Statistics and Applications {**ISSN 2452-7395 (online)**}
Volume 17 No. 2, 2019 (New Series), pp 55-64

Construction of Optimal Foldover Designs with the General Minimum Lower-Order Confounding

Faisal Atakora and Po Yang

Department of Statistics
University of Manitoba
Winnipeg, MB R3T 2N2 Canada

Received: December 26, 2018; Revised: July 16, 2019; Accepted: August 24, 2019

Abstract

Fractional factorial designs are widely used in industry and agriculture. Foldover fractional factorial designs can de-alias effects of interest so that the effects can be estimated without ambiguities. We consider optimal foldover designs using general minimum lower-order confounding criterion. A catalogue of 16- and 32-run optimal foldover designs is constructed and tabulated for practical use. A comparison is made between the general minimum lower-order confounding optimal foldover designs and other optimal foldover designs obtained using minimum aberration and clear effect criteria.

Key words: Foldover; Alias; Optimal designs; Aliased effect number pattern.

1. Introduction

Fractional factorial designs have been widely used in industry and agriculture. One problem an experimenter is likely to face by employing a fractional factorial design is that some effects may be aliased with others. This creates ambiguities about the analysis and estimation of the factors. Hence, sometimes there is the need for additional runs to clarify these ambiguities. Foldover is a follow-up method that is often used to solve the problem.

Foldover designs have been in the literature for many years. Some textbook, such as, Box et al. (2005) and Wu and Hamada (2009) studied foldover techniques. Montgomery and Runger (1996) considered the foldover plans that reverse the signs of one or two factors. They pointed out that for a resolution IV initial design, when changing the signs of a factor of interest, the combined foldover design can de-alias all the two-factor interactions that contain the factor; Li and Lin (2003) searched optimal foldover plans using minimum aberration optimality criterion

Corresponding Author: Po Yang
E-mail: Po.Yang@umanitoba.ca
Invited paper: Part of the special issue on Designs for Factorial Experiments and their Applications, guest edited by Rajender Parsad, Sudhir Gupta and Kashinath Chatterjee

and provided catalogues of optimal foldover designs. However, Wang et al. (2010) pointed out that some optimal foldover designs in Li and Lin (2003) may not be optimal under the clear effect optimality criterion. Uniformity criteria were also used to construct optimal foldover designs. Ou et al. (2011) provided some lower bounds of centered L_2-discrepancy, the symmetric L_2-discrepancy and the wrap-around L_2-discrepancy on combined foldover designs that can be used to evaluate the optimal foldover plans for two-level fractional factorial designs. The lower bounds were further improved by Ou et al. (2017). Qin et al. (2013) extended the results in Ou et al. (2011) to a set of asymmetric fractional factorials and obtained a new lower bound of centred L_2-discrepancy of combined designs, which can be used as a benchmark for searching optimal foldover plans. Recently, Li and Lin (2016) proposed to further improve foldover designs by allowing column permutations. Yang and Li (2019) investigated under what circumstances can a foldover design be improved by doing column permutations and provided some theoretical results. For more work related to optimal foldover designs, see Li and Mee (2002), Li and Jacroux (2007), and Ai et al. (2010).

Optimal designs have been studied extensively in the literature recently. The most popular optimality criterion for fractional factorial designs is minimum aberration (MA). However, Minimum aberration criterion is used to choose designs in cases where there is very small or lack of knowledge about the likely important effects. Recently, Zhang et al. (2008) proposed the general minimum lower-order confounding (GMC) criterion to select optimal designs. This criterion reveals the basic information of all effects aliased with other effects at varying severity degrees in a design. In particular, when an experimenter has a prior knowledge about which factors are more important, the GMC optimal designs are better than other optimal designs. It has been proved that the GMC criterion chooses optimal designs in a more elaborate and explicit manner than the existing ones, such as minimum aberration, clear effects, and maximum capacity criteria. A lot of research has been done for constructing general minimum lower-order confounding designs. In particular, Zhang and Cheng (2010), Cheng and Zhang (2010), and Li et al. (2011) provided construction theory for two-level unblocked GMC designs, and Zhang and Mukerjee (2009) and Zhao et al. (2013) constructed two-level blocked GMC designs.

