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Tests for Equality of Hazard Quantile Functions

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Abstract

In this paper, we propose two tests for testing equality of hazard quantile functions of two populations. The test statistics are based on estimators of the quantile density function. Limiting distribution of both these test statistics has been derived. The power of the new tests is computed through simulations for uncensored and censored observations. The new tests are compared with two existing tests available in literature. Procedures have been illustrated on real data.

Key words: Quantile function; Quantile density function; Kernel density estimator; Hazard rate; Hazard quantile function.

Reminiscences

Isha had known Prof. Aloke Dey for almost 36 years. She could walk into his office anytime to discuss statistics, official and even personal problems. He was a good listener, had a great sense of humor and was kind of a quick reference point on government rules and regulations. Three of us have fond memories of our association with Prof. Aloke Dey - a gentle person - went too soon.

1. Introduction

In survival analysis, the hazard rate is a basic reliability measure. It is studied as failure rate in reliability, force of mortality in demography or actuarial science, intensity function in stochastic processes and age specific failure rate in epidemiology. Sometimes, interest may be in comparing the hazard rates of two populations. Chikkagoudar and Shuster (1974) proposed the locally most powerful test for testing equality of hazard rates of two populations. Kochar (1979) provided distribution free test based on U-statistics and Kochar (1981) proposed a test based on linear function of order statistics for testing equality of hazard rates. For the same problem, Cheng (1985) proposed a test based on ranks.

Quantile based approach is popular now a days. The reliability analysis based on quantiles provides an alternate methodology for statistical analysis when cumulative distribution function (cdf) or probability density function (pdf) is not available in a closed form. Examples of such distributions are Generalised Lambda distribution (GLD) (Karian and Dudewicz

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(2000)), Skew logistic distribution (Gilchrist (2000)) and Davies distribution (Hankin and Lee (2006)). Maladan and Sankaran (2020) proposed a new family of distributions by using transformation in context of quantiles.

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two independent random samples from two populations with distribution functions $F(x)$ and $G(x)$, survival functions $\bar{F}(x)$ and $\bar{G}(x)$, pdfs $f(x)$ and $g(x)$, hazard rate functions $h_1(x) = \dfrac{f(x)}{\bar{F}(x)}$ and $h_2(x) = \dfrac{g(x)}{\bar{G}(x)}$, respectively. The quantile function for the first population is denoted by $Q_1(u)$ and defined as

$$Q_1(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\},\ 0 < u < 1. \tag{1}$$

From (1), it is seen that

$$F(Q_1(u)) = u. \tag{2}$$

Differentiating (2), we get the quantile density function for the first population as

$$q_1(u) = \frac{d}{du} Q_1(u) = \frac{1}{f(Q_1(u))}. \tag{3}$$

Note that the quantile density function as defined in (3) is not a density function in the usual sense but is reciprocal of density function at corresponding quantile function. Nair and Sankaran (2009a) presented various reliability measures *viz*, hazard rate, mean residual life function, variance residual life function and percentile residual life function in terms of quantiles. The hazard quantile function for the first population is given by

$$H_1(u) = h_1(Q_1(u)) = \frac{f(Q_1(u))}{\bar{F}(Q_1(u))} = ((1-u)q_1(u))^{-1}. \tag{4}$$

Hazard quantile function is the hazard function at the corresponding quantile function. The quantile function, quantile density function and hazard quantile function for second population are denoted by $Q_2(u)$, $q_2(u)$ and $H_2(u)$, respectively.

Many ageing concepts *viz* increasing failure rate, increasing failure rate average, new better than used and new better than used in expectation have been defined in terms of quantiles by Kumar and Nair (2011). Nair and Sankaran (2009b) studied estimation of the hazard quantile function based on right censored data. Peng and Fine (2007) provided tests for equality of cause specific hazard rates for competing risk data based on quantiles. Fan *et al.* (2020) proposed smooth kernel type estimator of quantile function for right-censored competing risks data.

We wish to test the null hypothesis of equality of hazard rate functions of two independent populations, that is

$$\begin{aligned} &H_0 : h_1(x) = h_2(x) \quad \text{for all } x \\ &\text{against the alternative} \\ &H_A : h_1(x) \leq h_2(x) \quad \text{for all } x \qquad (5) \\ &\text{with strict inequality in (5) with a positive probability.} \end{aligned}$$

Kochar (1979) showed that for increasing failure rate distributions, location-scale ordering of distribution functions leads to ordering of their corresponding hazard rates.

Above testing problem can be equivalently written in terms of hazard quantile functions as follows

$$\begin{aligned}
&H_0 : H_1(u) = H_2(u) \quad \text{for all } \ 0 < u < 1 \\
&\text{against the alternative} \\
&H_A : H_1(u) \leq H_2(u) \quad \text{for all } \ 0 < u < 1 \\
&\text{with strict inequality in (6) with a positive probability.}
\end{aligned} \tag{6}$$

From (4), it is noted that for all $0 < u < 1$

$$\begin{aligned}
H_1(u) = H_2(u) \quad &\text{iff} \quad q_1(u) = q_2(u), \\
H_1(u) \leq H_2(u) \quad &\text{iff} \quad q_1(u) \geq q_2(u).
\end{aligned} \tag{7}$$

