x1}C_{x2}$$
$$+\frac{(\alpha_1-1)}{2}\left\{C^2_{x1}+\left(1+C^2_{x1}\right)C^{*2}_{p1}\right\}+\frac{(\alpha_1+1)}{2}\left\{C^2_{x2}+\left(1+C^2_{x2}\right)C^{*2}_{p2}\right\}\Big]. \qquad (5.2)$$

Theorem 5.2. The mean squared error of the estimator $\bar{y}^{**}_{Power}$ to the first degree of approximation is given by

$$MSE\left(\bar{y}^{**}_{Power}\right)=\left(\frac{\bar{Y}^2}{n}\right)\Big[C^2_y+\left(1+C^2_y\right)C^{*2}_{p0}+\alpha_1^2\Big\{C^2_{x1}+C^2_{x2}-2\rho_{x1x2}C_{x1}C_{x2}$$
$$+\left(1+C^2_{x1}\right)C^{*2}_{p1}+\left(1+C^2_{x2}\right)C^{*2}_{p2}\Big\}-2\alpha_1\left(\rho_{yx2}C_yC_{x2}-\rho_{yx1}C_yC_{x1}\right)\Big] \qquad (5.3)$$

The optimum value of α_1 and the resulting minimum MSE of the estimator $\bar{y}^{**}_{Power}$ are given in the following theorem.

Theorem 5.3. The optimum value of α (for which the MSE $\left(\bar{y}^{**}_{Power}\right)$ in (5.3) is minimum) and the minimum MSE of the estimator $\bar{y}^{**}_{Poewr}$ are respectively given by

$$\alpha_1=\frac{C_y\left(\rho_{yx2}C_{x2}-\rho_{yx1}C_{x1}\right)}{\left[C^2_{x1}+C^2_{x2}-2\rho_{x1x2}C_{x1}C_{x2}+\left(1+C^2_{x1}\right)C^{*2}_{p1}+\left(1+C^2_{x2}\right)C^{*2}_{p2}\right]}$$
$$=\alpha_{10}\ \text{(say)} \qquad (5.4)$$

and

$$Min.MSE\left(\bar{y}^{**}_{Power}\right)=\frac{\bar{Y}^2}{n}\Bigg[C^2_y+\left(1+C^2_y\right)C^{*2}_{p0}-$$
$$\frac{C^2_y\left(\rho_{yx2}C_{x2}-\rho_{yx1}C_{x1}\right)^2}{\left\{C^2_{x1}+C^2_{x2}-2\rho_{x1x2}C_{x1}C_{x2}+\left(1+C^2_{x1}\right)C^{*2}_{p1}+\left(1+C^2_{x2}\right)C^{*2}_{p2}\right\}}\Bigg] \qquad (5.5)$$

Proof is simple so omitted.

5.1 Efficiency comparison

5.1.1. When the scalar α_1 does not coincide exactly with its optimum value α_{10}

From (1.6) and (5.6)

$$MSE\left(\bar{y}^{**}_{Power}\right)-V\left(\bar{y}_{BBB}\right)=\left(\frac{\bar{Y}^2}{n}\right)\left[C^2_y+\left(1+C^2_y\right)\left(C^{*2}_{p0}-C^2_p\right)\right.$$

$$+\alpha_1^2\left\{C_{x1}^2+C_{x2}^2-2\rho_{x1x2}C_{x1}C_{x2}+\left(1+C_{x1}^2\right)C_{p1}^{*2}+\left(1+C_{x2}^2\right)C_{p2}^{*2}\right\}$$
$$-2\alpha_1\left(\rho_{yx_2}C_yC_{x_2}-\rho_{yx_1}C_yC_{x_1}\right)\Big]$$

which is less than zero if

$$\alpha_1^2A-2\alpha_1B+C<0$$

i.e. if

$$\frac{B_1-\sqrt{B_1^2-A_1C_1}}{A_1}<\alpha_1<\frac{B_1+\sqrt{B_1^2-A_1C_1}}{A_1} \tag{5.6}$$

where

$$A_1=\left[C_{x1}^2+C_{x2}^2-2\rho_{x1x2}C_{x1}C_{x2}+\left(1+C_{x1}^2\right)C_{p1}^{*2}+\left(1+C_{x2}^2\right)C_{p2}^{*2}\right],$$
$$B_1=\left[\rho_{yx_2}C_yC_{x_2}-\rho_{yx_1}C_yC_{x_1}\right],$$
$$C_1=\left[C_y^2+\left(1+C_y^2\right)\left(C_{p0}^{*2}-C_p^2\right)\right].$$

Thus, we state the following theorem.

Theorem 5.4. The proposed estimator $\bar{y}_{Power}^{**}$ is more efficient than the Bar-Lev et al's (2004) estimator $\bar{y}_{(BBB)}$ as long as the condition (5.6) is satisfied.

Further from (1.19) and (5.3) we have

$$V\left(\bar{y}_{ST2}\right)-MSE\left(\bar{y}_{Power}^{**}\right)=\frac{\bar{Y}^2}{n}\Big[2\alpha_1C_y\left(\rho_{yx_2}C_{x_2}-\rho_{yx_1}C_{x_1}\right)$$
$$-\alpha_1^2\left\{C_{x1}^2+C_{x2}^2-2\rho_{x1x2}C_{x1}C_{x2}+\left(1+C_{x1}^2\right)C_{p1}^{*2}+\left(1+C_{x2}^2\right)C_{p2}^{*2}\right\}\Big]$$

which is non negative if

$$\alpha_1\left(2\alpha_{10}-\alpha_1\right)>0$$

i.e. if $\alpha_1\left(\alpha_1-2\alpha_{10}\right)<0$

i.e. if $\left|\left(\alpha_1-\alpha_{10}\right)\right|<\left|\alpha_{10}\right|$ (5.7)

where α_{10} is given by (5.4) .

Thus, we state the following theorem.

Theorem 5.5. The proposed estimator $\bar{y}_{Poewr}^{**}$ is more efficient than the Tarray and Singh's (2014) estimator $\bar{y}_{(ST1)}$ as long as the condition (5.7) is satisfied.

Further from (4.8) and (5.3) we have

$$MSE\left(\bar{y}_{Ratio}^{**}\right)-MSE\left(\bar{y}_{Power}^{**}\right)=\left(\frac{\bar{Y}^2}{n}\right)\left[\left(1-\alpha_1^2\right)A_1-2\left(1-\alpha_1\right)B_1\right]$$
$$=\left(\frac{\bar{Y}^2A_1}{n}\right)\left[\left(1-\alpha_1^2\right)-2\left(1-\alpha_1\right)\frac{B_1}{A_1}\right]$$

$$=\left(\frac{\bar{Y}^2 A_1}{n}\right)\left[\left(1-\alpha_1^2\right)-2\left(1-\alpha_1\right)\alpha_{10}\right]$$

which is positive if

$$\left(1-\alpha_1^2-2\alpha_{10}+2\alpha_1\alpha_{10}\right)>0$$

i.e. if $\left(\alpha_1^2-2\alpha_1\alpha_{10}-1+2\alpha_{10}\right)<0$

i.e. if $\left(\alpha_1-\alpha_{10}\right)^2<\left(1-\alpha_{10}\right)^2$

i.e. if $\left|\left(\alpha_1-\alpha_{10}\right)\right|<\left|\alpha_{10}\right|$ (5.8)

and $\alpha_{10}=\dfrac{B_1}{A_1}$ is same as given by (5.4).

Thus, we established the following theorem.

Theorem 5.6. The proposed estimator $\bar{y}^{**}_{Power}$ is more efficient than the proposed ratio type estimator $\bar{y}^{**}_{Ratio}$ as long as the condition (5.8) is satisfied.

5.1.2 When the optimum value α_{10} of the scalar α_1 is exactly known

$$V\left(\bar{y}_{(BBB)}\right)-Min.MSE\left(\bar{y}^{**}_{Power}\right)=\bar{Y}^2\left[\left(1+C_y^2\right)\left(C_p^2-C_{p0}^{*2}\right)\right.$$
$$\left.+\frac{C_y^2\left[\rho_{yx_2}C_{x_2}-\rho_{yx_1}C_{x_2}\right]^2}{\left[C_{x_1}^2+C_{x_2}^2-2\rho_{x_1x_2}C_{x_1}C_{x_2}+\left(1+C_{x_1}^2\right)C_{p1}^{*2}+\left(1+C_{x_2}^2\right)C_{p2}^{*2}\right]}\right]$$

which is non negative if

$$C_p^2>C_{p0}^{*2} \qquad (5.9)$$

Thus, we state the following theorem.