Some optimality criteria, such as minimum aberration and clear effects criteria, have been used to finding optimal foldover designs. Most of the optimal foldover designs in the literature are obtained from the minimum aberration designs tabulated in Chen et al. (1993). Zhang et al. (2008) obtained a catalogue of optimal GMC designs which are more useful comparing to MA designs when experimenters have prior knowledge about the important factors. The objective of this research is to find the optimal foldover plans using the general minimum lower-order confounding criterion from the GMC designs tabulated in Zhang et al. (2008). We then compare our results to the optimal foldover designs under the minimum aberration and clear effects criteria.

In section 2, we introduce aliased effect number pattern and general minimum lower-order confounding criterion. Minimum aberration and clear effects criteria will also be introduced. Some relationships between these criteria are mentioned as well as their limitations and drawbacks. The relationships of aliased effect number patterns between initial designs and combined foldover designs are presented in section 3. In section 4, we present optimal foldover designs obtained using general minimum lower-order confounding criterion, minimum aberration, and clear effects criteria. The comparison between those optimal foldover designs are made.

2. GMC Criterion, Foldover Designs and Other Optimality Criteria

Let n be the number of factors of a design and p be the number of generators. For a given 2^{n-p} design, we can find how ith-order effects and jth-order effects of the design are aliased with each other. When ith-order effects is aliased with k jth-order effects, we say that the *degree* of ith-order effects being aliased with jth-order effects is k. The number of ith-order effects that are aliased with jth-order effects at degree k is denoted by ${}_i^{\#}C_j^{(k)}$, which tells us how severe ith-order effects are aliased with jth-order effects. At the same time, it also tells how many ith-order effects are aliased with jth-order effects. It reveals the general aliasing that exists among effects. All the ${}_i^{\#}C_j^{(k)}$'s in a design forms a set

$$\{{}_i^{\#}C_j^{(k)}, i = 0, 1, 2, ..., n, j = 0, 1, 2, ...n, k = 0, 1, 2, ..., \binom{n}{j}\}.$$

The smaller the degree at which an ith-order effect is aliased with other effects, the less difficult it becomes in estimating the effect. Moreover, since the total number of i-th order effects of a design is $\binom{n}{i}$, the smaller the ${}_i^{\#}C_j^{(0)}$, that is, the smaller the number of i-th order effects that are not aliased j-th order effects, the more the severity of the confounding between ith-order effects and jth-order effects. On the other way, when the value of ${}_i^{\#}C_j^{(0)}$ is greater, the less severity of ith-order effects are aliased with jth-order effects.

Since for different k, ${}_i^{\#}C_j^{(k)}$s are not equally important. As the degree k increases, the severity of aliasing increases. Thus Zhang et al. (2008) arranged ${}_i^{\#}C_j^{(k)}$s as

$${}_i^{\#}C_j = ({}_i^{\#}C_j^{(0)}, {}_i^{\#}C_j^{(1)}, ..., {}_i^{\#}C_j^{(v)}), \tag{1}$$

where $v = \binom{n}{j}$. Equation (1) gives the total number of ith-order effects aliased with jth-order effects at various degrees starting from the least to the greatest in terms of severity.

Example 1. Consider the design with generators $4 = 12$ and $5 = 13$. The defining relation is I=124=135=2345. Since all the main effects are aliased with two-factor interactions, ${}_1^{\#}C_2^0$=0. Note that the number of main effects that are aliased with one and two two-factor interactions are 4 and 1, respectively, hence ${}_1^{\#}C_2^1$=4 and ${}_1^{\#}C_2^2$=1. Moreover, there are no main effects that are aliased with three or more two-factor interactions, hence ${}_1^{\#}C_2^k$=0 for $k \geq 3$. Therefore ${}_1^{\#}C_2 = (0, 4, 1)$

Similarly, there are four two-factor interactions that are not aliased with other two-factor interactions, we get ${}_2^{\#}C_2^0$=4; there are six two-factor interactions that are aliased with only one two-factor interactions, hence ${}_2^{\#}C_2^1$=6; there are no two-factor interactions that are aliased with other two or more two-factor interactions, hence ${}_2^{\#}C_2^k$=0 for $k \geq 2$. Therefore ${}_2^{\#}C_2 = (4, 6)$. One can find other ${}_i^{\#}C_j s$ similarly.