Hence, from (6) and (7), it is clear that testing for equality of hazard rates is equivalent to testing for equality of quantile density functions. Hence, we will propose tests for testing

$$\begin{aligned}
&H_0 : q_1(u) = q_2(u) \quad \text{for all } \ 0 < u < 1 \\
&\text{against the alternative} \\
&H_A : q_1(u) \geq q_2(u) \quad \text{for all } \ 0 < u < 1 \\
&\text{with strict inequality in (8) with a positive probability.}
\end{aligned} \tag{8}$$

In Section 2, we discuss few preliminaries that are needed to define and study the properties of test statistics. Two examples are given where distribution functions can not be expressed in closed forms but quantile functions have nice forms. We also discuss the estimator of quantile density function proposed by Soni *et al.* (2012). In Section 3, two test statistics - a supremum type and an integral type have been proposed for testing the equality of hazard quantile functions against the alternative that they are ordered. The statistics are based on estimators of quantile density functions due to Soni *et al.* (2012). Asymptotic distribution of two test statistics is discussed. The tests can be used when observations are uncensored or censored. Simulations are carried out in Section 4 for comparing power of the proposed tests with those suggested by Kochar (1979) and Cheng (1985). In Section 5, a real data set is considered to illustrate the utility of the tests proposed by us. The proofs of the Theorems and three Tables showing power comparisons are given in the Appendix.

2. Preliminaries

Two examples for which distribution function can not be written in a closed form but quantile function has a closed form, are discussed in Section 2.1. Estimator of quantile density function proposed by Soni *et al.* (2012) is discussed in Section 2.2.

2.1. Examples

(i) **Davies Distribution (Davies(C, λ_1, λ_2))** with $C > 0,\ \lambda_1 > 0,\ \lambda_2 > 0$ was given by Hankin and Lee (2006). The quantile function, the quantile density function and the hazard quantile function for $0 < u < 1$ are

$$Q_D(u, C, \lambda_1, \lambda_2) = \frac{Cu^{\lambda_1}}{(1-u)^{\lambda_2}}, \tag{9}$$

$$q_D(u, C, \lambda_1, \lambda_2) = \frac{Cu^{\lambda_1 - 1}(\lambda_1(1-u) + \lambda_2 u)}{(1-u)^{\lambda_2+1}}, \tag{10}$$

$$H_D(u, C, \lambda_1, \lambda_2) = \frac{(1-u)^{\lambda_2}}{Cu^{\lambda_1 - 1}(\lambda_1(1-u) + \lambda_2 u)}. \tag{11}$$

(ii) **The Generalized Lambda Distribution** (GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$)) was introduced by Ramberg and Schmeiser (1974) and further discussed by Karian and Dudewicz (2000). The quantile function, the quantile density function and the hazard quantile function for $0 < u < 1$ are given below:

$$Q_{GL}(u, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{(u^{\lambda_3} - (1-u)^{\lambda_4})}{\lambda_2}, \tag{12}$$

$$q_{GL}(u, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1}}{\lambda_2}, \tag{13}$$

$$H_{GL}(u, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left((1-u)\frac{(\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1})}{\lambda_2}\right)^{-1}. \tag{14}$$

The parameters $\lambda_1, \lambda_2, \lambda_3,$ and λ_4 can assume real values, but some restrictions on these parameters have been imposed for defining a valid distribution. The possible eight regions of parameter values for which GLD is a valid distribution have been listed in Karian and Dudewicz (2000). Table 1 gives two sets of choices of parameters λ_2, λ_3 and λ_4 of GLD with λ_1 taking any real value. These choices ensure that observations always have support on the positive real line.

Table 1: Considered regions and corresponding supports of GLD

	Regions	Supports
1.	$\lambda_2 > 0,\ \lambda_3 > 1,\ \lambda_4 > 0$	$\left(\lambda_1 - \frac{1}{\lambda_2}, \lambda_1 + \frac{1}{\lambda_2}\right)$
2.	$\lambda_2 < 0,\ \lambda_3 > 1,\ \lambda_4 < -1$	$\left(\lambda_1 - \frac{1}{\lambda_2}, \infty\right)$

Here λ_1 controls the left tail, λ_2 controls the right tail and C is the scale parameter.

These distributions will be used for simulation studies in Section 4.

2.2. Quantile density estimator

Estimators of the quantile density function were proposed by Parzen (1979), Csörgo (1981), Falk (1986), Jones (1992), Cheng and Parzen (1997) and Soni *et al.* (2012). The wavelet based estimator of quantile density function was proposed by Chesneau *et al.* (2016). This estimator behaved well in tails.

The estimator of $q_1(u)$ given by Soni *et al.* (2012), based on random sample $X_1, X_2, \ldots, X_n$ from $F(x)$ is

$$\hat{q}_1(u) = \frac{1}{h(n)} \int_0^1 \frac{K(\frac{t-u}{h(n)})}{f_n(\hat{Q}_1(t))} dt \tag{15}$$

where $f_n(x)$ is a kernel density estimator of $f(x)$ with $h(n)$ as bandwidth.

$\hat{Q}_1(u) = \inf\{x : F_n(x) \geq u\}, 0 < u < 1$ is the empirical estimator of the quantile function $Q(u)$ based on empirical distribution function $F_n(x)$. The kernel $K(.)$ is a density function satisfying regularity conditions (Prakasa Rao (1983))

Estimator proposed by Soni *et al.* (2012) performs better than those given by Jones (1992) in terms of mean square error. Soni *et al.* (2012) proved the following results for fixed u, where $0 < u < 1$:

(R1) $\hat{q}_1(u)$ is a consistent estimator of $q_1(u)$,

(R2) as $n \to \infty$, $\dfrac{\sqrt{n}h(n)(\hat{q}_1(u) - q_1(u))}{\sigma_{1n}(u)}$ is asymptotically normal with mean zero and variance 1, where

$\sigma^2_{1n}(u) = E(\int_0^1 dK^*_n(u,t)F_n(\hat{Q}_1(t)))^2$ with $K^*_n(u,t) = K(\frac{t-u}{h(n)})q_1(t)$.