Theorem 5.7. The proposed power transformation ratio-type estimator $\bar{y}^{**}_{power}$ (at its optimum condition *i.e.* when $\alpha_1=\alpha_{10}$) is better than Bar-Lev et al (2004) estimator $\bar{y}_{(BBB)}$ if $C_p^2>C_{p0}^{*2}$.

Further from (1.19) and (5.5) we have

$$V\left(\bar{y}_{(ST2)}\right)-Min.MSE\left(\bar{y}^{**}_{Power}\right)$$
$$=\frac{S_Y^2}{n}\frac{\left(\rho_{yx_2}C_{x_2}-\rho_{yx_1}C_{x_1}\right)}{\left[C_{x_1}^2+C_{x_2}^2-2\rho_{x_1x_2}C_{x_1}C_{x_2}+\left(1+C_{x_1}^2\right)C_{p1}^{*2}+\left(1+C_{x_2}^2\right)C_{p2}^{*2}\right]}$$

$$>0 \text{ provided } \rho_{yx_1}C_{x_1}\neq\rho_{yx_2}C_{x_2} \qquad (5.10)$$

Thus, we state the following theorem.

Theorem 5.8. The proposed power transformation ratio-type estimator $\bar{y}^{**}_{power}$ (at its optimum condition *i.e.* when $\alpha_1 = \alpha_{10}$) is better than Tarray and Singh's (2014) estimator $\bar{y}_{(ST2)}$ unless $\rho_{yx_1} C_{x_1} \neq \rho_{yx_2} C_{x_2}$, the case where both the estimator $\bar{y}_{(ST2)}$ and $\bar{y}^{**}_{power}$ are equally efficient.

Next from (4.8) and (5.5) we have

$$MSE\left(\bar{y}^{**}_{Ratio}\right) - Min.MSE\left(\bar{y}^{**}_{Power}\right) = \frac{\bar{Y}^2}{n} \frac{\left(A_1 - B_1\right)^2}{A_1} > 0 \text{ provided } A_1 \neq B_1. \quad (5.11)$$

Thus, we state the following theorem

Theorem 5.9. The proposed estimator $\bar{y}^{**}_{power}$ (at its optimum condition i.e. when $\alpha_1 = \alpha_{10}$) is more efficient than the proposed ratio type estimator $\bar{y}^{**}_{Ratio}$ unless $A_1 = B_1$, the case where both estimators $\bar{y}^{**}_{Ratio}$ and $\bar{y}^{**}_{power}$ are equally efficient.

5.2 Relative efficiency of the power transformation ratio type estimator

To see the performance of the proposed estimator $\bar{y}^{**}_{power}$ we computed the percent relative efficiency of the proposed estimator $\bar{y}^{**}_{power}$ with respect to Bar-Lev et al (2004) estimator $\bar{y}_{(BBB)}$ by using the formula

$$PRE\left(\bar{y}_{(BBB)}, \bar{y}^{**}_{Power}\right) = \frac{V\left(\bar{y}_{(BBB)}\right)}{MSE\left(\bar{y}^{**}_{Power}\right)} \times 100$$

(5.12)

We have also written the code to find the values of the parameter $C_y, C_{x_1}, C_{x_2}, C_{\gamma}, C_{\gamma_1}, C_{\gamma_2}, \rho_{yx_1}, \rho_{yx_2}, \rho_{x_1x_2}, \theta, \theta_1$ *and* θ_2 by keeping p, p_1 *and* p_2 each equal to 0.7. We changed the value of $C_y, C_{x_1}, C_{x_2}, C_{\gamma}, C_{\gamma_1}, C_{\gamma_2}$ between 0.1 to 0.5 with a step of 0.2. The values of θ, θ_1 *and* θ_2 were changed between 0 and 1 with a step of 0.5. The values ,ρ_{yx_2} *and* $\rho_{x_1x_2}$ were changed between 0.1 to 0.9 with a step of 0.2 and that of ρ_{yx_1} was changed between -0.9 to +0.9 with a step of 0.2. Findings are displayed in Table 5.1

Table 5.3: Descriptive statistics of the percent relative efficiency

Relative Efficiency	
Mean	496.26
Standard Error	14.66
Median	555.07
Standard	103.68

Deviation	
Sample Variance	10749.68
Kurtosis	–0.98
Skewness	–0.89
Range	296.27
Minimum	300.66
Maximum	596.93
Count	50

It is observed from the Table 5.3that the average percent relative efficiency is 496.26% with the standard deviation 103.68 with median 555.07%, minimum of 300.66% and maximum of 596.93% (see Table 5.1). It has been observed that there are 50 cases where the percent relative efficiency of the proposed ratio estimator remains between 300 to 600. It has been observed that a choice of larger values of θ, θ_1 and θ_2 may lead to inefficient results, thus the choice of these values is must while using the proposed ratio method in actual practice.

We have further computed the percent relative efficiency of the proposed estimator $\bar{y}^{**}_{power}$ with respect to Tarray and Singh (2014) estimator $\bar{y}_{(ST2)}$ by using the formula:

$$PRE\left(\bar{y}_{(ST2)}, \bar{y}^{**}_{Power}\right) = \frac{V\left(\bar{y}_{(ST2)}\right)}{MSE\left(\bar{y}^{**}_{Power}\right)} \times 100 \tag{5.12}$$

We have also written the code to find the values of the parameter $C_y, C_{x_1}, C_{x_2}, C_{\gamma,} C_{\gamma_1}, C_{\gamma_2}, \rho_{yx_1}, \rho_{yx_2}$ *and* $\rho_{x_1x_2}$ by keeping p, p_1 *and* p_2 each equal to 0.7. We changed the value of $C_y, C_{x_1}, C_{x_2}, C_{\gamma,} C_{\gamma_1}, C_{\gamma_2}$ between 0.1 to 0.5 with a step of 0.2. The values ,ρ_{yx_2} *and* $\rho_{x_1x_2}$ were changed between 0.1 to 0.9 with a step of 0.2 and that of ρ_{yx_1} was changed between -0.9 to +0.9 with a step of 0.2.

Findings are presented in Table 5.2.

Table 5.4: Descriptive statistics of the percent relative efficiency

Relative Efficiency	
Mean	408.09
Standard Error	13.94
Median	379.79
Standard Deviation	98.62
Sample Variance	9727.28
Kurtosis	–1.32

Skewness	0.43
Range	294.50
Minimum	300.37
Maximum	594.87
Count	50

Table 5.4 exhibits that the average percent relative efficiency is 408.09% with the standard deviation 98.62 with median 379.79%, minimum of 300.37% and maximum of 594.87% (see Table 5.2). It has been observed that there are 50 cases where the percent relative efficiency of the proposed ratio estimator remains between 300 to 600.

6. Conclusion

In this paper, taking clue from Odumade and Singh (2010), two new ratio-type and power transformation ratio-type estimators have been proposed and compared to BBB model and Tarray and Singh (2014) randomized response model. In the case of scrambled response unlike the repeated substitution method due to Srivastava (1967) and Garcia and Cebrian (1996) it has been observed in general enormity of percent relative efficiency of ratio estimator remains better than the power transformation ratio-type estimator.