Zhang et al. (2008) defined the *aliased effect number pattern* (AENP) as

$${}^{\#}C = ({}_1^{\#}C_1, {}_0^{\#}C_2, {}_1^{\#}C_2, {}_2^{\#}C_1, {}_2^{\#}C_2, {}_0^{\#}C_3, {}_1^{\#}C_3, {}_2^{\#}C_3, {}_3^{\#}C_1, {}_3^{\#}C_2, {}_3^{\#}C_3, {}_0^{\#}C_4, {}_1^{\#}C_4, ...) \tag{2}$$

The elements in ${}^{\#}C$ are placed in using a rule: If $max(i, j) < max(q, r)$, then ${}_i^{\#}C_j$ is placed before ${}_q^{\#}C_r$; if $max(i, j) = max(q, r)$ and $i < q$, then ${}_i^{\#}C_j$ is placed before ${}_q^{\#}C_r$; if $max(i, j) = max(q, r)$, $i = q$ and $j < r$, then ${}_i^{\#}C_j$ is placed before ${}_q^{\#}C_r$.

Suppose ${}^{\#}C(d_1)$ and ${}^{\#}C(d_2)$ are the aliased effect number patterns of two designs d_1 and d_2, respectively, and ${}^{\#}C_m$ is the mth component of ${}^{\#}C$. Let ${}^{\#}C_m$ be the first component for which ${}^{\#}C_m(d_1)$ and ${}^{\#}C_m(d_2)$ differ. If ${}^{\#}C_m(d_1) > {}^{\#}C_m(d_2)$ we say that d_1 has less general lower order confounding relative to d_2. A design is said to be a *general minimum lower-order confounding* (GMLOC or GMC) design, if it has minimum general lower order confounding relative to other designs.

A foldover design is obtained by reversing the signs of one or more factors of the initial design. The combination of the initial design and the foldover design is called a *combined foldover design*. The set of factors whose signs are revised in a foldover design is referred to as a *foldover plan*. For each initial design, there are many foldover plans. Li and Lin (2003) showed that any foldover plan is equivalent to a *core foldover plan*, which contains only generated factors. Thus, we only need to consider core foldover plans when studying combined foldover designs.

Eexample 2. Consider the combined foldover designs obtained from the initial design discussed in Example 1. The generated factors are 4 and 5. Thus, there are three core foldover plans 4, 5 and 45, where 45 indicates that the signs of both factors 4 and 5 are switched. The resulting three combined foldover designs are denoted as d_1, d_2 and d_3, respectively. The defining relations of d_1, d_2 and d_3 are $I = 135$, $I = 124$, and $I = 2345$, respectively. The AENPs of the three combined foldover designs first differ at ${}_1^{\#}C_2^1(d_1) = {}_1^{\#}C_2^1(d_2) = 2$ and ${}_1^{\#}C_2^1(d_3) = 5$. Since ${}_1^{\#}C_2^1(d_3)$ is larger than both ${}_1^{\#}C_2^1(d_1)$ and ${}_1^{\#}C_2^1(d_2)$, we obtain that 45 is the optimal foldover plan and the corresponding combined foldover design is the optimal combined foldover design.

Zhang et al. (2008) discussed the relationships between the GMC criterion and other criteria, such as minimum aberration and clear effects criteria. The minimum aberration criterion was introduced by Fries and Hunter (1980) and it has remained one of the popular criteria in choosing optimal designs when experimenters do not have information about the important effects. Define the length of a word as the number of factors in the word. The minimum aberration criterion depends on the *word length pattern* which is defined as $(A_1, A_2, A_3...)$, where $A_i, i = 1, 2, ...$, represent the number of length-i words in the defining relation of the design. For instance, for the design in Example 1, the word length pattern is (0, 0, 2, 1, 0...). The minimum aberration design can be obtained by sequentially minimizing the component of the word length pattern. Zhang et al. (2008) pointed out that the WLP is only related to ${}_i^{\#}C_0^1$, $i = 1, 2, ...$, and AENP is a more refined pattern than the WLP for judging designs.

