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A Discrete Analogue of Intervened Poisson Compounded Family of Distributions: Properties with Applications to Count Data

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Abstract

There are several discrete distributions have been developed in statistical literature. Even though, it is inadequate to analyse the real data produced from different fields through the various discrete distributions available in the existing literature. According to this motives, we have proposed a new family of discrete models called discrete intervened Poisson compounded (DIPc) family. A key feature of the proposed family is its hazrad rate function can take variety of shapes for distinct values of the parameters like decreasing, constant, bathtub shaped. Furthermore, several distributional characteristics are extensively studied for the particular distribuiton of DIPc family. Certain characterizations of the new distribution are obtained. An integer valued autoregressive process with the distribution as marginal is introduced. The unknown parameters of the distribution are estimated using different methods of estimation. Finally, we have explained the usefullness of the proposed family by using a real data set.

Key words: Characterizations; Exponential Intervened Poisson (EIP) distribution; Discrete Intervened Poisson (DEIP) distribution; INAR(1) process; Stress- strength parameter.

AMS Subject Classifications: 60E05, 62E10

1. Introduction

The intervened Poisson distribution (IPD) is introduced by Shanmugam (1985) which provides stochastic models to study the effect of such actions as they are closer to real life situations. The IPD is a modified version of zero truncated Poisson (ZTP) distribution, which is applicable in reliability analysis, queueing problems, epidemiological problems where ZTP fails. Jayakumar and Sankaran (2019) introduce a new family of distributions generated using IPD and this distributions helps to develop a rich class of families which contain Marshall and Olkin (1997) extended families of distribution. The intervened Poisson compounded (IP) family of continuous distributions is one among them. The cumulative distribution function

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(CDF) of IP family of distributions is given by

$$G(x;\lambda,\rho;\phi)=1-[\frac{e^{\lambda(1+\rho)\bar{F}(x;\phi)}-e^{\lambda\rho\bar{F}(x;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}]\ ;x\ \in\ \mathbb{R} \tag{1}$$

where $\bar{F}(x;\phi)$ is the CDF of base line continuous distribution and ϕ is the vector of the given model parameters.

Here we establish the discretization of continuous distribution. Discretization of a continuous lifetime model is an interesting and intuitively appealing approach to derive a lifetime model corresponding to the continuous one. Meanwhile, it is difficult or inconvenient to get samples from a continuous distribution in real life situations. In modelling, the observed values are actually discrete because they are measured to only a finite number of decimal places and cannot really constitute all points in a continuum. For example, in case of survival analysis, the number of days of survival for lung cancer patients since therapy are usually recorded in discrete values. In the recent, special role of discrete distributions are getting recognition in the field of reliability. In this way, one of the active areas of research is to model discrete data by developing discretized distributions.
Chakraborty (2015) surveyed different methods for generating discrete analogues of continuous probability distributions. One of the methods is described as follows:
Let X be a continuous random variable, then the discerte analogue Y of X can be derived by using the survival function as follows, $S(.)$ is the survival function of the random variable X, then

$$P(Y=y)=P(X\geq y)-P(X\geq y+1)=S(y)-S(y+1);y=0,1,2,3,.... \tag{2}$$

where $Y=\lfloor X\rfloor$ largest integer less than or equal to X. The first and easiest in this approach is the geometric distribution with pmf

$$p(x)=\theta^{x}-\theta^{x+1};x=0,1,2,...$$

which is derived by discretizing exponential distribution with survival function $S(x)=e^{-\lambda x};\lambda,x>0$ and $\theta=e^{-\lambda},(0<\theta<1)$.
Following this approach, discretization of some known continuous distributions for use as lifetime distribution was studied by different researchers. Nakagawa and Osaki (1975) proposed discrete Weibull distribution with pmf

$$P(Y=v)=q^{v^{\beta}}-q^{(v+1)^{\beta}},v=0,1,2,...;\beta>0,0<q<1. \tag{3}$$

Stein and Dattero (1984) presented another discretization of Weibull distribution. Roy (2003) proposed discrete normal distribution and also studied discrete Rayleigh distribution (Roy (2004)). Krishna and Pundir (2009) studied discrete Burr distribution, and obtained the discrete Pareto distribution as its particular case.

The discretization of a continuous distribution using this method retains the same functional form of the survival function. As a result, many reliability characteristics remain unchanged. As such there is enough motivation to use this technique of generating discretized version of continuous distribution with this approach to develop new discrete lifetime models corresponding to the existing continuous one. In this article, we propose a family of discrete univariate distributions using survival discretization method. Thus the objective of proposing Discrete Intervned Poisoon compounded (DIPc) family are to generate models for modelling probability distribution of count data and produce consistently superior fits than other developed discrete distributions in the existing literature.

The remaining parts of the article are as follows: Section 2 introduces the DIPc family and some statistical properties are derived. In Section 3, the special model of the proposed family is extensively studied. The expression for moments, stress - strength reliability are derived. Also, using the proposed distribution, an integer valued autoregressive process with the distribution as marginal is introduced. In Section 4, three characterizations of the new distribution are obtained and in Section 5, an extensive estimation and simulation study is conducted to investigate the behaviour of different estimation methods. The flexibility of the proposed model is illustrated by using a real data set in Section 6. Finally, some important remarks about the presented study are discussed in Section 7.

2. Genesis of the family

The random variable Y is said to follow Discrete Intervened Poisson compounded (DIPc) family, its probability mass function (pmf) is given by

$$P_Y(y;\lambda,\rho,\phi) = \left[\frac{e^{\lambda(1+\rho)\bar{F}(y;\phi)} - e^{\lambda\rho\bar{F}(y;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right] - \left[\frac{e^{\lambda(1+\rho)\bar{F}(y+1;\phi)} - e^{\lambda\rho\bar{F}(y+1;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]. \tag{4}$$