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Annexure

Table 1.1: The Percent Relative Efficiency of the Tarray and Singh (2014) estimator $\bar{y}_{(ST1)}$, with respect to Eichhorn and Hayre's (1983) estimator $\bar{y}_{(EH)}$

θ	P	C_γ	C_x	PRE
1.00	0.1	0.10	0.10	8996.82
1.50	0.1	0.10	0.15	5632.14
2.00	0.1	0.10	0.20	3753.53
2.50	0.1	0.10	0.40	1296.88
1.00	0.1	0.25	0.10	2493.11
1.50	0.1	0.25	0.15	2150.34
2.00	0.1	0.25	0.20	1816.14
2.50	0.1	0.25	0.40	964.46
1.00	0.1	0.50	0.10	714.55
1.50	0.1	0.50	0.15	687.20
2.00	0.1	0.50	0.20	653.70
2.50	0.1	0.50	0.40	511.73
1.00	0.1	0.75	0.10	339.61
1.50	0.1	0.75	0.15	334.65
2.00	0.1	0.75	0.20	328.22
2.50	0.1	0.75	0.40	295.81
1.00	0.1	1.00	0.10	204.52
1.50	0.1	1.00	0.15	203.28
2.00	0.1	1.00	0.20	201.65
2.50	0.1	1.00	0.40	192.76
1.00	0.1	1.25	0.10	141.29
1.50	0.1	1.25	0.15	140.98
2.00	0.1	1.25	0.20	140.56
2.50	0.1	1.25	0.40	138.18
1.00	0.1	1.50	0.10	106.76
1.50	0.1	1.50	0.15	106.72
2.00	0.1	1.50	0.20	106.68
2.50	0.1	1.50	0.40	106.40

Table 1.2: The Percent Relative Efficiency of the Tarray and Singh (2014) estimator $\bar{y}_{(ST1)}$ with respect to Bar-Lev et al's (2004) estimator $\bar{y}_{(BBB)}$

θ	P	C_γ	C_x	PRE
1.00	0.1	0.10	0.10	9472.82
1.50	0.1	0.10	0.15	4257.56
2.00	0.1	0.10	0.20	2479.42
2.50	0.1	0.10	0.40	1038.23
1.00	0.1	0.25	0.10	2728.67
1.50	0.1	0.25	0.15	1955.97
2.00	0.1	0.25	0.20	1458.49
2.50	0.1	0.25	0.40	811.33
1.00	0.1	0.50	0.10	790.60
1.50	0.1	0.50	0.15	683.47
2.00	0.1	0.50	0.20	603.03
2.50	0.1	0.50	0.40	462.48
1.00	0.1	0.75	0.10	376.62
1.50	0.1	0.75	0.15	340.49
2.00	0.1	0.75	0.20	315.93
2.50	0.1	0.75	0.40	277.17
1.00	0.1	1.00	0.10	227.00
1.50	0.1	1.00	0.15	208.62
2.00	0.1	1.00	0.20	197.43
2.50	0.1	1.00	0.40	183.84
1.00	0.1	1.25	0.10	156.88
1.50	0.1	1.25	0.15	145.28
2.00	0.1	1.25	0.20	138.77
2.50	0.1	1.25	0.40	133.05
1.00	0.1	1.50	0.10	118.56
1.50	0.1	1.50	0.15	110.23
2.00	0.1	1.50	0.20	105.81
2.50	0.1	1.50	0.40	103.01

Table 2.1: The Percent Relative Efficiency of the proposed estimator $\bar{y}^*_{Ratio}$ with respect to Bar-Lev et al's (2004) estimator $\bar{y}_{BBB}$ for the different choice of the parameters with $p = p_1 = p_2 = 0.7$

C_γ	$C_{\gamma 1}$	$C_{\gamma 2}$	C_y	C_{x1}	C_{x2}	ρ_{yx1}	ρ_{yx2}	ρ_{x1x2}	θ	θ_1	θ_2	PRE
0.3	0.3	0.3	0.3	0.3	0.1	-0.9	0.1	0.1	0.5	0.5	0.5	570.80
0.1	0.1	0.1	0.1	0.3	0.1	-0.9	0.3	0.1	0	0	0	524.70
0.1	0.1	0.1	0.1	0.3	0.3	-0.9	0.1	0.1	0	0	0	534.29
0.3	0.3	0.3	0.3	0.3	0.3	-0.9	0.1	0.1	0.5	0.5	0.5	580.39
0.1	0.1	0.1	0.1	0.3	0.5	-0.9	0.1	0.1	0	0	0	553.47
0.3	0.3	0.3	0.3	0.3	0.5	-0.9	0.1	0.1	0.5	0.5	0.5	599.57
0.5	0.5	0.5	0.5	0.5	0.5	-0.9	0.1	0.1	1	1	1	313.99
0.1	0.1	0.1	0.1	0.3	0.1	-0.7	0.1	0.1	0	0	0	524.70
0.3	0.3	0.3	0.3	0.3	0.1	-0.7	0.1	0.1	0.5	0.5	0.5	570.80
0.1	0.1	0.1	0.1	0.3	0.3	-0.7	0.1	0.1	0	0	0	534.29
0.3	0.3	0.3	0.3	0.3	0.3	-0.7	0.1	0.1	0.5	0.5	0.5	580.39
0.1	0.1	0.1	0.1	0.3	0.5	-0.7	0.1	0.1	0	0	0	553.47
0.3	0.3	0.3	0.3	0.3	0.5	-0.7	0.1	0.1	0.5	0.5	0.5	599.57
0.5	0.5	0.5	0.5	0.5	0.5	-0.7	0.1	0.1	1	1	1	313.99
0.1	0.1	0.1	0.1	0.3	0.1	-0.5	0.1	0.1	0	0	0	524.70
0.3	0.3	0.3	0.3	0.3	0.1	-0.5	0.1	0.1	0.5	0.5	0.5	570.80
0.1	0.1	0.1	0.1	0.3	0.3	-0.5	0.1	0.1	0	0	0	534.29
0.3	0.3	0.3	0.3	0.3	0.3	-0.5	0.1	0.1	0.5	0.5	0.5	580.39
0.1	0.1	0.1	0.1	0.3	0.5	-0.5	0.1	0.1	0	0	0	553.47
0.3	0.3	0.3	0.3	0.3	0.5	-0.5	0.9	0.1	0.5	0.5	0.5	599.57
0.5	0.5	0.5	0.5	0.5	0.5	-0.5	0.1	0.1	1	1	1	313.99
0.1	0.1	0.1	0.1	0.3	0.1	-0.3	0.1	0.1	0	0	0	524.70
0.3	0.3	0.3	0.3	0.3	0.1	-0.3	0.1	0.1	0.5	0.5	0.5	570.80
0.1	0.1	0.1	0.1	0.3	0.3	-0.3	0.3	0.1	0	0	0	534.29
0.3	0.3	0.3	0.3	0.3	0.3	-0.3	0.3	0.1	0.5	0.5	0.5	580.39
0.1	0.1	0.1	0.1	0.3	0.5	-0.3	0.9	0.9	0	0	0	553.47
0.3	0.3	0.3	0.3	0.3	0.5	-0.3	0.9	0.9	0.5	0.5	0.5	599.57
0.5	0.5	0.5	0.5	0.5	0.5	-0.3	0.9	0.9	1	1	1	313.99
0.1	0.1	0.1	0.1	0.3	0.1	-0.1	0.9	0.9	0	0	0	524.70
0.3	0.3	0.3	0.3	0.3	0.1	-0.1	0.9	0.9	0.5	0.5	0.5	570.80
0.1	0.1	0.1	0.1	0.3	0.3	-0.1	0.7	0.9	0	0	0	534.29
0.3	0.3	0.3	0.3	0.3	0.3	-0.1	0.7	0.9	0.5	0.5	0.5	580.39
0.1	0.1	0.1	0.1	0.3	0.5	-0.1	0.3	0.3	0	0	0	553.47
0.3	0.3	0.3	0.3	0.3	0.5	-0.1	0.3	0.3	0.5	0.5	0.5	599.57
0.5	0.5	0.5	0.5	0.5	0.5	-0.1	0.3	0.3	1	1	1	313.99
0.1	0.1	0.1	0.1	0.3	0.1	0.1	0.1	0.3	0	0	0	524.70
0.3	0.3	0.3	0.3	0.3	0.1	0.1	0.1	0.3	0.5	0.5	0.5	570.80
0.3	0.3	0.3	0.3	0.3	0.3	0.1	0.5	0.3	0.5	0.5	0.5	580.39
0.1	0.1	0.1	0.1	0.3	0.3	0.1	0.7	0.3	0	0	0	534.29
0.1	0.1	0.1	0.1	0.3	0.5	0.1	0.5	0.1	0	0	0	553.47
0.3	0.3	0.3	0.3	0.3	0.5	0.1	0.5	0.1	0.5	0.5	0.5	599.57
0.5	0.5	0.5	0.5	0.5	0.5	0.1	0.5	0.1	1	1	1	313.99
0.1	0.1	0.1	0.1	0.3	0.1	0.3	0.1	0.3	0	0	0	524.70
0.3	0.3	0.3	0.3	0.3	0.1	0.3	0.1	0.3	0.5	0.5	0.5	570.80
0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.1	0.5	0.5	0.5	580.39
0.1	0.1	0.1	0.1	0.3	0.3	0.3	0.5	0.1	0	0	0	534.29
0.5	0.5	0.5	0.5	0.5	0.5	0.3	0.9	0.1	1	1	1	313.99
0.1	0.1	0.1	0.1	0.3	0.5	0.3	0.1	0.3	0	0	0	553.47
0.3	0.3	0.3	0.3	0.3	0.5	0.3	0.1	0.3	0.5	0.5	0.5	599.57
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.1	0.1	1	1	1	313.99
0.1	0.1	0.1	0.1	0.3	0.5	0.5	0.3	0.1	0	0	0	553.47
0.3	0.3	0.3	0.3	0.3	0.5	0.5	0.3	0.1	0.5	0.5	0.5	599.57
0.1	0.1	0.1	0.1	0.3	0.3	0.5	0.9	0.7	0	0	0	534.29
0.3	0.3	0.3	0.3	0.3	0.3	0.5	0.9	0.7	0.5	0.5	0.5	580.39
0.1	0.1	0.1	0.1	0.3	0.1	0.5	0.9	0.7	0	0	0	524.70
0.3	0.3	0.3	0.3	0.3	0.1	0.5	0.9	0.7	0.5	0.5	0.5	570.80
0.1	0.1	0.1	0.1	0.3	0.1	0.7	0.9	0.1	0	0	0	524.70
0.3	0.3	0.3	0.3	0.3	0.1	0.7	0.9	0.1	0.5	0.5	0.5	570.80
0.1	0.1	0.1	0.1	0.3	0.3	0.7	0.3	0.3	0	0	0	534.29