Clear effects are effects that are not aliased with main effects and two-factor interactions. One of the drawbacks of the minimum aberration criterion is that, sometimes it is unable to maximize the number of some clear lower-order effects especially two-factor interactions. One of the criteria that takes care of this situation is the *clear effects criterion*. The optimal designs sequentially maximizes the number of clear main effects and the number of clear two-factor interactions. Zhang et al. (2008) pointed out that ${}_2^{\#}C_2^0$ is the number of clear main effects and ${}_2^{\#}C_2^0 - {}_1^{\#}C_2^1$ is the number of clear two-factor interactions. For more details about the relationship between the GMC criterion and MA and clear effects criteria, see Zhang et al. (2008).

3. Relationships Between AENPs

It is well known that when the signs of all the factors of an initial design are reversed, all the words containing odd number of factors disappear in the combined foldover design and all the words that contain even number of factors in the initial design are still in the combined foldover design. Let ${}_i^{\#}C_j^{(k)}(d)$ and ${}_i^{\#}C_j^{(k)}(c)$ denote the number of ith-order effects that is aliased with k jth-order effects in the initial design d and the combind foldover design c, respectively. Note that ${}_l^{\#}C_0^{(0)} = \binom{n}{l} - A_l$ or ${}_l^{\#}C_0^{(1)} = A_l$ by Theorem 2 in Zhang et al. (2008). From the well known result, we obtain some relationships between the AENPs of initial designs and combined foldover designs as shown in Result 1.

Result 1, Assume that a combined foldover design is obtained by reversing the signs of all the factors in the initial design. Then,

(1) if l is odd, then ${}_l^{\#}C_0^{(0)}(c) = \binom{n}{l}$ or ${}_l^{\#}C_0^{(1)}(c) = 0$.

(2) if $i + j$ is odd, then ${}_i^{\#}C_j^{(0)}(c) = r$, where r is the number of i-th order effects of the design d and ${}_i^{\#}C_j^{(k)}(c) = 0$ for $k = 1, 2, ...$

(3) if $i + j$ is even, then ${}_i^{\#}C_j^{(k)}(d) ={}_i^{\#} C_j^{(k)}(c)$ for $k = 0, 1, 2, ...$

Since some effects are de-aliased after folding an initial design, the number of **i**-order effects that are not aliased with any **j**-th order effects in a combined foldover design is always greater than or equal to that of its corresponding initial design. Therefore, we have Result 2.

Result 2. ${}_i^{\#}C_j^{(0)}(c) \geq{}_i^{\#} C_j^{(0)}(d)$ for any $i, j = 1, 2, ...$

The resolution of a design is defined as the length of the shortest word of the design. Note that the results 1 and 2 are true for designs with any resolution.

4. Optimal Foldover Designs

Zhang et al. (2008) presented a catalogue of 16- and 32-run GMC designs. We search optimal foldover designs from the designs in Zhang et al. (2008) using GMC criterion, and compare them with the optimal foldover designs obtained using MA and clear effects criteria. For each initial design, we consider all core foldover plans and calculate the AENP for each combined foldover design. Then the AENPs are compared and the optimal combined foldover designs are obtained. Similarly, the optimal MA foldover designs and the optimal clear effects foldover designs are also obtained. Tables 1 and 2 present the optimal foldover designs obtained from the 16- ($6 \leq n \leq 12$) and 32-run ($7 \leq n \leq 15$) designs, respectively, in Zhang et al. (2008).