The corresponding CDF of DIPc is obtained as

$$\begin{aligned} G_Y(y;\lambda,\rho,\phi) &= 1 - G_X(y;\lambda,\rho,\phi) + P_Y(y;\lambda,\rho,\phi) \\ &= 1 - \left[\frac{e^{\lambda(1+\rho)\bar{F}(y;\phi)} - e^{\lambda\rho\bar{F}(y;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right] + \left[\frac{e^{\lambda(1+\rho)\bar{F}(y;\phi)} - e^{\lambda\rho\bar{F}(y;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right] - \left[\frac{e^{\lambda(1+\rho)\bar{F}(y+1;\phi)} - e^{\lambda\rho\bar{F}(y+1;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right] \\ &= 1 - \left[\frac{e^{\lambda(1+\rho)\bar{F}(y+1;\phi)} - e^{\lambda\rho\bar{F}(y+1;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]; y \in \mathbb{N} \end{aligned} \tag{5}$$

where $\mathbb{N} = \{0, 1, 2, ...\}$, $(\lambda, \rho) \in (0, \infty)$ and $G_X(y;\lambda,\rho,\phi) = \left[\frac{e^{\lambda(1+\rho)\bar{F}(y;\phi)} - e^{\lambda\rho\bar{F}(y;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]$ is the CDF of X.
The survival function of DIPc family is given by

$$S_Y(y;\lambda,\rho,\phi) = \frac{e^{\lambda(1+\rho)\bar{F}(y+1;\phi)} - e^{\lambda\rho\bar{F}(y+1;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)} \; ; y \in \mathbb{N}. \tag{6}$$

The hazard rate and reverse hazard rate are

$$h_Y(y;\lambda,\rho,\phi) = 1 - \left[\frac{e^{\lambda\rho[\bar{F}(y+1;\phi)-\bar{F}(y,\phi)]}(e^{\lambda\bar{F}(y;\phi)}-1)}{e^{\lambda\bar{F}(y+1;\phi)}-1}\right] \tag{7}$$

and

$$r_Y(y;\lambda,\rho,\phi) = \frac{\left[e^{\lambda(1+\rho)\bar{F}(y;\phi)} - e^{\lambda\rho\bar{F}(y;\phi)}\right] - \left[e^{\lambda(1+\rho)\bar{F}(y+1;\phi)} - e^{\lambda\rho\bar{F}(y+1;\phi)}\right]}{[e^{\lambda\rho}(e^{\lambda}-1)] - \left[e^{\lambda(1+\rho)\bar{F}(y+1;\phi)} - e^{\lambda\rho\bar{F}(y+1;\phi)}\right]} \tag{8}$$

respectively.

2.1. Moments

Let the random variable $Y \sim DIPc(\lambda, \rho, \phi)$, then the r^{th} moment is given by

$$\mu'_r = \sum_{y=0}^{\infty}((y+1)^r - y^r)\left[\frac{e^{\lambda(1+\rho)\bar{F}(y+1;\phi)} - e^{\lambda\rho\bar{F}(y+1;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right] \ ; y \ \in \ \mathbb{N}. \tag{9}$$

Using the Equation 9, the mean and variance of DIPc can be obtained as follows, respectively,

$$\mu'_1 = \sum_{y=0}^{\infty}\left[\frac{e^{\lambda(1+\rho)\bar{F}(y+1;\phi)} - e^{\lambda\rho\bar{F}(y+1;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right] \tag{10}$$

and

$$variance = \sum_{y=0}^{\infty}(2y+1)\left[\frac{e^{\lambda(1+\rho)\bar{F}(y+1;\phi)} - e^{\lambda\rho\bar{F}(y+1;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right] - (\mu'_1)^2. \tag{11}$$

The index of dispersion (DI), (variance/ mean), determines whether the given distribution is suited for under, over or equi-dispersed data sets. If $DI > 1$, then distribution is overdispersed whereas $DI < 1$, then distribution is underdispersed. If $DI = 1$, then distribution is equidispersed.
The moment generating function of the distribution is given by

$$M_Y(t) = \sum_{y=0}^{t}\sum_{r=0}^{\infty}\frac{(yt)^r}{r!}\left[\frac{e^{\lambda(1+\rho)\bar{F}(y;\phi)} - e^{\lambda\rho\bar{F}(y;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right] - \left[\frac{e^{\lambda(1+\rho)\bar{F}(y+1;\phi)} - e^{\lambda\rho\bar{F}(y+1;\phi)}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]. \tag{12}$$

From the Equation (12), it can be obtained first four raw moments about the origin when $t = 0$. Also skewness and kurtosis based on moments can be computed by using the moment generating function.

3. Special model

In this section, we study a particular distribution of DIPc family to establish its viability. The main objective of establishing new model is to study the properties of the particular model of the presented family, to illustrate the flexibility of the developed family through real data sets.

3.1. Discrete Exponential Intervened Poisson (DEIP) distribution

Using the CDF of the exponential distribution, the pmf of DEIP can be formulated as

$$P(Y = y) = \frac{\left[e^{\lambda(1+\rho)e^{-\theta y}} - e^{\lambda\rho e^{-\theta y}}\right] - \left[e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}\right]}{e^{\lambda\rho}(e^{\lambda}-1)} \tag{13}$$

where $y = 0, 1, 2, \ldots$, $\lambda > 0$, $\rho \geq 0$, $\theta > 0$.

Theorem 1: The pmf of DEIP distribution is unimodal.

Proof: The pmf of DEIP is log concave, where $P(y+1;\lambda,\rho,\theta)/P(y;\lambda,\rho,\theta)$ is a decreasing function in y for all model parameters. As a direct consequence of log concavity, the DEIP is unimodal.

Figures 1 and 2 show the pmf and hazard rate plots of the DEIP model respectively. The pmf is unimodal and can be used to analyze positively skewed data set. Furthermore, the hazard rate can be either decreasing, constant, decreasing- constant and bathtubshaped. Therefore, the parameters of the DEIP model can be fixed to fit most data sets.

3.2. Structural Properties

The CDF of DEIP is given by

$$\begin{aligned} F(y;\lambda,\rho,\theta) =& P(Y \leq y) \\ =& 1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]. \end{aligned} \tag{14}$$

The survival function of DEIP is given by

$$S(y;\lambda,\rho,\theta)=\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]. \tag{15}$$

The hazard rate of DEIP distribution is

$$\begin{aligned} h(y)=P(Y=y|Y\geq y)=&\frac{P(Y=y)}{P(Y\geq y)} \\ =&\frac{\frac{\left[e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}\right]-\left[e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}\right]}{e^{\lambda\rho}(e^{\lambda}-1)}}{\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]} \\ =&\frac{\left[e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}\right]-\left[e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}\right]}{e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}} \\ =&\left[\frac{e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}}{e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}}\right]-1. \end{aligned} \tag{16}$$