Table 2.2: The Percent Relative Efficiency of the proposed ratio estimator $\bar{y}^{*}_{Ratio}$ with respect to Tarray and Singh's (2014) $\bar{y}_{ST1}$ estimator for different choices of the parameters with $p = p_1 = p_2 = 0.7$

C_γ	$C_{\gamma 1}$	$C_{\gamma 2}$	C_y	C_{x1}	C_{x2}	ρ_{yx1}	ρ_{yx2}	ρ_{x1x2}	PRE
0.1	0.1	0.1	0.1	0.3	0.1	-0.9	0.1	0.1	355.67
0.5	0.5	0.5	0.5	0.5	0.1	-0.9	0.7	0.5	315.54
0.5	0.5	0.5	0.5	0.3	0.1	-0.9	0.1	0.1	450.47
0.3	0.3	0.3	0.3	0.3	0.3	-0.9	0.5	0.5	302.12
0.1	0.1	0.1	0.1	0.3	0.1	-0.9	0.3	0.1	533.30
0.3	0.3	0.3	0.3	0.3	0.1	-0.9	0.3	0.1	576.75
0.3	0.3	0.3	0.3	0.5	0.5	-0.9	0.1	0.5	307.58
0.3	0.3	0.3	0.3	0.3	0.3	-0.5	0.3	0.1	587.94
0.5	0.5	0.5	0.5	0.5	0.1	-0.9	0.3	0.1	315.54
0.3	0.3	0.3	0.3	0.3	0.5	-0.9	0.9	0.7	321.30
0.5	0.5	0.5	0.5	0.3	0.1	-0.9	0.5	0.1	343.87
0.3	0.3	0.3	0.3	0.3	0.3	-0.9	0.7	0.5	302.12
0.3	0.3	0.3	0.3	0.5	0.5	-0.9	0.3	0.5	307.58
0.5	0.5	0.5	0.5	0.5	0.1	-0.9	0.3	0.9	315.54
0.3	0.3	0.3	0.3	0.3	0.5	-0.9	0.1	0.9	321.30
0.3	0.3	0.3	0.3	0.3	0.3	-0.9	0.1	0.1	587.94
0.3	0.3	0.3	0.3	0.3	0.3	-0.3	0.3	0.1	587.94
0.3	0.3	0.3	0.3	0.3	0.1	0.9	0.9	0.9	576.75
0.3	0.3	0.3	0.3	0.3	0.1	0.7	0.7	0.9	576.75
0.3	0.3	0.3	0.3	0.5	0.5	0.9	0.5	0.1	307.58
0.3	0.3	0.3	0.3	0.3	0.3	0.1	0.3	0.3	587.94
0.5	0.5	0.5	0.5	0.5	0.5	0.9	0.5	0.1	349.11
0.3	0.3	0.3	0.3	0.3	0.5	-0.9	0.7	0.1	321.30
0.5	0.5	0.5	0.5	0.3	0.5	-0.9	0.7	0.1	372.65
0.1	0.1	0.1	0.1	0.3	0.5	-0.9	0.7	0.1	386.04
0.3	0.3	0.3	0.3	0.3	0.5	-0.9	0.7	0.1	417.64
0.5	0.5	0.5	0.5	0.3	0.5	-0.9	0.7	0.1	480.84
0.3	0.3	0.3	0.3	0.3	0.3	-0.1	0.1	0.3	587.94
0.1	0.1	0.1	0.1	0.3	0.5	-0.9	0.7	0.1	566.87
0.3	0.3	0.3	0.3	0.3	0.3	-0.5	0.7	0.3	302.12
0.5	0.5	0.5	0.5	0.3	0.3	-0.5	0.7	0.3	353.46
0.1	0.1	0.1	0.1	0.3	0.3	-0.5	0.7	0.3	365.80
0.3	0.3	0.3	0.3	0.3	0.3	-0.5	0.7	0.3	397.39
0.5	0.5	0.5	0.5	0.3	0.3	-0.3	0.7	0.3	460.59
0.1	0.1	0.1	0.1	0.3	0.3	-0.3	0.7	0.3	544.49
0.3	0.3	0.3	0.3	0.3	0.3	-0.3	0.7	0.3	587.94
0.5	0.5	0.5	0.5	0.5	0.3	-0.3	0.7	0.3	326.73
0.5	0.5	0.5	0.5	0.5	0.1	0.5	0.3	0.9	315.54
0.5	0.5	0.5	0.5	0.3	0.1	0.7	0.3	0.9	343.87
0.1	0.1	0.1	0.1	0.3	0.1	0.7	0.3	0.9	355.67
0.3	0.3	0.3	0.3	0.3	0.1	0.7	0.5	0.5	387.27
0.3	0.3	0.3	0.3	0.3	0.3	-0.7	0.1	0.1	587.94
0.5	0.5	0.5	0.5	0.3	0.1	0.7	0.5	0.5	450.47
0.1	0.1	0.1	0.1	0.3	0.1	0.7	0.5	0.5	533.30
0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.5	0.3	587.94
0.3	0.3	0.3	0.3	0.3	0.1	0.7	0.5	0.5	576.75
0.1	0.1	0.1	0.1	0.3	0.1	-0.7	0.1	0.1	355.67
0.3	0.3	0.3	0.3	0.3	0.1	-0.7	0.1	0.1	387.27
0.5	0.5	0.5	0.5	0.3	0.1	-0.7	0.1	0.1	450.47

Table 3.1: The Percent Relative Efficiency of the proposed power transformation ratio-type estimator $\bar{y}^{*}_{Power}$ with respect to Bar-Lev et al's (2004) estimator $\bar{y}_{BBB}$ for different choices of the parameters with $p = p_1 = p_2 = 0.7$