Let $\hat{q}_2(u)$ denote the corresponding estimator of $q_2(u)$ based on a random sample $Y_1, Y_2, \ldots, Y_n$ from $G(x)$.

3. Test Statistics and Asymptotic Distribution

We propose two test statistics for testing H_0 against H_A. Let $\hat{q}_1(u)$ and $\hat{q}_2(u)$ be consistent estimators of $q_1(u)$ and $q_2(u)$ as discussed in Section 2.2. The difference $\hat{q}_1(u) - \hat{q}_2(u)$ is an empirical measure of departure from the null hypothesis. This difference is expected to be zero under the null hypothesis and non-negative under the alternative hypothesis.

First proposed test statistic T_1 is Kolmogorov-Smirnov type distance between $\hat{q}_1(u)$ and $\hat{q}_2(u)$ and is given as

$$T_1 = \sup_{0<u<1} (\hat{q}_1(u) - \hat{q}_2(u)). \tag{16}$$

Second proposed test statistic T_2 is Cramer-von Mises type difference, as given below

$$T_2 = \int_0^1 (\hat{q}_1(u) - \hat{q}_2(u)) d\Big(\frac{\hat{Q}_1(u) + \hat{Q}_2(u)}{2}\Big). \tag{17}$$

Test based on T_1, T_2 will reject H_0 in favour of H_A for large values of normalised versions of the statistics T_1 and T_2, respectively.

Next we consider a lemma needed to derive the asymptotic distributions of T_1 and T_2 under the null hypothesis.

D and $D[0,1]$ are equipped with the uniform norm $||.||$ and the product norm respectively. In the following lemma, weak convergence of the process $S_n(u)$ is established on D, where

$$S_n(u) = \{\sqrt{n}h(n)(\hat{q}_1(u) - q_1(u)), \sqrt{n}h(n)(\hat{q}_2(u) - q_2(u))\}.$$

Lemma 1: Let $B_1(q_1(u))$ and $B_2(q_1(u))$ be Brownian bridge processes with zero means. Then $S_n(u)$ converges in D to a 2-dimensional Gaussian Process $\{B_1(q_1(u)), B_2(q_2(u))\}$ as $n \to \infty$.

Proof: See the Appendix. □

The above lemma helps us in determining the asymptotic distribution of T_1 as established in Theorem 1 given below.

Theorem 1: Under H_0, as $n \to \infty$, $\sqrt{n}h(n)T_1$ converges in distribution to $\sup\limits_{0<u<1} (B_1(q_1(u)) - B_2(q_2(u)))$.

Proof: See the Appendix. □

Remark 1: A slight modification can be made to the test statistic T_1 as discussed below.

Suppose under H_0, $B_1(q_1(u)) - B_2(q_2(u)) = g(u)$, where $g(u)$ is difference of two Brownian process. Then variance of random variable $g(u)$ is given by

$$Var(g(u)) = Var(B_1(q_1(u))) + Var(B_2(q_2(u))) = \sigma_g^2(u) \cdots \text{(say)}.$$

If $\{W(t) : t \geq 0\}$ is a standard Brownian motion (Wiener process), then

$$g(u) \to W(\sigma_g^2(u)).$$

Under H_0, this gives for $0 < u < 1$ and $n \to \infty$,

$$\sqrt{n}h(n)(\hat{q}_1(u) - \hat{q}_2(u)) \to W(\sigma_g^2(u))$$

$$\Rightarrow \frac{\sqrt{n}h(n)(\hat{q}_1(u)-\hat{q}_2(u))}{\sigma_g(T)} \rightarrow W(\frac{\sigma_g^2(u)}{\sigma_g^2(T)})$$

where $\sigma_g^2(T) = \max_t[\sigma_g^2(t)]$ for $T \in (0,1)$. Note that $\frac{\sigma_g^2(u)}{\sigma_g^2(T)} \in (0,1)$.

Let $\hat{\sigma}_g(T)$ be a consistent estimator of $\sigma_g(T)$ and define

$$T_{1n}^* = \frac{\sqrt{n}h(n)(\sup_{0<u<1}(\hat{q}_1(u)-\hat{q}_2(u)))}{\hat{\sigma}_g(T)} = \frac{\sqrt{n}h(n)T_1}{\hat{\sigma}_g(T)}. \tag{18}$$

Theorem 2: Under H_0 ,

$$\lim_{n\to\infty} P[T_{1n}^* > b] = P[\sup_{0<u<1} W(u) > b] = 2(1-\Phi(b)) \tag{19}$$

where $\Phi(b)$ is the cdf of Standard Normal distribution at b.