The reverse hazard rate is

$$\begin{aligned} r(y)=&P(Y=y)/P(Y\leq y) \\ =&\frac{\left[e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}\right]-\left[e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}\right]}{\left[e^{\lambda\rho}(e^{\lambda}-1)\right]-\left[e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}\right]} \end{aligned} \tag{17}$$

and the second rate of failure of DEIP distribution is given by,

$$\begin{aligned} h^{**}(y)=&log\left[\frac{S(y)}{S(y+1)}\right] \\ =&log\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}}{e^{\lambda(1+\rho)e^{-\theta(y+2)}}-e^{\lambda\rho e^{-\theta(y+2)}}}\right]. \end{aligned} \tag{18}$$

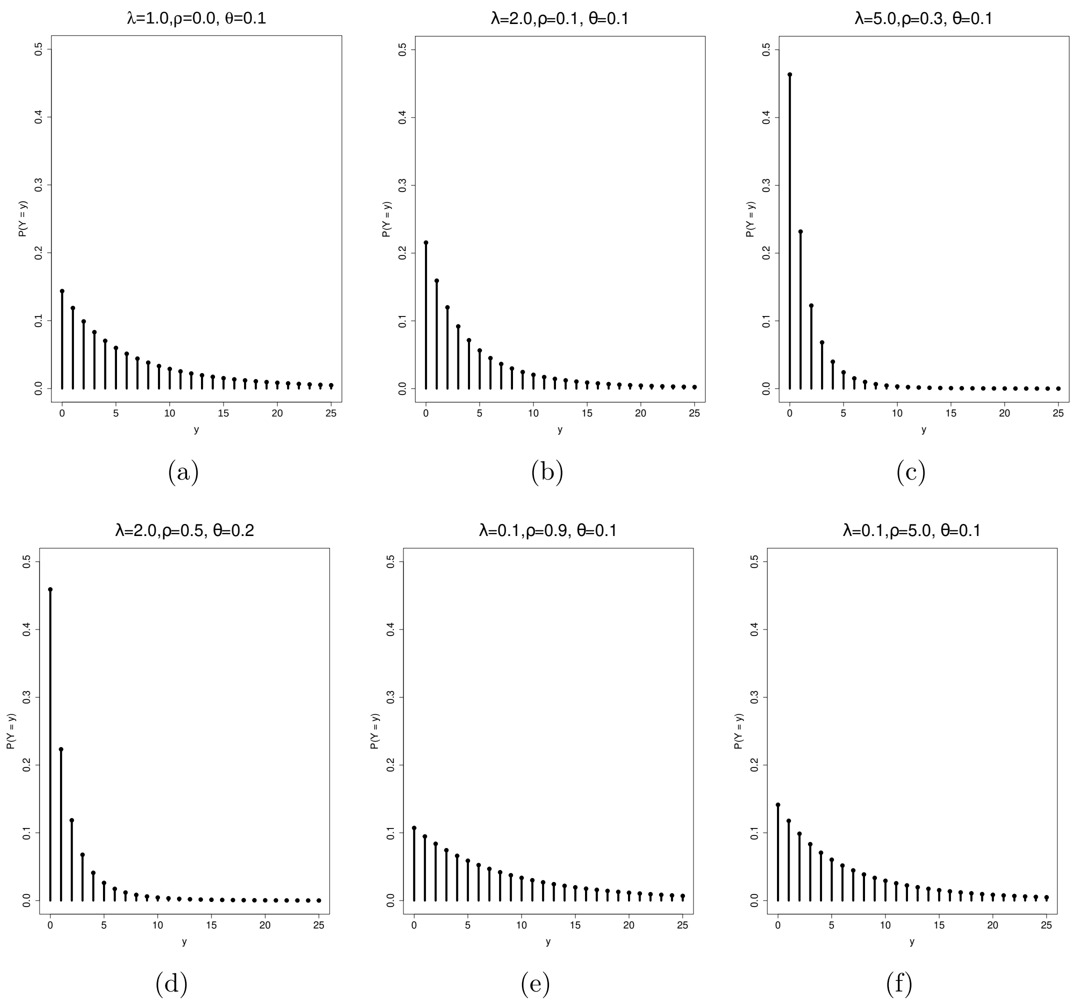

Figure 1: The pmf plots of DEIP(λ, ρ, θ) for different values of λ, ρ and θ

3.3. Recurrence relation for probabilities

The recurrence relation for generating probabilities of DEIP $(\lambda,\ \rho,\ \theta)$ is given by

$$\frac{p(y+1;\lambda,\rho,\theta)}{p(y;\lambda,\rho,\theta)} = \frac{\left[e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}\right] - \left[e^{\lambda(1+\rho)e^{-\theta(y+2)}} - e^{\lambda\rho e^{-\theta(y+2)}}\right]}{\left[e^{\lambda(1+\rho)e^{-\theta y}} - e^{\lambda\rho e^{-\theta y}}\right] - \left[e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}\right]} \tag{19}$$

3.4. Moments

The r^{th} moment of DEIP distribution is given by

$$\begin{aligned} E(Y^r) &= \sum_{y=0}^{\infty} y^r P(Y=y) \\ &= \sum_{y=0}^{\infty} \left[(y+1)^r - y^r\right] S(y). \end{aligned} \tag{20}$$