C_γ	$C_{\gamma 1}$	$C_{\gamma 2}$	C_y	C_{x1}	C_{x2}	ρ_{yx1}	ρ_{yx2}	ρ_{x1x2}	θ	θ_1	θ_2	α_0	PRE
0.3	0.3	0.3	0.3	0.1	0.1	-0.5	0.1	0.1	0.5	0.5	0.5	1.80	553.96
0.5	0.5	0.5	0.3	0.3	0.1	-0.5	0.1	0.1	1	1	1	0.53	553.28
0.3	0.3	0.3	0.3	0.5	0.1	-0.5	0.1	0.1	0.5	0.5	0.5	0.31	553.15
0.5	0.5	0.5	0.3	0.1	0.1	-0.3	0.1	0.1	1	1	1	1.20	552.16
0.3	0.3	0.3	0.3	0.3	0.1	-0.3	0.1	0.1	0.5	0.5	0.5	0.33	551.72
0.5	0.5	0.5	0.3	0.5	0.1	-0.3	0.1	0.1	1	1	1	0.19	551.64
0.3	0.3	0.3	0.3	0.1	0.1	-0.1	0.1	0.1	0.5	0.5	0.5	0.60	551.08
0.5	0.5	0.5	0.3	0.3	0.1	-0.1	0.1	0.1	1	1	1	0.13	550.88
0.3	0.3	0.3	0.3	0.5	0.1	-0.1	0.1	0.1	0.5	0.5	0.5	0.07	550.85
0.5	0.5	0.5	0.3	0.1	0.1	0.1	0.1	0.1	1	1	1	0.00	550.72
0.3	0.3	0.3	0.3	0.3	0.1	0.1	0.1	0.1	0.5	0.5	0.5	-0.07	550.76
0.5	0.5	0.5	0.3	0.5	0.1	0.1	0.1	0.1	1	1	1	-0.05	550.78
0.3	0.3	0.3	0.3	0.3	0.1	0.3	0.1	0.1	0.5	0.5	0.5	-0.27	551.36
0.5	0.5	0.5	0.3	0.5	0.1	0.3	0.1	0.1	1	1	1	-0.17	551.42
0.3	0.3	0.3	0.3	0.3	0.1	0.5	0.1	0.1	0.5	0.5	0.5	-0.47	552.68
0.5	0.5	0.5	0.3	0.5	0.1	0.5	0.1	0.1	1	1	1	-0.29	552.79
0.3	0.3	0.3	0.3	0.3	0.1	0.7	0.1	0.1	0.5	0.5	0.5	-0.67	554.72
0.5	0.5	0.5	0.3	0.5	0.1	0.7	0.1	0.1	1	1	1	-0.41	554.88
0.3	0.3	0.3	0.3	0.3	0.1	0.9	0.1	0.1	0.5	0.5	0.5	-0.87	557.48
0.3	0.3	0.3	0.3	0.5	0.1	0.9	0.1	0.1	0.5	0.5	0.5	-0.53	557.69
0.3	0.3	0.3	0.3	0.1	0.1	-0.9	0.3	0.1	0.5	0.5	0.5	3.60	563.68
0.3	0.3	0.3	0.5	0.1	0.1	-0.9	0.3	0.1	0.5	0.5	0.5	6.00	301.14
0.3	0.3	0.3	0.3	0.5	0.1	-0.9	0.3	0.1	0.5	0.5	0.5	0.58	559.01
0.3	0.3	0.3	0.3	0.5	0.1	-0.7	0.3	0.1	0.5	0.5	0.5	0.46	555.92
0.3	0.3	0.3	0.3	0.5	0.1	-0.5	0.3	0.1	0.5	0.5	0.5	0.34	553.54
0.3	0.3	0.3	0.3	0.5	0.1	-0.3	0.3	0.1	0.5	0.5	0.5	0.22	551.88
0.3	0.3	0.3	0.3	0.5	0.1	-0.1	0.3	0.1	0.5	0.5	0.5	0.10	550.95
0.3	0.3	0.3	0.3	0.5	0.1	0.1	0.3	0.1	0.5	0.5	0.5	-0.02	550.73
0.3	0.3	0.3	0.3	0.5	0.1	0.3	0.3	0.1	0.5	0.5	0.5	-0.14	551.24
0.3	0.3	0.3	0.3	0.5	0.1	0.5	0.3	0.1	0.5	0.5	0.5	-0.26	552.46
0.3	0.3	0.3	0.3	0.5	0.1	0.7	0.3	0.1	0.5	0.5	0.5	-0.38	554.40
0.3	0.3	0.3	0.3	0.5	0.1	0.9	0.3	0.1	0.5	0.5	0.5	-0.50	557.07
0.3	0.3	0.3	0.3	0.1	0.1	-0.9	0.5	0.1	0.5	0.5	0.5	4.20	568.36
0.3	0.3	0.3	0.5	0.1	0.1	-0.9	0.5	0.1	0.5	0.5	0.5	7.00	314.14
0.5	0.5	0.5	0.3	0.3	0.1	-0.9	0.5	0.1	1	1	1	1.07	560.96
0.3	0.3	0.3	0.5	0.5	0.3	-0.7	0.9	0.1	0.5	0.5	0.5	1.24	311.58
0.5	0.5	0.5	0.5	0.1	0.3	-0.5	0.9	0.1	1	1	1	16.00	529.14
0.3	0.3	0.3	0.5	0.1	0.3	-0.5	0.9	0.1	0.5	0.5	0.5	16.00	364.19
0.5	0.5	0.5	0.5	0.1	0.3	-0.5	0.9	0.1	1	1	1	16.00	377.35
0.3	0.3	0.3	0.3	0.5	0.3	-0.5	0.9	0.1	0.5	0.5	0.5	0.62	568.45
0.5	0.5	0.5	0.5	0.5	0.3	-0.5	0.9	0.1	1	1	1	1.04	300.18
0.3	0.3	0.3	0.3	0.1	0.5	-0.9	0.1	0.1	0.5	0.5	0.5	4.20	592.36
0.5	0.5	0.5	0.5	0.1	0.5	-0.9	0.1	0.1	1	1	1	7.00	338.14
0.3	0.3	0.3	0.3	0.3	0.5	-0.9	0.1	0.1	0.5	0.5	0.5	1.07	584.96
0.5	0.5	0.5	0.5	0.3	0.5	-0.9	0.1	0.1	1	1	1	1.78	317.58
0.3	0.3	0.3	0.3	0.5	0.5	-0.9	0.1	0.1	0.5	0.5	0.5	0.60	583.72
0.5	0.5	0.5	0.5	0.1	0.3	-0.7	0.7	0.1	1	1	1	14.00	469.14
0.3	0.3	0.3	0.5	0.1	0.3	-0.7	0.7	0.1	0.5	0.5	0.5	14.00	304.19
0.5	0.5	0.5	0.5	0.1	0.3	-0.7	0.7	0.1	1	1	1	14.00	317.35
0.3	0.3	0.3	0.3	0.1	0.1	-0.5	0.1	0.1	0.5	0.5	0.5	1.80	553.96

Table 3.2: The Percent Relative Efficiency of the proposed power transformation ratio-type estimator $\bar{y}^{*}_{Power}$ with respect to Tarray and Singh (2014) estimator $\bar{y}_{ST1}$ for different choices of the parameters with $p = p_1 = p_2 = 0.7$