Proof: The proof follows from Section 7.4 of Durrett (2019). □

In the next theorem, we find the null asymptotic distribution of

$$T_{2n}^* = \sqrt{n}h(n)\int_0^1 (\hat{q}_1(u)-\hat{q}_2(u))d(\frac{\hat{Q}_1(u)+\hat{Q}_2(u)}{2}) = \sqrt{n}h(n)T_2. \tag{20}$$

Theorem 3: Under H_0, T_{2n}^* converges in distribution to a normal random variable with mean zero and variance σ^2 as $n \to \infty$ where

$$\sigma^2 = Var(\int (B_1(q_1(u)) - B_2(q_2(u)))d[\frac{Q_1(u)+Q_2(u)}{2}]).$$

Proof: The proof follows using Hadamard differentiability and functional delta method (Ref. van der Vaart and Wellner (1996); Theorem 3.9.4). For details, see the Appendix. □

In the sequel, T_{1n}^* will be referred to as the supremum statistic and T_{2n}^* as the integral statistic.

4. Simulations

A simulation study has been carried out to verify the asymptotic distribution of test statistics under H_0 and to compute size and power of the standardized versions of statistics T_{1n}^* and T_{2n}^*. The data are generated from GLD, Davies and exponential distributions with sample size $n = 25, 50, 100$. For censored data, censoring distribution is chosen so as to ensure 20% censoring. The chosen bandwidths are $0.15, 0.19, 0.25$ (Soni *et al.* (2012)) for GLD and exponential distributions and 0.85 for Davies distribution. Variances of T_{1n}^* and T_{2n}^*, are estimated by taking 5000 bootstrap samples from the underlying distribution and then T_{1n}^* and T_{2n}^* are calculated for each sample. The kernels used for estimation of quantile density functions are

(i) Triangular: $K(u) = (1 - |u|)I(|u| \leq 1)$ and

(ii) Epanechnikov: $K(u) = .75(1-u^2)I(|u| \leq 1)$ (the optimal kernel (Prakasa Rao (1983))).

4.1. Asymptotic distribution

Simulations are used to verify asymptotic distribution of the proposed statistics under H_0. Test Statistics have been calculated by considering GLD(1,1,2,1) distribution. Both graphical and testing procedures have been employed to test the normality. Figures 1 and 2 show Q-Q plots of standardized versions of T^*_{1n} and T^*_{2n} for $n = 25$ and these plots indicate normality of the statistics.

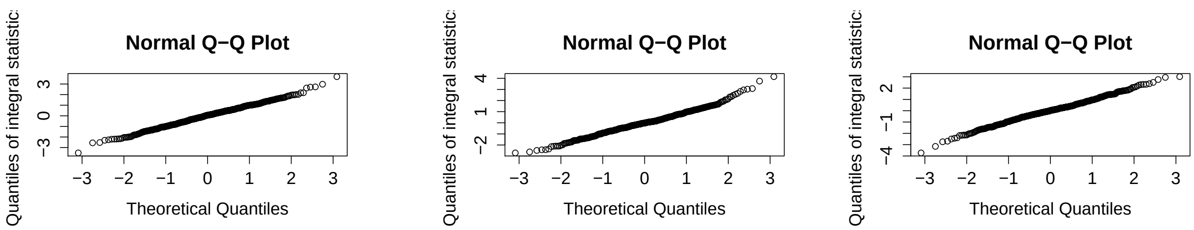


Figure 1: Q-Q plot of Integral statistic for $n = 25$, $h(n) = 0.15, 0.19, 0.25$

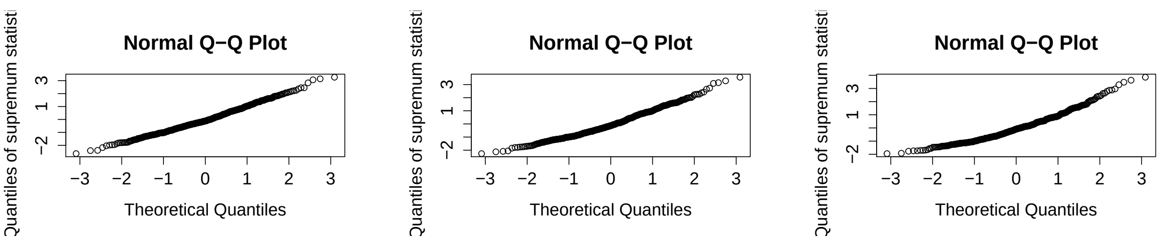


Figure 2: Q-Q plot of Supremum statistic for $n = 25$, $h(n) = 0.15, 0.19, 0.25$

Kolmogorov-Smirnov goodness of fit statistic is used to test the hypothesis that the simulated distributions of two test statistics are asymptotically normal. Table 2 shows p - values of Kolmogorov-Smirnov test statistic for $n = 25$ and bandwidth $h(n) = 0.15, 0.19, 0.25$.

Table 2: p - values of Kolmogorov-Smirnov Test

$h(n)$	Int Statistic	Sup Statistic
0.15	0.257	0.559
0.19	0.612	0.978
0.25	0.934	0.257

Hence from Q-Q plots (Figures 1 and 2) and Kolmogorov-Smirnov goodness of fit test, we conclude that standardized versions of both T^*_{1n} and T^*_{2n} follow Standard Normal distribution for $n \geq 25$. In the following subsection, we compute size and power of supremum and integral statistics when observations are uncensored and censored.

4.2. Calculation of estimates of size and power of tests

Power of the tests based on supremum and integral statistics and those given by Kochar (1979) and Cheng (1985) have been computed for GLD distribution with parameters in the regions listed in Table 1.