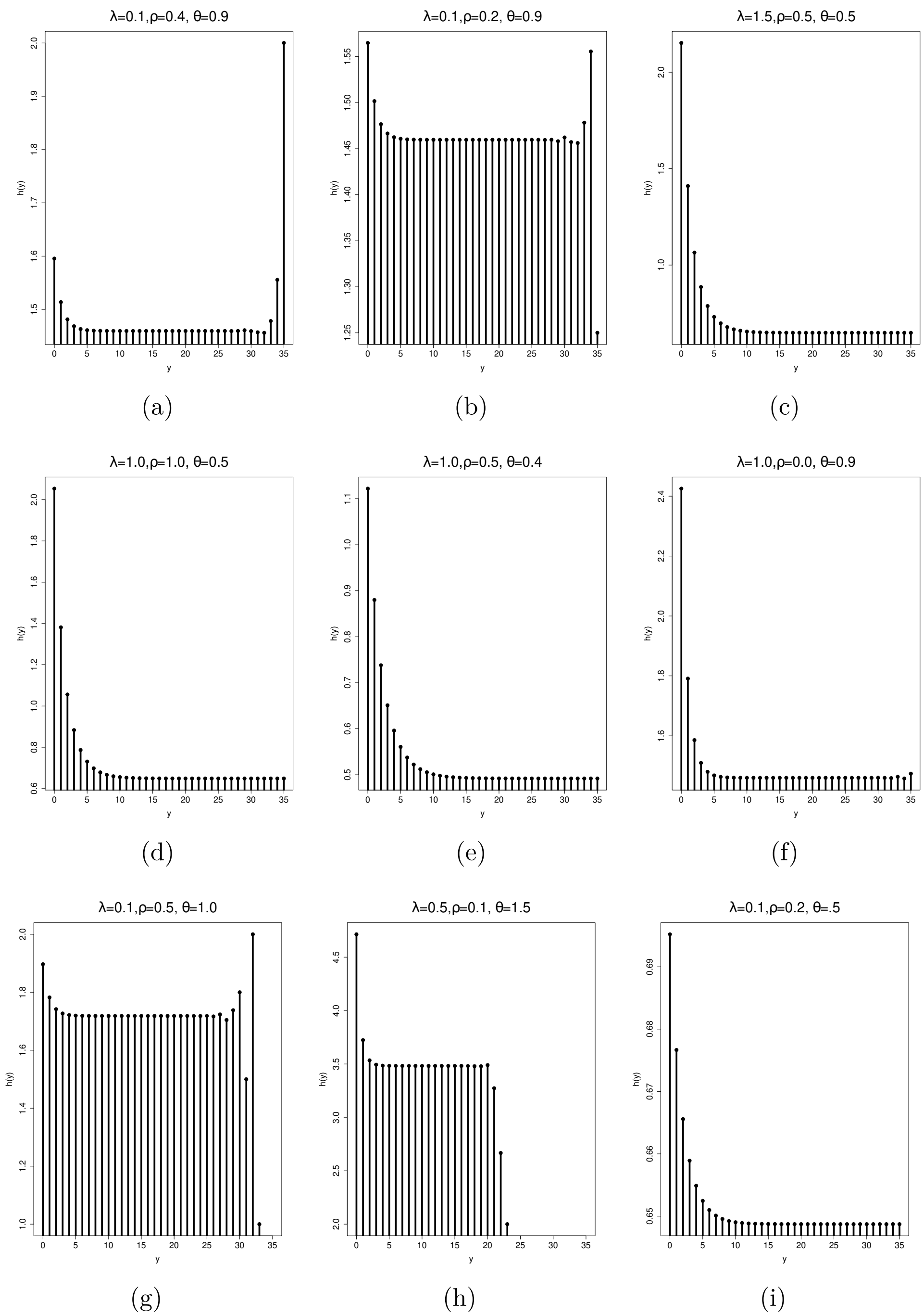

Figure 2: Hazard rate plots of DEIP(λ, ρ, θ) for different values of λ, ρ and θ

$$
\begin{aligned}
E(Y) &= \sum_{y=0}^{\infty} S(y) \\
&= \sum_{y=0}^{\infty} \left[\frac{e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)} \right].
\end{aligned} \tag{21}
$$

Now

$$\begin{aligned} E(Y^2) &= \sum_{y=0}^{\infty}(2y+1)S(y) \\ &= \sum_{y=0}^{\infty}(2y+1)\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]. \end{aligned} \tag{22}$$

$$\begin{aligned} V(Y) =& E(Y^2) - E(Y) \\ =& \sum_{y=0}^{\infty}(2y+1)S(y) - \sum_{y=0}^{\infty}S(y) \\ =& \sum_{y=0}^{\infty}(2y+1)\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right] - \left[\sum_{y=0}^{\infty}\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]\right]^2. \end{aligned} \tag{23}$$

Table 1 shows mean and variance (given in parenthesis) of DEIP distribution using given values of λ, ρ and θ. For fixed θ, as ρ increases mean and variance decreases. Also, as λ increases, mean and variance decreases. From the Table 1, it can also seen that mean is always less than variance for different set of the parameters λ, ρ and θ. Therefore, DEIP is suited better for modelling over-dispersed data.

3.5. Stress-Strength analysis

The stress - strength analysis is used in mechanical component analysis and the stress - strength parameter R measures component reliabilty. Let the random variable Y be strength of a component which is subjected to a random stress Z. For a detailed review of stress- strength models, one may refer Choudhary *et al.* (2021). The stress-strength model defined in discrete case as,

$$P(Y > Z) = \sum_{y=0}^{\infty} p_Y(y)F_Z(y). \tag{24}$$

Let Y and Z be independent stress and strength random variables from $Y \sim \text{DEIP}(\lambda_1, \rho_1, \theta_1)$ and $Z \sim \text{DEIP}(\lambda_2, \rho_2, \theta_2)$ respectively. Also p_Y and F_Y denote the pmf and CDF of the distribution respectively.

Then the stress - strength parameter for the model DEIP is given by,

$$\begin{aligned} R = P(Y > Z) =& \sum_{y=0}^{\infty}\left[\frac{e^{\lambda_1\rho_1 e^{-\theta_1 y}}\left[e^{\lambda_1 e^{-\theta y}} - 1\right] - e^{\lambda_1\rho_1 e^{-\theta_1(y+1)}}\left[e^{\lambda_1 e^{-\theta_1(y+1)}} - 1\right]}{e^{\lambda_1\rho_1}(e^{\lambda_1}-1)}\right] \\ &\times\left[1 - \left(\frac{e^{\lambda_2(1+\rho_2)e^{-\theta_2(y+1)}} - e^{\lambda_2\rho_2 e^{-\theta_2(y+1)}}}{e^{\lambda_2\rho_2}(e^{\lambda_2}-1)}\right)\right] \\ =& \delta(\lambda_1, \rho_1, \theta_1, \lambda_2, \rho_2, \theta_2). \end{aligned} \tag{25}$$

Obviously, the solution of the summation in Equation (25) cannot be obtained explicitly. That is, there is no closed form expression of $\delta(.)$, therefore, we resort to numerical method