In Tables 1 and 2, the first column lists the initial designs in Zhang et al. (2008); the second column lists additional columns, which represent the generators of each design, from the design matrix (Table 1 in Chen et al. Chen *et al.* (1993)); the third column lists AENP of the initial design. To save space, we list only $({}_1^{\#}C_2\,,\ {}_2^{\#}C_2)$. The fourth and fifth columns represent the optimal foldover plans based on GMC criterion and AENP of the corresponding optimal combined foldover designs, respectively. The sixth column shows the word length pattern (A_4, A_5, A_6) of the optimal foldover designs chosen based on MA criterion. We do not show A_3 since it is zero for any optimal foldover designs in the table. To save space, the corresponding optimal foldover plans are not listed. The last column presents the number of clear main effects c_1 and the number of clear two-factor interactions c_2 of the optimal foldover designs based on clear effects criterion. Again, to save space, the corresponding optimal foldover plans are not listed.

For example, for 16-run design 7.3.1, the AENP of the initial design is $({}^{\#}_{1}C_2; {}^{\#}_{2}C_2) = (7; 0, 0, 21)$. To save space, we write it as $(7; 0^2, 21)$. The optimal foldover plans based on GMC criterion are $5, 6, 7, 56, 57, 67, 567$ and the AENP of the corresponding optimal foldover designs is $(7; 6, 12, 3)$. The word length pattern of the optimal foldover designs based on MA criterion is $(A_4, A_5, A_6) = (3, 0, 0)$. For the optimal foldover designs obtained using clear effects criterion, the number clear main effects is $c_1 = 7$ and the number of clear two-factor interactions is $c_2 = 6$.

For the optimal GMC foldover designs, we find that most of the 16-run designs considered have only one optimal foldover plan except for designs 6.2.1, 7.3.1 and 8.4.1; for 32-run designs, most of them have more than one optimal foldover plans. One can see that ${}^{\#}_{i}C^{(0)}_{j}$ of all the optimal foldover designs are the same or larger than that of the corresponding initial designs.

Although the optimal foldover plans of MA foldover designs are not listed, MA and GMC criteria choose the same foldover plans as optimal for all the designs considered except for the 16-run design 8.4.1 and 32-run designs 9.4.3, 11.6.1, 12.7.1, 12.7.2, 13.8.1 and 15.10.1, for which, the two criteria choose completely different optimal foldover plans. This shows that the GMC criterion can choose completely different foldover plans as the optimal from MA criterion. For example, for design 8.4.1, eight foldover plans 5, 6, 7, 8, 567, 568, 578, and 678 are chosen as optimal according to GMC criterion while six foldover plans 56, 57, 58, 67, 68, and 78 are selected as optimal according to MA criterion. The $AENP's$ of the optimal foldover designs chosen by GMC and MA criteria are $(8; 7, 0, 21)$ and $(8; 0, 24, 0, 4)$, respectively. Clearly, the foldover designs chosen by GMC criterion has seven clear two-factor interactions and the optimal foldover designs selected by MA criterion has no clear two-factor interactions. Therefore, GMC criterion chooses better designs than MA criterion in terms of estimation of effects. For 32-run designs, we search optimal designs for 13 designs. The two criteria choose complete different optimal foldover plans for six designs 9.4.3, 11.6.1, 12.7.1, 12.7.2, 13.8.1 and 15.10.1. For each of the six designs, there are more than one optimal MA foldover plans. We present only one for each design here, they are 67, $69\underline{10}$, $67\underline{10}\,\underline{12}$, $67\underline{12}$, $67\underline{11}\,\underline{13}$, and $67\underline{11}\,\underline{14}\,\underline{15}$, respectively. The results show that when the number of runs becomes larger, the two criteria tends to choose different optimal foldover plans.

For the optimal foldover designs chosen by the clear effects criterion, their optimal foldover plans always include the ones chosen by the GMC criterion. In fact, the two criteria choose the same optimal foldover plans for all the designs except for the 16-run design 9.5.1 and 32-run designs 9.4.1, 11.6.2, 12.7.2, and 13.8.1. For instance, for design 9.5.1, the GMC criterion chooses 589 as the optimal foldover plan while the clear effect criterion chooses 5 and 589 as the optimal foldover plans. The $AENP's$ of the optimal foldover designs obtained by foldover plans 5 and 589 are $(9; 8, 0, 0, 28)$ and $(9; 8, 24, 0, 4)$, respectively. Even though both foldover designs have the same number of clear main effects and two-factor interactions, the former have 24 two-factor interactions aliased with only 1 two-factor interaction whiles the latter have 28 two-factor interactions aliased with 3 two-factor interactions. The alias structure in latter is more severe than in the former, thereby making it less preferable. In general, when the run size of a design is large, the GMC criterion tends to choose less optimal foldover plans than the clear effects criterion.