C_γ	$C_{\gamma 1}$	$C_{\gamma 2}$	C_y	C_{x1}	C_{x2}	ρ_{yx1}	ρ_{yx2}	ρ_{x1x2}	α_0	PRE
0.3	0.3	0.3	0.3	0.5	0.1	0.3	0.1	0.1	-0.17	537.18
0.3	0.3	0.3	0.3	0.1	0.1	0.5	0.1	0.1	-1.20	361.90
0.3	0.3	0.3	0.3	0.1	0.1	0.5	0.1	0.1	-1.20	537.91
0.3	0.3	0.3	0.3	0.3	0.1	0.5	0.1	0.1	-0.47	362.42
0.3	0.3	0.3	0.3	0.3	0.1	0.5	0.1	0.1	-0.47	538.43
0.3	0.3	0.3	0.3	0.5	0.1	0.5	0.1	0.1	-0.29	362.53
0.3	0.3	0.3	0.3	0.5	0.1	0.5	0.1	0.1	-0.29	538.55
0.3	0.3	0.3	0.3	0.1	0.1	0.7	0.1	0.1	-1.80	363.70
0.3	0.3	0.3	0.3	0.1	0.1	0.7	0.1	0.1	-1.80	539.71
0.3	0.3	0.3	0.3	0.3	0.1	0.7	0.1	0.1	-0.67	364.46
0.3	0.3	0.3	0.3	0.3	0.1	0.7	0.1	0.1	-0.67	540.47
0.3	0.3	0.3	0.3	0.5	0.1	0.7	0.1	0.1	-0.41	364.62
0.3	0.3	0.3	0.3	0.5	0.1	0.7	0.1	0.1	-0.41	540.63
0.3	0.3	0.3	0.3	0.1	0.1	0.9	0.1	0.1	-2.40	366.22
0.3	0.3	0.3	0.3	0.1	0.1	0.9	0.1	0.1	-2.40	542.23
0.3	0.3	0.3	0.3	0.3	0.1	0.9	0.1	0.1	-0.87	367.22
0.3	0.3	0.3	0.3	0.3	0.1	0.9	0.1	0.1	-0.87	543.23
0.3	0.3	0.3	0.3	0.5	0.1	0.9	0.1	0.1	-0.53	367.43
0.3	0.3	0.3	0.3	0.5	0.1	0.9	0.1	0.1	-0.53	543.44
0.3	0.3	0.3	0.3	0.1	0.1	-0.9	0.3	0.1	3.60	373.42
0.3	0.3	0.3	0.3	0.1	0.1	-0.9	0.3	0.1	3.60	549.43
0.5	0.5	0.5	0.5	0.1	0.1	-0.9	0.3	0.1	6.00	311.26
0.3	0.3	0.3	0.3	0.5	0.1	-0.9	0.3	0.1	0.58	368.75
0.3	0.3	0.3	0.3	0.5	0.1	-0.9	0.3	0.1	0.58	544.77
0.3	0.3	0.3	0.3	0.1	0.1	-0.7	0.3	0.1	3.00	369.46
0.3	0.3	0.3	0.3	0.1	0.1	-0.7	0.3	0.1	3.00	545.47
0.3	0.3	0.3	0.3	0.5	0.1	-0.7	0.3	0.1	0.46	365.66
0.3	0.3	0.3	0.3	0.5	0.1	-0.7	0.3	0.1	0.46	541.67
0.3	0.3	0.3	0.3	0.5	0.1	-0.5	0.3	0.1	0.34	363.28
0.3	0.3	0.3	0.3	0.5	0.1	-0.5	0.3	0.1	0.34	539.30
0.3	0.3	0.3	0.3	0.5	0.1	-0.3	0.3	0.1	0.22	361.62
0.3	0.3	0.3	0.3	0.5	0.1	-0.3	0.3	0.1	0.22	537.64
0.3	0.3	0.3	0.3	0.5	0.1	-0.1	0.3	0.1	0.10	360.69
0.5	0.5	0.5	0.5	0.1	0.3	0.5	0.5	0.1	5.00	308.26
0.3	0.3	0.3	0.3	0.3	0.3	0.5	0.5	0.1	0.00	368.46
0.3	0.3	0.3	0.3	0.3	0.3	0.5	0.5	0.1	0.00	544.47
0.3	0.3	0.3	0.3	0.1	0.3	0.7	0.5	0.1	2.40	374.22
0.3	0.3	0.3	0.3	0.1	0.3	0.7	0.5	0.1	2.40	550.23
0.3	0.3	0.3	0.3	0.3	0.3	0.9	0.5	0.1	-0.40	369.90
0.3	0.3	0.3	0.3	0.3	0.3	0.9	0.5	0.1	-0.40	545.91
0.3	0.3	0.3	0.3	0.1	0.3	-0.9	0.7	0.1	9.00	361.45
0.5	0.5	0.5	0.5	0.1	0.3	-0.9	0.7	0.1	15.00	383.26
0.3	0.3	0.3	0.3	0.1	0.3	-0.9	0.7	0.1	9.00	449.46
0.5	0.5	0.5	0.5	0.1	0.3	-0.9	0.7	0.1	15.00	424.92
0.5	0.5	0.5	0.5	0.1	0.3	-0.9	0.7	0.1	15.00	508.26
0.3	0.3	0.3	0.3	0.5	0.3	-0.5	0.9	0.1	0.62	554.21
0.5	0.5	0.5	0.5	0.5	0.3	-0.5	0.9	0.1	1.04	310.30
0.3	0.3	0.3	0.3	0.5	0.3	-0.3	0.9	0.1	0.50	374.81
0.3	0.3	0.3	0.3	0.5	0.3	-0.3	0.9	0.1	0.50	550.82
0.5	0.5	0.5	0.5	0.5	0.3	-0.3	0.9	0.1	0.84	300.90

Table 4.1: The Percent Relative Efficiency of the proposed ratio-type estimator $\bar{y}^{**}_{Ratio}$ with respect to Bar-Lev et al (2004) estimator $\bar{y}_{BBB}$ for different choices of the parameters with $p = p_1 = p_2 = 0.7$

C_γ	$C_{\gamma 1}$	$C_{\gamma 2}$	C_y	C_{x1}	C_{x2}	ρ_{yx1}	ρ_{yx2}	ρ_{x1x2}	α_0	PRE
0.3	0.3	0.3	0.3	0.1	0.1	-0.9	0.5	0.1	0.5	531.97
0.3	0.3	0.3	0.3	0.3	0.1	-0.9	0.5	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	-0.9	0.5	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	-0.7	0.5	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	-0.7	0.5	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	-0.7	0.5	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	-0.5	0.5	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	-0.5	0.5	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	-0.5	0.5	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	-0.3	0.5	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	-0.3	0.5	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	-0.3	0.5	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	-0.1	0.5	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	-0.1	0.5	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	-0.1	0.5	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	0.1	0.5	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	0.1	0.5	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	0.1	0.5	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	0.3	0.5	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	0.3	0.5	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	0.3	0.5	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	0.5	0.5	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	0.5	0.5	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	0.5	0.5	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	0.7	0.5	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	0.7	0.5	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	0.7	0.5	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	0.9	0.5	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	0.9	0.5	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	0.9	0.5	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	-0.9	0.7	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	-0.9	0.7	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	-0.9	0.7	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	-0.7	0.7	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	-0.7	0.7	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	-0.7	0.7	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	-0.5	0.7	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	-0.5	0.7	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	-0.5	0.7	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	-0.3	0.7	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	-0.3	0.7	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	-0.3	0.7	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	-0.1	0.7	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	-0.1	0.7	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	-0.1	0.7	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	0.1	0.7	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	0.1	0.7	0.1	0.5	539.91
0.3	0.3	0.3	0.3	0.5	0.1	0.1	0.7	0.1	0.5	555.91
0.3	0.3	0.3	0.3	0.1	0.1	0.3	0.7	0.1	0.5	531.91
0.3	0.3	0.3	0.3	0.3	0.1	0.3	0.7	0.1	0.5	539.91

Table 4.2: The Percent Relative Efficiency of the proposed ratio-type estimator $\bar{y}^{**}_{Ratio}$ with respect to Tarray and Singh (2014) estimator $\bar{y}_{ST2}$ for different choices of the parameters with $p = p_1 = p_2 = 0.7$