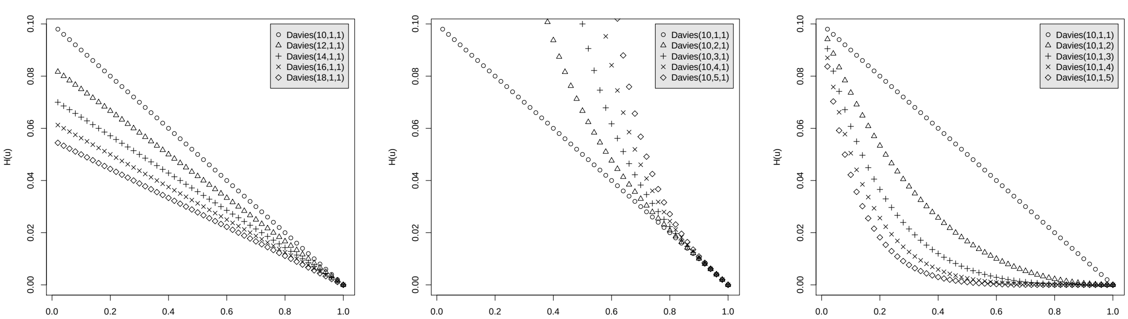
Table 3 depicts size of all tests for a sample of size 25, for uncensored data. For calculating size of the tests, considered distribution is GLD (1,1,2,1).

Table 3: Size of all tests

		$h(n)$		
n	Statistics	0.15	0.19	0.25
25	Sup	0.030	0.050	0.049
	Int	0.048	0.050	0.051
	Kochar	0.044	0.044	0.044
	Cheng	0.050	0.050	0.050

For the calculation of power of the tests, we first consider Davies distribution with quantile, quantile density and hazard quantile function as mentioned in (9), (10), and (11) respectively. The selection of parameters, which will lead to the ordering of hazard quantile functions is explained below through Figure 3(a)-3(c).

Figure 3 (a) shows the hazard quantile functions for Davies (10,1,1), Davies (10,2,1), Davies (10,3,1), Davies (10,4,1) and Davies (10,5,1), that is, λ_1 is changing. Figure 3(b) plots the hazard quantile functions for Davies (10,1,1), Davies (10,1,2), Davies (10,1,3), Davies (10,1,4) and Davies (10,1,5), that is, λ_2 varies. Figure 3(c) displays the hazard quantile functions for Davies (10,1,1), Davies (12,1,1), Davies (14,1,1), Davies (16,1,1) and Davies(18,1,1), that is, scale parameter C is varied.



(a) varying C **(b) varying λ_1** **(c) varying λ_2**

Figure 3: Hazard quantile functions for Davies distribution

Figures 3(a)-3(c) lead to the conclusions that

(i) for $C \leq C^*$, that is, when scale parameters are ordered, $H_D(u, C^*, \lambda_1, \lambda_2) \leq H_D(u, C, \lambda_1, \lambda_2)$;

(ii) if $\lambda_1 \leq \lambda_1^*$, that is, shape parameters are ordered, then
$H_D(u, C, \lambda_1, \lambda_2) \leq H_D(u, C, \lambda_1^*, \lambda_2)$;

(iii) when $\lambda_2 \leq \lambda_2^*$,
$H_D(u, C, \lambda_1, \lambda_2^*) \leq H_D(u, C, \lambda_1, \lambda_2)$.

Tables 4 and 5 give the power of four statistics for comparing the hazard quantile functions of two Davies distributions for $h(n) = 0.85$. In these tables, shape parameter λ_2 of Davies distribution is varied and in Table 5, departure in shape parameter λ_2 is reduced. In body of Tables 4 and 5, the first (second) value corresponds to power when Triangular (Epanechnikov) kernel is used for the estimation of quantile density function.

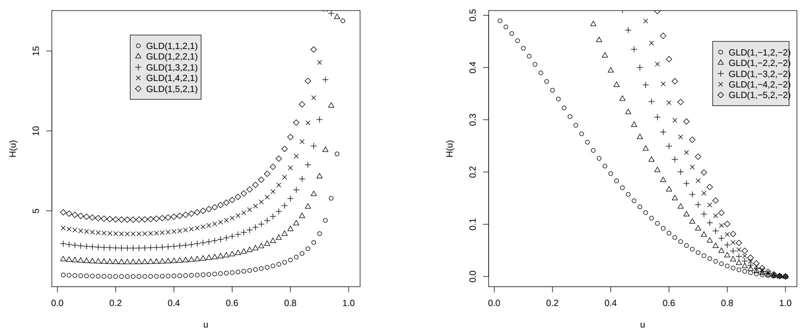
Table 4: Power comparison - Davies(10,1,1) *vs* Davies(10,1,2)(Uncensored case)

		n		
$h(n)$	Statistics	25	50	100
.85	Sup	0.186(0.260)	0.476(0.594)	0.886(0.902)
	Int	0.676(0.636)	0.838(0.848)	0.980(0.978)
	Kochar	0.508	0.786	0.972
	Cheng	0.566	0.754	0.938

Table 5: Power comparison - Davies(10,1,1) *vs* Davies(10,1,1.5)(Uncensored case)

		n		
$h(n)$	Statistics	25	50	100
.85	Sup	0.154(0.157)	0.212(0.238)	0.574(0.596)
	Int	0.286(0.490)	0.466(0.492)	0.646(0.696)
	Kochar	0.276	0.354	0.672
	Cheng	0.364	0.422	0.634

The next distribution of interest is GLD, with quantile, quantile density and hazard quantile function as mentioned in (12), (13), and (14) respectively. Selection of parameters of GLD, required for ordering of hazard quantile functions is explained through Figures 4(a) and 4(b). Figure 4(a) plots the hazard quantile functions of GLD (1,1,2,1), GLD (1,2,2,1), GLD (1,3,2,1), GLD (1,4,2,1) and GLD (1,5,2,1) and Figure 4(b) displays the hazard quantile functions of GLD (1,-1,2,-2), GLD (1,-2,2,-2), GLD (1,-3,2,-2), GLD (1,-4,2,-2) and GLD (1,-5,2,-2). Note that in both the figures, only scale parameter has been changed and all other parameters are same.