Acknowledgments

The authors would like to express their sincere thanks to the anonymous referee and the editor for their valuable comments and suggestions which led to significant improvement in the presentation and clarity of the paper. Atakora's research was supported by Manitoba Graduate Scholarships. Yang's research was supported by the National Sciences and Engineering Research Council of Canada.

References

Ai, M., Xu, X. and Wu, C. F. J. (2010). Optimal blocking and foldover plans for regular two-level designs. *Statistica Sinica*, **20**, 183–207.

Box, G. E. P., Hunter, J. S. and Hunter, W. G. (2005). *Statistics for Experimenters: Design, Innovation, and Discovery*. Wiley, New York, NY.

Chen, J., Sun, D. X. and Wu, C. F. J. (1993). A catalogue of two-level and three-level fractional factorial designs with small runs. *International Statistical Review*, **61**(131–145).

Cheng, Y. and Zhang, R. C. (2010). On construction of general minimum lower order confounding 2^{n-m} designs with $N/4 + 1 \leq n \leq 9N/32$. *Journal of Statistical Planning and Inference*, **140**, 2384–2394.

Fries, A. and Hunter, W. G. (1980). Minimum aberration 2^{k-p} designs. *Technometrics*, **22**(4), 601–608.

Li, H. and Mee, R. (2002). Better foldover fractions for resolution III 2^{k-p} designs. *Technometrics*, **44**, 278–283.

Li, P., Zhao, S. and Zhang, R. C. (2011). A theory on constructing 2^{n-m} designs with general minimum lower order confounding. *Statistica Sinica*, **21**, 1571–1589.

Li, R. and Jacroux, M. (2007). Optimal foldover plans for blocked 2^{m-k} fractional factorial designs. *Journal of Statistical Planning and Inference*, **137**, 2439–2452.

Li, W. and Lin, D. (2003). Optimal foldover plans for two-level fractional factorial designs. *Technometrics*, **45**(2), 142–149.

Li, W. and Lin, D. K. J. (2016). A note on foldover of 2^{k-p} designs with column permutations. *Technometrics*, **58**(4), 508–512.

Montgomery, D. and Runger, G. (1996). Foldovers of 2^{k-p} resolution IV experimental designs. *Journal of Quality Technology*, **28**(4), 446–450.

Ou, Z., Chatterjee, K. and Qin, H. (2011). Lower bounds of various discrepancies on combined designs. *Metrika*, **74**, 109–119.

Ou, Z., Qin, H. and Chatterjee, K. (2017). Some new lower bounds to various discrepancies on combined designs. *Communications in Statistics*, **46**, 3244–3254.

Qin, H., Chatterjee, K. and Ou, Z. (2013). A lower bound for the centred L_2-discrepancy on combined designs under the asymmetric factorials. *Statistics*, **47**, 992–1002.

Wang, B., McLeod, R. G. and Brewster, J. F. (2010). A note on the selection of optimal foldover plans for 16- and 32-run fractional factorial designs. *Journal of Statistical Planning and Inference*, **140**, 1497–1500.

Wu, J. and Hamada, M. (2009). *Experiments: Planning, Analysis and Parameter Design Optimization*. Wiley, New York, NY.

Yang, P. and Li, W. (2019). Some properties of foldover designs with column permutations. *Metrika*, **82**, 705–717.

Zhang, R. C. and Cheng, Y. (2010). General minimum lower order confounding designs: an overview and a construction theory. *Journal of Statistical Planning and Inference*, **140**, 1719–1730.

Zhang, R. C., Li, P., Zhao, S. and Ai, M. (2008). A general minimum lower-order confounding criterion for two-level regular designs. *Statistica Sinica*, **18**, 1689–1705.

Zhang, R. C. and Mukerjee, R. (2009). General minimum lower order confounding in block designs using complementary sets. *Statistica Sinica*, **19**, 1787–1802.