C_γ	$C_{\gamma 1}$	$C_{\gamma 2}$	C_y	C_{x1}	C_{x2}	ρ_{yx1}	ρ_{yx2}	ρ_{x1x2}	PRE
0.5	0.5	0.5	0.5	0.1	0.1	-0.9	0.1	0.1	348.97
0.5	0.5	0.5	0.5	0.3	0.1	-0.9	0.1	0.1	356.97
0.5	0.5	0.5	0.5	0.5	0.1	-0.9	0.1	0.1	372.97
0.5	0.5	0.5	0.5	0.1	0.1	-0.7	0.1	0.1	348.97
0.5	0.5	0.5	0.5	0.3	0.1	-0.7	0.1	0.1	356.97
0.5	0.5	0.5	0.5	0.5	0.1	-0.7	0.1	0.1	372.97
0.5	0.5	0.5	0.5	0.1	0.1	-0.5	0.1	0.1	348.97
0.5	0.5	0.5	0.5	0.3	0.1	-0.5	0.1	0.1	356.97
0.5	0.5	0.5	0.5	0.5	0.1	-0.5	0.1	0.1	372.97
0.5	0.5	0.5	0.5	0.1	0.1	-0.3	0.1	0.1	348.97
0.5	0.5	0.5	0.5	0.3	0.1	-0.3	0.1	0.1	356.97
0.5	0.5	0.5	0.5	0.5	0.1	-0.3	0.1	0.1	372.97
0.5	0.5	0.5	0.5	0.1	0.1	-0.1	0.1	0.1	348.97
0.5	0.5	0.5	0.5	0.3	0.1	-0.1	0.1	0.1	356.97
0.5	0.5	0.5	0.5	0.5	0.1	-0.1	0.1	0.1	372.97
0.5	0.5	0.5	0.5	0.1	0.1	0.1	0.1	0.1	348.97
0.5	0.5	0.5	0.5	0.3	0.1	0.1	0.1	0.1	356.97
0.5	0.5	0.5	0.5	0.5	0.1	0.1	0.1	0.1	372.97
0.5	0.5	0.5	0.5	0.1	0.1	0.3	0.1	0.1	348.97
0.5	0.5	0.5	0.5	0.3	0.1	0.3	0.1	0.1	356.97
0.5	0.5	0.5	0.5	0.5	0.1	0.3	0.1	0.1	372.97
0.5	0.5	0.5	0.5	0.1	0.1	0.5	0.1	0.1	348.97
0.5	0.5	0.5	0.5	0.3	0.1	0.5	0.1	0.1	356.97
0.5	0.5	0.5	0.5	0.5	0.1	0.5	0.1	0.1	372.97
0.5	0.5	0.5	0.5	0.1	0.1	0.7	0.1	0.1	348.97
0.5	0.5	0.5	0.5	0.3	0.1	0.7	0.1	0.1	356.97
0.5	0.5	0.5	0.5	0.5	0.1	0.7	0.1	0.1	372.97
0.5	0.5	0.5	0.5	0.1	0.1	0.9	0.1	0.1	348.97
0.5	0.5	0.5	0.5	0.3	0.1	0.9	0.1	0.1	356.97
0.5	0.5	0.5	0.5	0.5	0.1	0.9	0.1	0.1	372.97
0.5	0.5	0.5	0.5	0.3	0.3	0.5	0.1	0.1	364.97
0.5	0.5	0.5	0.5	0.5	0.3	0.5	0.1	0.1	380.97
0.5	0.5	0.5	0.5	0.3	0.3	0.7	0.1	0.1	364.97
0.5	0.5	0.5	0.5	0.5	0.3	0.7	0.1	0.1	380.97
0.5	0.5	0.5	0.5	0.3	0.3	0.9	0.1	0.1	364.97
0.5	0.5	0.5	0.5	0.5	0.3	0.9	0.1	0.1	380.97
0.5	0.5	0.5	0.5	0.5	0.5	-0.9	0.1	0.1	396.97
0.5	0.5	0.5	0.5	0.5	0.5	-0.7	0.1	0.1	396.97
0.5	0.5	0.5	0.5	0.5	0.5	-0.5	0.1	0.1	396.97
0.5	0.5	0.5	0.5	0.5	0.5	-0.3	0.1	0.1	396.97
0.5	0.5	0.5	0.5	0.5	0.5	-0.1	0.1	0.1	396.97
0.5	0.5	0.5	0.5	0.5	0.5	0.1	0.1	0.1	396.97
0.5	0.5	0.5	0.5	0.5	0.5	0.3	0.1	0.1	396.97
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.1	0.1	396.97
0.5	0.5	0.5	0.5	0.5	0.5	0.7	0.1	0.1	396.97
0.5	0.5	0.5	0.5	0.5	0.5	0.9	0.1	0.1	396.97
0.5	0.5	0.5	0.5	0.5	0.3	-0.3	0.1	0.1	380.97
0.5	0.5	0.5	0.5	0.3	0.3	-0.1	0.1	0.1	364.97
0.5	0.5	0.5	0.5	0.5	0.3	-0.1	0.1	0.1	380.97
0.5	0.5	0.5	0.5	0.3	0.3	0.1	0.1	0.1	364.97

Table 5.1: The Percent Relative Efficiency of the proposed estimator $\bar{y}^{**}_{Power}$ with respect to Bar-Lev-et al (2004) estimator $\bar{y}_{BBB}$ for different choices of parameter with $p = p_1 = p_2 = 0.7$

C_γ	$C_{\gamma 1}$	$C_{\gamma 2}$	C_y	C_{x1}	C_{x2}	ρ_{yx1}	ρ_{yx2}	ρ_{x1x2}	θ	θ_1	θ_2	α_0	PRE
0.5	0.5	0.5	0.5	0.1	0.5	-0.9	0.5	0.1	1	1	1	17.00	553.66
0.5	0.5	0.5	0.5	0.3	0.5	-0.9	0.5	0.1	1	1	1	2.89	339.77
0.5	0.5	0.5	0.5	0.1	0.5	-0.7	0.5	0.1	1	1	1	16.00	520.66
0.3	0.3	0.3	0.3	0.3	0.5	-0.7	0.5	0.1	0.5	0.5	0.5	1.53	574.53
0.5	0.5	0.5	0.5	0.3	0.5	-0.7	0.5	0.1	1	1	1	2.56	323.44
0.5	0.5	0.5	0.5	0.1	0.5	-0.5	0.5	0.1	1	1	1	15.00	320.48
0.3	0.3	0.3	0.3	0.3	0.5	-0.5	0.5	0.1	0.5	0.5	0.5	1.33	569.37
0.5	0.5	0.5	0.5	0.3	0.5	-0.5	0.5	0.1	1	1	1	2.22	309.10
0.5	0.5	0.5	0.5	0.1	0.5	-0.3	0.5	0.1	1	1	1	14.00	460.66
0.3	0.3	0.3	0.3	0.3	0.5	-0.3	0.5	0.1	0.5	0.5	0.5	1.13	564.93
0.5	0.5	0.5	0.5	0.1	0.5	-0.1	0.5	0.1	1	1	1	13.00	433.66
0.3	0.3	0.3	0.3	0.3	0.5	-0.1	0.5	0.1	0.5	0.5	0.5	0.93	561.21
0.5	0.5	0.5	0.5	0.1	0.5	0.1	0.5	0.1	1	1	1	12.00	408.66
0.3	0.3	0.3	0.3	0.3	0.5	0.1	0.5	0.1	0.5	0.5	0.5	0.73	558.21
0.3	0.3	0.3	0.3	0.1	0.5	0.3	0.5	0.1	0.5	0.5	0.5	6.60	596.93
0.5	0.5	0.5	0.5	0.1	0.5	0.3	0.5	0.1	1	1	1	11.00	385.66
0.3	0.3	0.3	0.3	0.3	0.5	0.3	0.5	0.1	0.5	0.5	0.5	0.53	555.93
0.3	0.3	0.3	0.3	0.1	0.5	0.5	0.5	0.1	0.5	0.5	0.5	6.00	589.37
0.5	0.5	0.5	0.5	0.1	0.5	0.5	0.5	0.1	1	1	1	10.00	364.66
0.3	0.3	0.3	0.3	0.1	0.5	0.7	0.5	0.1	0.5	0.5	0.5	5.40	582.53
0.5	0.5	0.5	0.5	0.1	0.5	0.7	0.5	0.1	1	1	1	9.00	345.66
0.3	0.3	0.3	0.3	0.3	0.5	0.7	0.5	0.1	0.5	0.5	0.5	0.13	553.53
0.3	0.3	0.3	0.3	0.1	0.5	0.9	0.5	0.1	0.5	0.5	0.5	4.80	576.41
0.5	0.5	0.5	0.5	0.1	0.5	0.9	0.5	0.1	1	1	1	8.00	328.66
0.3	0.3	0.3	0.3	0.3	0.5	0.9	0.5	0.1	0.5	0.5	0.5	-0.07	553.41
0.3	0.3	0.3	0.3	0.1	0.5	-0.9	0.7	0.1	0.5	0.5	0.5	13.20	310.33
0.5	0.5	0.5	0.5	0.1	0.5	-0.9	0.7	0.1	1	1	1	22.00	578.18
0.5	0.5	0.5	0.5	0.1	0.5	-0.9	0.7	0.1	1	1	1	22.00	579.48
0.3	0.3	0.3	0.3	0.3	0.5	-0.9	0.7	0.1	0.5	0.5	0.5	2.07	591.81
0.5	0.5	0.5	0.5	0.3	0.5	-0.9	0.7	0.1	1	1	1	3.44	371.44
0.5	0.5	0.5	0.5	0.1	0.5	-0.7	0.7	0.1	1	1	1	21.00	535.18
0.3	0.3	0.3	0.3	0.3	0.5	-0.7	0.7	0.1	0.5	0.5	0.5	1.87	584.73
0.5	0.5	0.5	0.5	0.1	0.5	-0.5	0.7	0.1	1	1	1	20.00	494.18
0.3	0.3	0.3	0.3	0.3	0.5	-0.5	0.7	0.1	0.5	0.5	0.5	1.67	578.37
0.5	0.5	0.5	0.5	0.3	0.5	-0.5	0.7	0.1	1	1	1	2.78	334.10
0.3	0.3	0.3	0.3	0.5	0.5	-0.5	0.7	0.1	0.5	0.5	0.5	0.72	566.33
0.5	0.5	0.5	0.5	0.5	0.5	-0.5	0.7	0.1	1	1	1	1.20	300.66
0.3	0.3	0.3	0.3	0.3	0.5	-0.3	0.7	0.1	0.5	0.5	0.5	1.47	572.73
0.5	0.5	0.5	0.5	0.3	0.5	-0.3	0.7	0.1	1	1	1	2.44	318.44
0.3	0.3	0.3	0.3	0.5	0.5	-0.3	0.7	0.1	0.5	0.5	0.5	0.60	562.37
0.5	0.5	0.5	0.5	0.1	0.5	-0.1	0.7	0.1	1	1	1	18.00	588.66
0.3	0.3	0.3	0.3	0.5	0.5	-0.1	0.7	0.1	0.5	0.5	0.5	0.48	559.13
0.3	0.3	0.3	0.3	0.3	0.5	0.1	0.7	0.1	0.5	0.5	0.5	1.07	563.61
0.3	0.3	0.3	0.3	0.5	0.5	0.1	0.7	0.1	0.5	0.5	0.5	0.36	556.61
0.3	0.3	0.3	0.3	0.3	0.5	0.3	0.7	0.1	0.5	0.5	0.5	0.87	560.13
0.3	0.3	0.3	0.3	0.5	0.5	0.3	0.7	0.1	0.5	0.5	0.5	0.24	554.81
0.3	0.3	0.3	0.3	0.3	0.5	0.5	0.7	0.1	0.5	0.5	0.5	0.67	557.37
0.3	0.3	0.3	0.3	0.3	0.5	0.7	0.7	0.1	0.5	0.5	0.5	0.47	555.33
0.3	0.3	0.3	0.3	0.3	0.5	0.9	0.7	0.1	0.5	0.5	0.5	0.27	554.01
0.3	0.3	0.3	0.3	0.5	0.5	0.9	0.7	0.1	0.5	0.5	0.5	-0.12	553.73