(a) Region 1 **(b) Region 2**

Figure 4: Hazard quantile functions for GLD in regions 1 and 2

For $0 < u < 1$, Figures 4(a) and 4(b) depict the following:

(i) In Region 1, for $\lambda_2 \leq \lambda_2^*$, it is observed that

$$H_{GL}(u, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \leq H_{GL}(u, \lambda_1, \lambda_2^*, \lambda_3, \lambda_4);$$

(ii) In Region 2, for $\lambda_2 \leq \lambda_2^*$, it is seen that

$$H_{GL}(u, \lambda_1, \lambda_2^*, \lambda_3, \lambda_4) \leq H_{GL}(u, \lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

Tables 6 and 7 (given in Appendix A.2) give the power for GLD in the Region 1 (Table 1) for Triangular and Epanechnikov kernels in censored as well uncensored case.

We consider GLD(1,1,2,1), GLD(1,2,2,1) in Table 6 and GLD(1,1,2,1), GLD(1,1.2,2,1) in Table 7 wherein departure in scale parameter λ_2 is reduced. The censoring variables have been generated from uniform distribution such that percentage of censoring in both cases is 20 and values in bold font are for censored case.

Table 8 (Appendix A.2) gives the power of our proposed test statistics for testing the equality of hazard quantile functions of EXP(1) and EXP(2) in censored as well as uncensored case. The censoring variables are distributed as EXP(.25) and EXP(0.5) respectively which ensure 20 percentage of censoring. In Tables 6-8, values in parentheses correspond to Epanechnikov kernel.

On the basis of values in Tables 3-5 and 6-8 (given in Appendix A.2), it can be concluded that

(i) for all test statistics and $n \geq 25$, size of tests ≤ 0.05 (level of significance);

(ii) power is not affected by choice of kernel considered;

(iii) power increases with an increase in sample size in uncensored as well as censored cases;

(iv) when observations are from GLD, both the proposed test statistics give higher power than Cheng's and Kochar's test statistics;

(v) when the parameters of GLD are in Region 1 (Table 1), integral statistic has more power than supremum statistic in uncensored case. Integral statistic is able to discriminate small departure in scale parameter λ_2;

(vi) when observations are from GLD, integral statistic is performing better than the supremum statistic in censored case;

(vii) when observations follow Davies distribution, the integral statistic has more power than all other test statistics;

(viii) when observations follow exponential distribution, supremum statistic performs better than integral statistic in censored case;

(ix) Cheng's and Kochar's statistics have more power than newly proposed test statistics when the underlying distribution is exponential.

5. Real Data

Data set of 101 patients with advanced acute myelogenous leukemia reported to International Bone Marrow Transplant Registry is considered (Source: Klein and Moeschberger (1997)). Fifty one of these patients had received an autologous bone marrow transplant in which high doses of chemotherapy and their own bone marrow were reinfused to replace their destroyed immune system. Fifty patients had an allogoneic bone marrow transplant where marrow from an HLA (Histocompatibility Leukocyte Antigen) matched sibling was used to replenish their immune systems. An important issue in bone marrow transplantation is the comparison of hazard quantile functions for these two methods. We compare hazard quantile functions of two techniques through their quantile density functions. Since test statistics proposed by us are for equal sample sizes, we randomly remove one observation from first sample.

Plots of quantile density functions are given in Figure 5. Solid line indicates estimate of quantile density function for auto transplant data and dotted one shows an estimate of quantile density function for allo transplant data. This figure shows that two quantile density functions are ordered. For supremum and integral statistics, p-values are 0.01 and 0.03 respectively. This leads to rejection of null hypothesis at 5 percent level of significance. Hence, it can be concluded that auto transplant technique is more effective than allogenic transplant technique.

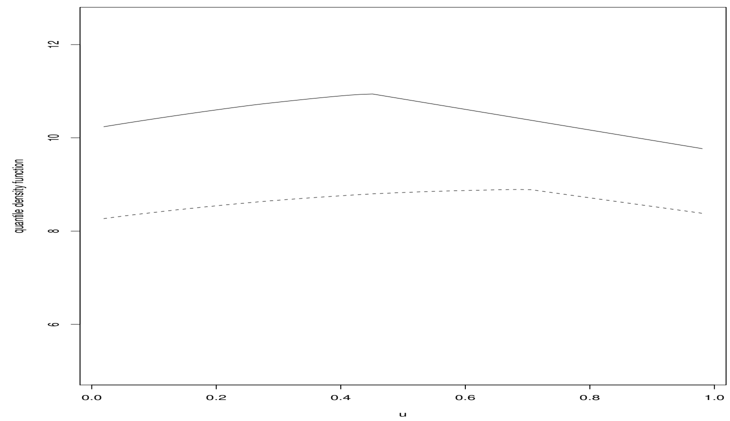


Figure 5: Graph of quantile density functions of autologous(auto) transplant and allogenic(allo) transplant

6. Conclusion

In this paper, we propose two tests based on consistent estimators of quantile density functions for testing equality of two hazard functions or equivalently, the hazard quantile functions, against the alternative that they are ordered. The tests have limiting normal distributions. Numerical studies show that all the tests attain their size. The supremum and the integral tests have better power than the tests proposed by Kochar and Cheng for some alternatives. However, it should be noted that tests by Kochar, Cheng and others can not be used when the observations are censored. But both the tests proposed in this paper can be used for censored data as well. Our tests perform well for families of distributions when closed form of distribution function is not available but explicit form of the quantile function is known.