Zhao, S., Li, P., Zhang, R. and Karunamuni, R. (2013). Construction of blocked two-level regular designs with general minimum lower order confounding. *Statistica Sinica*, **143**, 1082–1090.

Table 1: Optimal foldover designs for 16-run initial designs

Initial designs			Optimal foldover designs			
Design	Additional Columns	AENP $({}^{\#}_{1}C_2;\ {}^{\#}_{2}C_2)$	OFP	ANEP $({}^{\#}_{1}C_2;\ {}^{\#}_{2}C_2)$	WLP (A_4, A_5, A_6)	CE c_1, c_2
6.2.1	7,14	$6; 0, 12, 3$	5,6,56	$6; 9, 6$	$(1, 0, 0)$	6,9
6.2.3	6,12	$1, 4, 1; 9, 6$	56	$6; 9, 6$	$(1, 0, 0)$	6,9
6.2.4	3,6	$6; 15$	56	$6; 15$	$(0, 0, 1)$	6,15
7.3.1	7,11,14	$7; 0^2, 21$	5,...,567[1]	$7; 6, 12, 3$	$(3, 0, 0)$	7,6
7.3.3	6,10,12	$1, 0, 6; 6, 12, 3$	567	$7; 6, 12, 3$	$(3, 0, 0)$	7,6
7.3.5	3,6,12	$0, 5, 2; 9, 12$	567	$7; 9, 12$	$(2, 0, 1)$	7,9
8.4.1	7,11,13,14	$8; 0^3, 28$	5,..., 678[2]	$8; 7, 0, 21$	$(6, 0, 0)$	8,7
8.4.2	3,5,7,14	$2, 0, 6; 0, 24, 0, 4$	568	$8; 13, 12, 3$	$(3, 4, 0)$	8,13
8.4.3	3,7,11,14	$1, 6, 0, 1; 7, 0, 21$	58	$8; 13, 12, 3$	$(3, 4, 0)$	8,13
8.4.4	6,10,12,14	$1, 0^2, 7; 7, 0, 21$	567	$8; 7, 0, 21$	$(7, 0, 0)$	8,7
8.4.5	3,7,12,14	$0, 4, 4; 4, 18, 16$	57	$8; 4, 18, 6$	$(5, 0, 2)$	8,4
9.5.1	3,7,11,13,14	$0, 8, 0^2, 1; 8, 0^2, 28$	589	$9; 8, 24, 0, 4$	$(6, 8, 0)$	9,8
9.5.2	3,6,7,11,14	$0, 2, 5, 2; 2, 12, 18, 4$	56	$9; 2, 12, 18, 4$	$(10, 0, 4)$	9,2
9.5.3	3,6,10,12,14	$0, 2, 0, 6, 1; 2, 12, 18, 4$	5678	$9; 2, 12, 18, 4$	$(10, 0, 4)$	9,2
9.5.4	3,7,9,12,14	$0^2, 9; 0, 18, 18$	578	$9; 0, 18, 18$	$(9, 0, 6)$	9,0
9.5.5	3,6,7,12,14	$0^2, 6, 3; 0, 18, 18$	568	$9; 0, 18, 18$	$(9, 0, 6)$	9,0
10.6.1	3,6,7,11,13,14	$0^2, 8, 0, 2; 0, 16, 0, 24, 5$	56	$10; 0, 16, 0, 24, 5$	$(18, 0, 8)$	10,0
10.6.3	3,5,6,10,12,14	$0^2, 3, 4, 3; 0, 6, 27, 12$	56789	$10; 0, 6, 27, 12$	$(16, 0, 12)$	10,0
10.6.4	3,6,7,12,14,15	$0^3, 10; 0^2, 45$	568$\underline{10}$	$10; 0^2, 45$	$(15, 0, 15)$	10,0
11.7.1	3,6,7,11,12,13,14	$0^3, 8, 3; 0^2, 24, 16, 15$	569	$11; 0^2, 24, 16, 15$	$(26, 0, 24)$	11,0
11.7.2	3,5,6,7,11,13,14	$0^3, 8, 0, 3; 0^2, 24, 16, 15$	567	$11; 0^2, 24, 16, 15$	$(26, 0, 24)$	11,0
11.7.3	3,5,6,7,9,12,14	$0^3, 5, 6; 0^2, 15, 40$	5679$\underline{10}$	$11; 0^2, 15, 40$	$(25, 0, 27)$	11,0
12.8.1	3,6,7,9,11,12,13,14	$0^4, 12; 0^3, 48, 0, 18$	568$\underline{10}$	$12; 0^3, 48, 0, 18$	$(39, 0, 48)$	12,0