Table 5.2: The Percent Relative Efficiency of the proposed power transformation ratio-type estimator $\bar{y}^{**}_{Power}$ with respect to Tarray and Singh's (2014) $\bar{y}_{ST2}$ estimator for the different choices of the parameters with $p = p_1 = p_2 = 0.7$

C_γ	C_{γ_1}	C_{γ_2}	C_y	C_{x_1}	C_{x_2}	ρ_{yx_1}	ρ_{yx_2}	$\rho_{x_1x_2}$	α_0	PRE
0.5	0.5	0.5	0.5	0.1	0.5	0.1	0.7	0.1	17.00	354.87
0.5	0.5	0.5	0.5	0.1	0.5	0.1	0.7	0.1	17.00	378.89
0.5	0.5	0.5	0.5	0.1	0.5	0.3	0.7	0.1	16.00	307.86
0.5	0.5	0.5	0.5	0.1	0.5	0.3	0.7	0.1	16.00	321.87
0.5	0.5	0.5	0.5	0.1	0.5	0.3	0.7	0.1	16.00	345.89
0.5	0.5	0.5	0.5	0.1	0.5	0.5	0.7	0.1	15.00	314.89
0.3	0.3	0.3	0.3	0.1	0.5	-0.9	0.9	0.1	16.20	300.37
0.1	0.1	0.1	0.1	0.1	0.5	-0.9	0.9	0.1	5.40	314.26
0.3	0.3	0.3	0.3	0.1	0.5	-0.9	0.9	0.1	16.20	329.97
0.3	0.3	0.3	0.3	0.1	0.5	-0.9	0.9	0.1	16.20	380.70
0.1	0.1	0.1	0.1	0.1	0.5	-0.7	0.9	0.1	5.20	312.14
0.3	0.3	0.3	0.3	0.1	0.5	-0.7	0.9	0.1	15.60	310.89
0.3	0.3	0.3	0.3	0.1	0.5	-0.7	0.9	0.1	15.60	361.62
0.1	0.1	0.1	0.1	0.1	0.5	-0.5	0.9	0.1	5.00	310.10
0.3	0.3	0.3	0.3	0.1	0.5	-0.5	0.9	0.1	15.00	343.26
0.1	0.1	0.1	0.1	0.1	0.5	-0.3	0.9	0.1	4.80	308.14
0.3	0.3	0.3	0.3	0.1	0.5	-0.3	0.9	0.1	14.40	325.62
0.5	0.5	0.5	0.5	0.1	0.5	-0.1	0.9	0.1	23.00	580.86
0.1	0.1	0.1	0.1	0.1	0.5	-0.1	0.9	0.1	4.60	306.26
0.5	0.5	0.5	0.5	0.1	0.5	-0.1	0.9	0.1	23.00	594.87
0.3	0.3	0.3	0.3	0.1	0.5	-0.1	0.9	0.1	13.80	308.70
0.5	0.5	0.5	0.5	0.1	0.5	0.1	0.9	0.1	22.00	535.86
0.1	0.1	0.1	0.1	0.1	0.5	0.1	0.9	0.1	4.40	304.46
0.5	0.5	0.5	0.5	0.1	0.5	0.1	0.9	0.1	22.00	549.87
0.5	0.5	0.5	0.5	0.1	0.5	0.1	0.9	0.1	22.00	573.89
0.5	0.5	0.5	0.5	0.1	0.5	0.3	0.9	0.1	21.00	492.86
0.1	0.1	0.1	0.1	0.1	0.5	0.3	0.9	0.1	4.20	302.74
0.5	0.5	0.5	0.5	0.1	0.5	0.3	0.9	0.1	21.00	506.87
0.5	0.5	0.5	0.5	0.1	0.5	0.3	0.9	0.1	21.00	530.89
0.5	0.5	0.5	0.5	0.1	0.5	0.5	0.9	0.1	20.00	451.86
0.1	0.1	0.1	0.1	0.1	0.5	0.5	0.9	0.1	4.00	301.10
0.5	0.5	0.5	0.5	0.1	0.5	0.5	0.9	0.1	20.00	465.87
0.5	0.5	0.5	0.5	0.1	0.5	0.5	0.9	0.1	20.00	489.89
0.5	0.5	0.5	0.5	0.1	0.5	0.7	0.9	0.1	19.00	412.86
0.5	0.5	0.5	0.5	0.1	0.5	0.7	0.9	0.1	19.00	426.87
0.5	0.5	0.5	0.5	0.1	0.5	0.7	0.9	0.1	19.00	450.89
0.5	0.5	0.5	0.5	0.1	0.5	0.9	0.9	0.1	18.00	375.86
0.5	0.5	0.5	0.5	0.1	0.5	0.9	0.9	0.1	18.00	389.87
0.5	0.5	0.5	0.5	0.1	0.5	0.9	0.9	0.1	18.00	413.89
0.5	0.5	0.5	0.5	0.1	0.5	-0.9	0.7	0.1	22.00	535.86
0.1	0.1	0.1	0.1	0.1	0.5	-0.9	0.7	0.1	4.40	304.46
0.5	0.5	0.5	0.5	0.1	0.5	-0.9	0.7	0.1	22.00	549.87
0.5	0.5	0.5	0.5	0.1	0.5	-0.9	0.7	0.1	22.00	573.89
0.5	0.5	0.5	0.5	0.1	0.5	-0.7	0.7	0.1	21.00	492.86
0.1	0.1	0.1	0.1	0.1	0.5	-0.7	0.7	0.1	4.20	302.74
0.5	0.5	0.5	0.5	0.1	0.5	-0.7	0.7	0.1	21.00	506.87
0.5	0.5	0.5	0.5	0.1	0.5	-0.7	0.7	0.1	21.00	530.89
0.5	0.5	0.5	0.5	0.1	0.5	-0.5	0.7	0.1	20.00	451.86
0.1	0.1	0.1	0.1	0.1	0.5	-0.5	0.7	0.1	4.00	301.10
0.5	0.5	0.5	0.5	0.1	0.5	-0.5	0.7	0.1	20.00	465.87