Table 1: Mean and Variance of DEIP for different values of λ, ρ and θ

$\theta = 0.5$					
λ |ρ	0.25	0.5	1.0	2.0	3.0
0.50	1.2141	1.1259	0.9703	0.7261	0.5487
	(3.1536)	(2.9273)	(2.5116)	(1.8282)	(1.3212)
0.75	1.0734	0.9604	0.7721	0.5074	0.3403
	(2.7956)	(2.4901)	(1.9633)	(1.2059)	(0.7405)
1.00	0.9468	0.8182	0.6157	0.3585	0.2156
	(2.4604)	(2.1002)	(1.5200)	(0.7918)	(0.4192)
2.00	0.5620	0.4276	0.2540	0.0972	0.04007
	(1.3905)	(1.0006)	(0.5234)	(0.1564)	(0.0535)
$\theta = 1.0$					
λ |ρ	0.25	0.5	1.0	2.0	3.0
0.50	0.4388	0.4008	0.3348	0.2342	0.1645
	(0.7198)	(0.6619)	(0.5573)	(0.3901)	(0.2702)
0.75	0.3787	0.3309	0.25309	0.1489	0.0884
	(0.6282)	(0.5517)	(0.4225)	(0.2436)	(0.1391)
1.00	0.3255	0.2722	0.1908	0.0948	0.0477
	(0.5440)	(0.4556)	(0.3166)	(0.1503)	(0.0712)
2.00	0.1714	0.1206	0.0604	0.0156	0.0041
	(0.2860)	(0.1969)	(0.0929)	(0.0211)	(0.0051)
$\theta = 3.0$					
λ |ρ	0.25	0.5	1.0	2.0	3.0
0.50	0.0363	0.0322	0.0253	0.0157	0.0097
	(0.0387)	(0.0344)	(0.0273)	(0.0170)	(0.0106)
0.75	0.0299	0.0250	0.0175	0.0085	0.0042
	(0.0321)	(0.0269)	(0.0189)	(0.0093)	(0.0045)
1.00	0.0246	0.0193	0.0120	0.0046	0.0017
	(0.0264)	(0.0209)	(0.0131)	(0.0050)	(0.0019)
2.00	0.0106	0.0066	0.0025	0.0003	0.0005
	(0.0116)	(0.0072)	(0.0027)	(0.0004)	(0.0006)

to calculate the system reliability.

To find the maximum likelihood (ML) estimator of the system reliability, we consider $Y_i, i = (1, 2, ...n)$ and $Z_j, j = (1, 2, ..., m)$ two independent samples from DEIP$(\lambda_1, \rho_1, \theta_1)$ and DEIP$(\lambda_2, \rho_2, \theta_2)$ respectively. Then the likelihood function is given by,

$$\begin{aligned} L =& \prod_{i=1}^{n} P(Y = y_i) \prod_{j=1}^{m} P(Y = z_j) \\ =& e^{-n\lambda_1\rho_1}(e^{\lambda} - 1)^{-n} \prod_{i=1}^{n} [L_1 - L_2] \times e^{-m\lambda_2\rho_2}(e^{\lambda_2} - 1)^{-m} \prod_{j=1}^{m} [L_3 - L_4] \end{aligned} \tag{26}$$

where,
$L_1 = e^{\lambda_1(1+\rho_1)e^{-\theta_1 y_i}} - e^{\lambda_1\rho_1 e^{-\theta y_i}}$, $L_2 = e^{\lambda_1(1+\rho_1)e^{-\theta(y_i+1)}} - e^{\lambda_1\rho_1 e^{-\theta_1(y_i+1)}}$
$L_3 = e^{\lambda_2(1+\rho_2)e^{-\theta_2 z_j}} - e^{\lambda_2\rho_2 e^{-\theta z_j}}$ and $L_4 = e^{\lambda_2(1+\rho_2)e^{-\theta_2(z_j+1)}} - e^{\lambda_2\rho_2 e^{-\theta_2(z_j+1)}}$. In order to obtain the ML estimators of $\lambda_1, \rho_1, \theta_1, \lambda_2, \rho_2$ and θ_2, we first derive the log-likelihood (LogL) function by taking the logarithm of Equation (26). Then, we take the derivatives of the logL function with respect to the parameters of interest and obtain the likelihood equations. The solutions of these equations cannot be obtained in closed form, and the estimates of the unknown parameters are found by using numerical methods with the help of **R** programming. Then by using the invariance property of ML estimators, the ML estimate of system reliability is obtained as

$$\hat{R} = \delta(\hat{\lambda}_1, \hat{\rho}_1, \hat{\theta}_1, \hat{\lambda}_2, \hat{\rho}_2, \hat{\theta}_2).$$

Some numerical results of R are reported in Table 2 using DEIP distribution for the parameters $\lambda_1 = \lambda_2 = \rho_1 = \rho_2 = 0.5$. It is clear that R decreases(increases) when θ_1 increases (θ_2 increases).

Table 2: Some numerical results of R for different values of θ_1 and θ_2

θ_1—θ_2	0.1	0.5	0.9	1.0
0.1	0.5475	0.6027	0.6093	0.6100
0.5	0.3105	0.5995	0.7066	0.7229
0.9	0.2705	0.5482	0.6707	0.6906
1.0	0.2653	0.5404	0.6651	0.6854

3.6. Infinite divisibility

The famous structural property of infinite divisibility of the distribution is an interesting area to the researchers. Such a characteristic has a close relation to the Central Limit Theorem and waiting time distributions. According to Steutel and van Harn (2003), if $p_x, x \in \mathbb{N}_0$ is infinitely divisible, then $p_x \leq e^{-1}$ for all $x \in \mathbb{N}$. Also from Theoreom 3.2 of Steutel and van Harn (2003), if for atleast one case for which p_x is greater than $1/e$, then pmf cannot be compound Poisson and hence it cannot be infinitely divisible. In DEIP distribution, $\lambda = 3$, $\rho = 0.6$ and $\theta = 0.1$, then $p_0 = 0.3776 > e^{-1} = 0.367$. Therefore we can conclude that DEIP distribution is not infinitely divisible. The classes of self-decomposable and stable distributions are subclasses of infinitely divisible distributions, in their discrete concepts. So in this case, DEIP distribution can be neither self-decomposable nor stable in general.