Note: [1]The complete set is: 5,6,7,56,57,67,567. [2]The complete set is: 5,6,7,8,567,568,578,678.

Table 2: Optimal foldover designs for 32-Run initial designs

Initial designs			Optimal foldover designs			
Design	Additional Columns	AENP $({}^{\#}_{1}C_2;\ {}^{\#}_{2}C_2)$	OFP	ANEP $({}^{\#}_{1}C_2;\ {}^{\#}_{2}C_2)$	WLP (A_4, A_5, A_6)	CE c_1, c_2
7.2.1	7,30	$7; 15, 6$	6, 67	$7; 21$	(0, 1, 0)	7, 21
8.3.1	7,11,30	$8; 13, 12, 3$	6, 7, 67, 68 78, 678	$8; 22, 6$	(1, 2, 0)	8, 22
9.4.1	7,11,13,30	$9; 15, 0, 21$	67, 68, 69, 78 79, 89, 6789	$9; 21, 12, 3$	(3, 3, 0)	9, 21
9.4.2	7,11,19,30	$9; 8, 24, 0, 4$	67, 68, 78 679, 689, 789	$9; 24, 12$	(2, 4, 0)	9, 24
9.4.3	14,22,26,28	$9; 8, 0^2, 28$	6, 7, 8, 9, 678 679, 689, 789	$9; 15, 0, 21$	(6, 0, 0)	9, 15
10.5.1	7,11,19,29,30	$10; 0, 40, 0^2, 5$	67, 68, ..., $6789\underline{10}$[1]	$10; 24, 18, 3$	(4, 8, 0)	10, 24
11.6.1	7,11,14,22,26 28	$11; 0^2, 24, 16, 15$	68, 69, ..., $679\underline{10}\,\underline{11}$[2]	$11; 12, 18, 21, 4$	(10, 0, 16)	11, 12
11.6.2	7,11,14,19,25 28	$11; 0^2, 15, 40$	$68\underline{10}$, $69\underline{11}$,..., $6789\underline{10}\,\underline{11}$[3]	$11; 10, 30, 15$	(10, 0, 16)	11, 10
12.7.1	7,11,13,14,22 26, 28	$12; 0^3, 48, 0, 18$	6, 7, 8, ..., $789\underline{11}\,\underline{12}$[4]	$12; 11, 0, 24, 16, 15$	(15, 0, 32)	12, 11
12.7.2	7,11,13,14,19 25, 28	$12; 0^3, 36, 30$	$\underline{10}$, $\underline{11}$, $\underline{12}$, $\underline{10}\,\underline{11}\,\underline{12}$	$12; 11, 0, 24, 16, 15$	(16, 0, 30)	12, 11
13.8.1	7,11,13,14,19 22,26,28	$13; 0^4, 60, 18$	$\underline{10}$	$13; 12, 0^2, 48, 0, 18$	(23, 0, 56)	13, 12
14.9.1	7,11,13,14,19 21,22,26,28	$14; 0^5, 84, 7$	6, 7, 8, ..., $689\underline{11}\,\underline{12}\,\underline{14}$[5]	$14; 13, 0^3, 60, 18$	(33, 0, 96)	14, 13
15.10.1	7,11,13,14,19 21,22,25,26,28	$15; 0^6, 105$	6, 7, 8, ..., $\underline{10}\,\underline{11}\,\underline{12}\,\underline{13}\,\underline{14}\,\underline{15}$[6]	$15; 14, 0^4, 84, 7$	(45, 0, 160)	15, 14

Note: [1] - [6]The complete sets can be obtained upon request.