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Appendix

A.1: Proofs

Proof:[**Lemma 1**] For arbitrary real numbers λ_1 and λ_2, we consider $T_n(u) = \sqrt{n}h(n)(\lambda_1(\hat{q}_1(u) - q_1(u)) + \lambda_2(\hat{q}_2(u) - q_2(u)))$.

On using central limit theorem, $T_n(u)$ converges in distribution to $N(0, \sigma_n^2(u))$ as $n \to \infty$ where $\sigma_n^2(u) = \lambda_1^2\sigma_{1n}^2(u) + \lambda_2^2\sigma_{2n}^2(u)$.

Using Cramer Wold device, as $n \to \infty$, we get

$S_n(u) = \{\sqrt{n}h(n)(\hat{q}_1(u) - q_1(u), \hat{q}_2(u) - q_2(u))\} \to$ Gaussian process $N(0, \Sigma_n)$ where

$$\Sigma_n = \begin{bmatrix} \sigma_{1n}^2(u) & 0 \\ 0 & \sigma_{2n}^2(u) \end{bmatrix}.$$

For a finite set of numbers $u_1, ..., u_n$ and arbitrary $\lambda_{1i}, \lambda_{2i}$, $\sum_{i=1}^{n}(\lambda_{1i}\hat{q}_1(u_i) + \lambda_{2i}\hat{q}_2(u_i))$ is sum of independent random variables. Using central limit theorem for the univariate independent random variables and Cramer Wold device, we conclude that the finite dimensional distribution of process $\{S_n(u)\}$ converges weakly to that of a 2-dimensional Gaussian process.

It is well known that the sequences $\sqrt{n}h(n)(\hat{q}_1(u) - q_1(u))$ and $\sqrt{n}h(n)(\hat{q}_2(u) - q_2(u))$ converge weakly in $(D[0,1], .)$ to $B(q_1(u))$ and $B(q_2(u))$ respectively, where $B(q_1(u))$ and $B(q_2(u))$ are Brownian Bridge processes with zero means. Thus, two sequences $\sqrt{n}h(n)(\hat{q}_1(u) - q_1(u))$ and $\sqrt{n}h(n)(\hat{q}_2(u) - q_2(u))$ are asymptotically tight which implies that the process $\{S_n(u)\}$ is also asymptotically tight using (Lemma 1.4.3 and Theorem 1.5.4, van der Vaart and Wellner (1996)).

Distribution of $S_n(u)$ is established using Theorem 1.5.4 of van der Vaart and Wellner (1996). Hence, we conclude that the finite dimensional distribution of the process $\{S_n(u)\}$ converges weakly to that of a 2-dimensional Gaussian process $\{B_1(q_1(u)), B_2(q_2(u))\}$. □

Proof: [**Theorem 1**] From Lemma 1, we have

$$\sqrt{n}h(n)\{(\hat{q}_1(u) - q_1(u)), (\hat{q}_2(u) - q_2(u))\} \xrightarrow{\mathbf{L}} \{B_1(q_1(u)), B_2(q_2(u))\}$$

where B_i are Brownian bridge processes with zero means. Using continuous mapping theorem,
$\sup_{0<u<1} \sqrt{n}h(n)(\hat{q}_1(u) - \hat{q}_2(u))$ converges to $\sup_{0<u<1} (B_1(q_1(u)) - B_2(q_2(u)))$ as $n \to \infty$. □

Proof: [**Theorem 3**] The proof follows using Hadamard differentiability and functional delta method (Theorem 3.9.4, Van der Vaart and Wellner (1996)). Let $BV_1[0,1]$ denote the set of cadlag functions of total variation bounded by M (finite). The map

$\phi(A, B) = \int_0^1 AdB$ from $D[0,1] \times BV_1[0,1]$ to the real line is Hadamard differentiable (using Lemma 3.9.17 of van der Vaart and Wellner (1996)). The Hadamard derivative of

$\phi(A,B)$ is

$$\phi_{(A,B)}(\alpha,\beta) = \int_0^1 A d\beta + \int_0^1 \alpha dB \tag{21}$$

where $\int A d\beta$ is defined via integration by parts if β is not of bounded variation.

Let $A = q_1(u) - q_2(u)$, $B = \frac{Q_1(u)+Q_2(u)}{2}$, $\alpha = B_1(q_1(u)) - B_2(q_2(u))$ and $\beta = B_3(\frac{Q_1(u)+Q_2(u)}{2})$ where B_3 is a Brownian bridge process with mean zero.