3.7. Application in first order integer valued autoregressive (INAR(1)) process

There has been a growing interest in discrete-valued time series models and several models for stationary processes with discrete marginal distributions have been proposed in the literature. A simple model for a stationary sequence of integer-valued random variables with lag-one dependence is given and is referred to as the integer-valued autoregressive of order one (INAR(1)) process. It is widely used to model the time series of counts in different applied sciences such as actuarial, finance and medical sciences. The INAR(1) process differs from the first-order autoregressive, shortly AR(1), process by applying the binomial thinning operator. The first INAR(1) process was introduced by McKenzie (1985) based on the Poisson innovations and is called as INAR(1)P.

$$Y_t = \alpha \ o \ Y_{t-1} + \epsilon_t, t \in \mathbb{Z} \tag{27}$$

where $\alpha \in (0,1)$ and ϵ_t is an innovation process with mean $E(\epsilon_t) = \mu_\epsilon$ and variance $Var(\epsilon_t) = \sigma^2_{\epsilon_t}$.
According to Steutel and van Harn (1979), the binomial thinning operator "o" is defined as

$$\alpha \ o \ Y_t = \sum_{j=i}^{Y_t} Z_j \tag{28}$$

where Z_j is the Bernoulli random variable with $P(Z_j = 1) = p = 1 - P(Z_j = 0)$. The one-step transition probability of INAR(1) process is

$$P(Y_t = k|Y_{t-1} = l) = \sum_{\substack{i=1 \\ k,l \geq 0}}^{min(k,l)} P(B_l^p = i)P(\epsilon_t = k - i) \tag{29}$$

where $B_n^p \sim$ Binomial(n, p) and $p \in (0,1)$.
Following the results of McKenzie (1985) and Al-Osh and Alzaid (1987), we propose an INAR(1) process with DEIP innovations by assuming that the $\{\epsilon_t\}_{t \in \mathbb{Z}}$ innovations follow DEIP distribution, given in Equation(13). Thus, one-step transition probability of INAR(1) DEIP process is given by

$$\begin{aligned} P(Y_t = k|Y_{t-1} = l) = \sum_{i=1}^{min(k,l)} \binom{l}{i} \alpha^i (1-\alpha)^{l-i} \times \\ \frac{\left[e^{\lambda(1+\rho)e^{-\theta(k-i)}} - e^{\lambda\rho e^{-\theta(k-i)}}\right] - \left[e^{\lambda(1+\rho)e^{-\theta(k-i+1)}} - e^{\lambda\rho e^{-\theta(k-i+1)}}\right]}{e^{\lambda\rho}(e^\lambda - 1)}. \end{aligned} \tag{30}$$

The mean and variance of the Y_t process are respectively given by,

$$E(Y_t) = \frac{\mu_\epsilon}{1-\alpha} \tag{31}$$

$$V(Y_t) = \frac{\alpha\mu_\epsilon + \sigma_\epsilon^2}{1-\alpha^2}. \tag{32}$$

The mean and variance of the INAR(1)DEIP process can be computed by replacing μ_ϵ and σ_ϵ in Equation(31) and Equation(32) with Equation (21) and Equation(23) respectively. The conditional expectation and variance of INAR(1) DEIP process are given, respectively, as (see Weiß (2018) and Al-Osh and Alzaid (1988))

$$E(Y_t|Y_{t-1}) = pY_{t-1} + \mu_\epsilon \tag{33}$$

and

$$V(Y_t|Y_{t-1}) = p(1-p)Y_{t-1} + \sigma_\epsilon^2 \tag{34}$$

where μ_ϵ and σ_ϵ^2 are given in Equation(21) and Equation(23).

4. Characterizations

Characterizations of distributions is an important research area which has attracted the attention of many researchers. The problem of characterizing a distribution is an important problem, where an investigator is vitally interested to know if their model follows the right distribution. Thus, various characterization results have been reported in the literature. These characterizations have been established in different directions. In this section, we obtain three characterizations of DEIP distribution based on: (i) the hazard rate function and (ii) the reverse hazard rate function and (iii) conditional expectation of certain function of the random variable.

4.1. Characterization based on hazard rate function

Proposition 2: Let $Y:\Omega\to\mathbb{N}$ be a random variable. The pmf of Y is in Equation(13) if and only if its hazard rate function satisfies the difference equation

$$h(k+1)-h(k)=\left[\frac{e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}}{e^{\lambda(1+\rho)e^{-\theta(k+2)}}-e^{\lambda\rho e^{-\theta(k+2)}}}\right]-\left[\frac{e^{\lambda(1+\rho)e^{-\theta k}}-e^{\lambda\rho e^{-\theta k}}}{e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}}\right], \tag{35}$$

$k\in\mathbb{N}$, with the boundary condition $h(0)=\frac{e^{\lambda\rho(e^{-\theta}-1)}(e^{\lambda}-1)}{e^{\lambda e^{-\theta}}-1}-1$.

Proof: If Y has pmf in Equation(13), then clearly Equation(35) holds. Now, if Equation(35) holds, then for every $y\in\mathbb{N}$ we have

$$\sum_{k=1}^{y-1}h(k+1)-h(k)=\sum_{k=1}^{y-1}\left[\frac{e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}}{e^{\lambda(1+\rho)e^{-\theta(k+2)}}-e^{\lambda\rho e^{-\theta(k+2)}}}\right]-\left[\frac{e^{\lambda(1+\rho)e^{-\theta k}}-e^{\lambda\rho e^{-\theta k}}}{e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}}\right].$$

$$h(y)-h(0)=\left[\frac{e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}}{e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}}\right]-\left[\frac{e^{\lambda\rho(e^{-\theta}-1)}(e^{\lambda}-1)}{e^{\lambda e^{-\theta}}-1}\right].$$

In view of the fact that $h(0)=\frac{e^{\lambda\rho(e^{-\theta}-1)}(e^{\lambda}-1)}{e^{\lambda e^{-\theta}}-1}-1$, from the last equation we have

$$h(y)=\left[\frac{e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}}{e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}}-1\right]$$

which in view of Equation(16), implies Y has pmf in Equation(13).

4.2. Characterization based on reverse hazard rate function

Proposition 3: Let $Y:\Omega\to\mathbb{N}$ be a random variable. The pmf of Y is in Equation(13) if and only if its reverse hazard rate function satisfies the difference equation

$$\begin{aligned} r(k+1)-r(k)=&\frac{\left[e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}\right]-\left[e^{\lambda(1+\rho)e^{-\theta(k+2)}}-e^{\lambda\rho e^{-\theta(k+2)}}\right]}{\left[e^{\lambda\rho}(e^{\lambda}-1)\right]-\left[e^{\lambda(1+\rho)e^{-\theta(k+2)}}-e^{\lambda\rho e^{-\theta(k+2)}}\right]}\\ &-\frac{\left[e^{\lambda(1+\rho)e^{-\theta k}}-e^{\lambda\rho e^{-\theta k}}\right]-\left[e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}\right]}{\left[e^{\lambda\rho}(e^{\lambda}-1)\right]-\left[e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}\right]} \end{aligned} \tag{36}$$

with the boundary condition $r(0) = 1$.

Proof: If Y has pmf in Equation(13), then clearly Equation(36) holds. Now, if Equation(36) holds, then for every $y \epsilon \mathbb{N}$ we have

$$\sum_{k=1}^{y-1} r(k+1) - r(k) = \sum_{k=1}^{y-1} \frac{\left[e^{\lambda(1+\rho)e^{-\theta(k+1)}} - e^{\lambda\rho e^{-\theta(k+1)}}\right] - \left[e^{\lambda(1+\rho)e^{-\theta(k+2)}} - e^{\lambda\rho e^{-\theta(k+2)}}\right]}{\left[e^{\lambda\rho}(e^{\lambda}-1)\right] - \left[e^{\lambda(1+\rho)e^{-\theta(k+2)}} - e^{\lambda\rho e^{-\theta(k+2)}}\right]} - \frac{\left[e^{\lambda(1+\rho)e^{-\theta k}} - e^{\lambda\rho e^{-\theta k}}\right] - \left[e^{\lambda(1+\rho)e^{-\theta(k+1)}} - e^{\lambda\rho e^{-\theta(k+1)}}\right]}{\left[e^{\lambda\rho}(e^{\lambda}-1)\right] - \left[e^{\lambda(1+\rho)e^{-\theta(k+1)}} - e^{\lambda\rho e^{-\theta(k+1)}}\right]}. \tag{37}$$

Or,

$$r(y) - r(0) = \frac{\left[e^{\lambda(1+\rho)e^{-\theta y}} - e^{\lambda\rho e^{-\theta y}}\right] - \left[e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}\right]}{\left[e^{\lambda\rho}(e^{\lambda}-1)\right] - \left[e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}\right]} - 1.$$

In view of the fact that $r(0) = 1$, from the last equation we have

$$r(y) = \frac{\left[e^{\lambda(1+\rho)e^{-\theta y}} - e^{\lambda\rho e^{-\theta y}}\right] - \left[e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}\right]}{\left[e^{\lambda\rho}(e^{\lambda}-1)\right] - \left[e^{\lambda(1+\rho)e^{-\theta(y+1)}} - e^{\lambda\rho e^{-\theta(y+1)}}\right]}$$

which in view of Equation(17), implies Y has pmf in Equation(13).

4.3. Characterization in terms of the conditional expectation of certain function of the random variable

Proposition 4: Let $Y : \Omega \to \mathbb{N}$ be a random variable. The pmf of Y is in Equation(13) if and only if

$$E\left\{\frac{\left[e^{\lambda(1+\rho)e^{-\theta Y}} - e^{\lambda\rho e^{-\theta Y}}\right] + \left[e^{\lambda(1+\rho)e^{-\theta(Y+1)}} - e^{\lambda\rho e^{-\theta(Y+1)}}\right]}{\left[e^{\lambda\rho}(e^{\lambda}-1)\right]} | Y > k\right\} = \frac{e^{\lambda(1+\rho)e^{-\theta(k+1)}} - e^{\lambda\rho e^{-\theta(k+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}. \tag{38}$$

Proof: If Y has pmf Equation(13), then LHS of Eqution(38) will be

$$(1-F(k))^{-1}\sum_{y=k+1}^{\infty}\frac{\left[e^{\lambda(1+\rho)e^{-\theta Y}}-e^{\lambda\rho e^{-\theta Y}}\right]+\left[e^{\lambda(1+\rho)e^{-\theta(Y+1)}}-e^{\lambda\rho e^{-\theta(Y+1)}}\right]}{[e^{\lambda\rho}(e^{\lambda}-1)]}\times$$
$$\frac{\left[e^{\lambda(1+\rho)e^{-\theta Y}}-e^{\lambda\rho e^{-\theta Y}}\right]-\left[e^{\lambda(1+\rho)e^{-\theta(Y+1)}}-e^{\lambda\rho e^{-\theta(Y+1)}}\right]}{[e^{\lambda\rho}(e^{\lambda}-1)]}$$
$$=(1-F(k))^{-1}\sum_{y=k+1}^{\infty}\left(\frac{e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right)^2-\left(\frac{e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right)^2 \tag{39}$$
$$=\left(\frac{e^{\lambda\rho}(e^{\lambda}-1)}{e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}}\right)\left(\frac{e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right)^2$$
$$=\frac{e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}.$$

Conversely, if Equation (38) holds, then

$$\sum_{y=k+1}^{\infty}\frac{\left[e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}\right]+\left[e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}\right]}{[e^{\lambda\rho}(e^{\lambda}-1)]}f(y)$$
$$=\sum_{y=k+1}^{\infty}\frac{\left[e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}\right]+\left[e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}\right]}{[e^{\lambda\rho}(e^{\lambda}-1)]}\times$$
$$\frac{\left[e^{\lambda(1+\rho)e^{-\theta Y}}-e^{\lambda\rho e^{-\theta Y}}\right]-\left[e^{\lambda(1+\rho)e^{-\theta(Y+1)}}-e^{\lambda\rho e^{-\theta(Y+1)}}\right]}{[e^{\lambda\rho}(e^{\lambda}-1)]}$$
$$=\sum_{y=k+1}^{\infty}\left(\frac{e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right)^2-\left(\frac{e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right)^2 \tag{40}$$
$$=\left(\frac{e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right)^2$$
$$=(1-F(k))\left(\frac{e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right)$$
$$=(1-F(k+1)+f(k+1))\left(\frac{e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right).$$

From Equation (39), we also have,

$$\sum_{y=k+2}^{\infty}\frac{\left[e^{\lambda(1+\rho)e^{-\theta y}}-e^{\lambda\rho e^{-\theta y}}\right]+\left[e^{\lambda(1+\rho)e^{-\theta(y+1)}}-e^{\lambda\rho e^{-\theta(y+1)}}\right]}{[e^{\lambda\rho}(e^{\lambda}-1)]}f(y)$$
$$=(1-F(k+1))\left(\frac{e^{\lambda(1+\rho)e^{-\theta(k+2)}}-e^{\lambda\rho e^{-\theta(k+2)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right). \tag{41}$$

Now, subtracting Equation (41) from Equation (40), we arrive at

$\left(\frac{e^{\lambda(1+\rho)e^{-\theta(k+2)}}-e^{\lambda\rho e^{-\theta(k+2)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right)f(k+1)=$

$(1-F(k+1))\left(\frac{\left(e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}\right)-\left(e^{\lambda(1+\rho)e^{-\theta(k+2)}}-e^{\lambda\rho e^{-\theta(k+2)}}\right)}{e^{\lambda\rho}(e^{\lambda}-1)}\right).$

$h(y)=\frac{f(k+1)}{1-F(k+1)}=\left(\frac{\left(e^{\lambda(1+\rho)e^{-\theta(k+1)}}-e^{\lambda\rho e^{-\theta(k+1)}}\right)-\left(e^{\lambda(1+\rho)e^{-\theta(k+2)}}-e^{\lambda\rho e^{-\theta(k+2)}}\right)}{e^{\lambda(1+\rho)e^{-\theta(k+2)}}-e^{\lambda\rho e^{-\theta(k+2)}}}\right)$

which, in view of Equation (16), implies that Y has pmf in Equation (13).