Using Lemma 1 and delta method, we get for large n and under H_0

$$\begin{aligned}\sqrt{n}h(n)T_2 &\to \phi_{(q_1(u)-q_2(u),\frac{\hat{Q}_1(u)+\hat{Q}_2(u)}{2})}(B_1(q_1(u)) - B_2(q_2(u)), B_3(\frac{Q_1(u)+Q_2(u)}{2})) \\ &= \int (B_1(q_1(u)) - B_2(q_2(u)))d\Big(\frac{(Q_1(u)+Q_2(u)}{2}\Big),\end{aligned} \tag{22}$$

since the first term in (22) is zero under H_0 for large n.

Hence, the limiting random variable is normally distributed with mean zero and variance

$$\sigma^2 = Var(\int (B_1(q_1(u)) - B_2(q_2(u)))d(\frac{(Q_1(u)+Q_2(u)}{2})). \tag{23}$$

□

A.2: Tables

Table 6: Power comparison for GLD(1,1,2,1) *vs* GLD(1,2,2,1)

		h(n)		
n	Statistics	0.15	0.19	0.25
25	Sup uncensored	0.390(0.636)	0.288(0.614)	0.310(0.824)
	Sup censored	**0.15(0.168)**	**0.18(0.153)**	**0.250(0.266)**
	Int uncensored	0.984(1.000)	0.966(1.000)	0.978(1.000)
	Int censored	**0.262(0.247)**	**0.347(0.365)**	**0.457(0.428)**
	Kochar	0.356	0.356	0.356
	Cheng	0.146	0.146	0.146
50	Sup uncensored	0.422(0.806)	0.712(0.948)	0.836(0.948)
	Sup censored	**0.305(0.585)**	**0.389(0.444)**	**0.491(0.584)**
	Int uncensored	1.000(1.000)	1.000(1.000)	1.000(1.000)
	Int censored	**0.565(0.283)**	**0.767(0.793)**	**0.862(0.923)**
	Kochar	0.524	0.524	0.524
	Cheng	0.146	0.146	0.146
100	Sup uncensored	1.000(1.000)	1.000(1.000)	1.000(1.000)
	Sup censored	**0.496(0.638)**	**0.773(0.82)**	**0.951(0.963)**
	Int uncensored	1.000(1.000)	1.000(1.000)	1.000(1.000)
	Int censored	**0.972(0.981)**	**0.992(0.987)**	**0.997(0.998)**
	Kochar	0.798	0.798	0.798
	Cheng	0.160	0.160	0.160

Table 7: Power comparison for GLD(1,1,2,1) *vs* GLD(1,1.2,2,1)

		h(n)		
n	Statistics	0.15	0.19	0.25
25	Sup uncensored	0.086(0.102)	0.076 (0.080)	0.122 (0.084)
	Sup censored	**0.061(0.078)**	**0.095(0.100)**	**0.086(0.100)**
	Int uncensored	0.196(0.182)	0.260(0.214)	0.304(0.308)
	Int censored	**0.111(0.133)**	**0.127(0.194)**	**0.170(0.193)**
	Kochar	0.102	0.102	0.102
	Cheng	0.109	0.109	0.109
50	Sup uncensored	0.086(0.104)	0.130(0.16)	0.182(0.142)
	Sup censored	**0.122(0.122)**	**0.142(0.165)**	**0.157(0.205)**
	Int uncensored	0.636(0.58)	0.688(0.588)	0.804(0.804)
	Int censored	**0.266(0.343)**	**0.361(0.401)**	**0.539(0.548)**
	Kochar	0.200	0.200	0.200
	Cheng	0.110	0.110	0.110
100	Sup uncensored	0.146(0.118)	0.188(0.222)	0.322(0.358)
	Sup censored	**0.232(0.241)**	**0.283(0.316)**	**0.369(0.405)**
	Int uncensored	0.974(0.968)	0.982(0.986)	0.992(0.996)
	Int censored	**0.712(0.756)**	**0.805(0.819)**	**0.915(0.941)**
	Kochar	0.301	0.301	0.301
	Cheng	0.119	0.119	0.119

Table 8: Power comparison for EXP(1) *vs* EXP(2)

		h(n)		
n	Statistics	0.15	0.19	0.25
25	Sup uncensored	0.270(0.310)	0.400(0.230)	0.350(0.290)
	Sup censored	**0.126(0.106)**	**0.106(0.170)**	**0.186(0.242)**
	Int uncensored	0.570(0.510)	0.620(0.600)	0.600(0.650)
	Int censored	**0.086(0.118)**	**0.122(0.198)**	**0.108(0.144)**
	Kochar	0.694	0.694	0.694
	Cheng	0.740	0.740	0.740
50	Sup uncensored	0.540(0.360)	0.590(0.360)	0.550(0.570)
	Sup censored	**0.240(0.244)**	**0.238(0.192)**	**0.386(0.356)**
	Int uncensored	0.700(0.690)	0.700(0.730)	0.780(0.810)
	Int censored	**0.216(0.154)**	**0.246(0.240)**	**0.254 (0.222)**
	Kochar	0.926	0.926	0.926
	Cheng	0.939	0.939	0.939
100	Sup uncensored	0.380(0.670)	0.490(0.320)	0.620(0.710)
	Sup censored	**0.370(0.250)**	**0.476(0.366)**	**0.538(0.386)**
	Int uncensored	0.810(0.780)	0.900(0.830)	0.830(0.910)
	Int censored	**0.304(0.296)**	**0.172(0.240)**	**0.284(0.338)**
	Kochar	0.998	0.998	0.998
	Cheng	0.999	0.999	0.999