5. Estimation and simulation

In this section, some estimation methods are discussed. In particular, we considered the following estimation methods: Maximum likelihood (ML) estimation, ordinary least square (OLS) estimation, weighted least square (WLS) estimation and Cramer-von Mises (CVM) estimation.

5.1. Maximum likelihood estimation

We apply method of ML estimation for estimating the parameter vector $\beta=(\lambda,\rho,\theta)^T$ of DEIP distribution. Let $(y_1,y_2,...,y_n)$ be a random sample of size n, drawn from DEIP (λ,ρ,θ) distribution.
The log likelihood function is given below

$$\begin{aligned}logL=&-n(\lambda\rho+log(e^{\lambda}-1))+\\&\sum_{j=1}^{n}log\left\{e^{\lambda\rho e^{-\theta y_j}}\left[e^{\lambda e^{-\theta y_j}}-1\right]-e^{\lambda\rho e^{-\theta(y_j+1)}}\left[e^{\lambda e^{-\theta(y_j+1)}}-1\right]\right\}.\end{aligned}\tag{42}$$

By differentiating Equation 42 with respect to the parameters λ, ρ and θ, we get non linear likelihood equations as follows.

$$\begin{aligned}\frac{\partial logL}{\partial\lambda}=&-n\rho-\frac{ne^{\lambda}}{e^{\lambda}-1}+\\&\sum_{j=1}^{n}\frac{(\rho+1)e^{-\theta y_j}A_1-\rho e^{-\theta y_j}A_2-(\rho+1)e^{-\theta(y_j+1)}A_3+\rho e^{-\theta(y_j+1)}A_4}{A_1-A_2-A_3+A_4}.\end{aligned}\tag{43}$$

$$\frac{\partial logL}{\partial\rho}=-n\lambda+\sum_{j=1}^{n}\frac{\lambda e^{-\theta y_j}(A_1+A_2)-\lambda e^{-\theta(y_j+1)}(A_3-A_3)}{A_1-A_2-A_3+A_4}.\tag{44}$$

$$\frac{\partial logL}{\partial\theta}=\sum_{j=1}^{n}\frac{\lambda y_j e^{-\theta y_j}(A_2\rho-A_1(\rho+1))+\lambda(y_j+1)e^{-\theta(y_j+1)}(A_3(\rho+1)-A_4\rho)}{A_1-A_2-A_3+A_4}\tag{45}$$

where $A_1=e^{\lambda(\rho+1)e^{-\theta y_j}}$, $A_2=e^{\lambda\rho e^{-\theta y_j}}$, $A_3=e^{\lambda(\rho+1)e^{-\theta(y_j+1)}}$ and $A_4=e^{\lambda\rho e^{-\theta(y_j+1)}}$.
These Equations(43–45) cannot be solved analytically, therefore an iterative procedure like Newton Raphson is required to solve them numerically. The solutions of likelihood Equations (43–45) provide ML estimators of $\beta=(\lambda,\rho,\theta)^T$, say $\hat{\beta}=(\hat{\lambda},\hat{\rho},\hat{\theta})^T$.

The conditions for maximum are obtained as:

Let $g_1(\lambda;\rho,\theta,y)$ denote the function on the right hand side (RHS) of Equation (43) where ρ and θ are the true values of the parameters. Then there exist atleast one root for $g_1(\lambda;\rho,\theta,y)=0$ for $\lambda\in(0,\infty)$ and the solution is unique when

$$\sum_{j=1}^{n}\frac{e^{-2\theta y_j}[(1+\rho)^2A_1-\rho^2A_2]-e^{-2\theta(y_j+1)}[(1+\rho)^2A_3-\rho^2A_4]}{A_1-A_2-A_3+A_4}<\frac{ne^{\lambda}}{(e^{\lambda}-1)^2}+$$

$$\sum_{j=1}^{n}\frac{(\rho[e^{-\theta(y_j+1)}A_4-e^{-\theta y_j}A_2]-(1+\rho)[e^{-\theta(_jy+1)}A_3-e^{-\theta y_j}A_1])(e^{-\theta y_j}((1+\rho)A_1-\rho A_2)-e^{-\theta(y_j+1)}((1+\rho)A_3-\rho A_4))}{(A_1-A_2-A_3+A_4)^2}.$$

Let $g_2(\rho;\lambda,\theta,y)$ denote the function on the right hand side (RHS) of Equation (44) where λ and θ are the true values of the parameters. Then there exist atleast one root for $g_2(\rho;\lambda,\theta,y)=0$ for $\rho\in(0,\infty)$ when

$$-n+\sum_{j=1}^{n}\frac{e^{-\theta y}(1+e^{\lambda e^{-\theta y}})-e^{-\theta(y+1)}(e^{\lambda e^{-\theta(y+1)}}-1)}{e^{\lambda e^{-\theta y}}-e^{\lambda e^{-\theta(y+1)}}}>0$$

and the solution is unique when

$$\sum_{j=1}^{n}\frac{\lambda^2e^{-2\theta y_j}(A_2+A_4)-\lambda^2e^{-2\theta(y_j+1)}(A_3-A_1)}{A_1-A_2-A_3+A_4}<\sum_{j=0}^{n}\frac{(\lambda e^{-\theta y_j}(A_2+A_1)-\lambda e^{-\theta(y_j+1)}(A_3-A_4))^2}{(A_1-A_2-A_3+A_4)^2}.$$

Let $g_3(\theta;\lambda,\rho,y)$ denote the function on the right hand side (RHS) of Equation (45) where ρ and θ are the true values of the parameters. Then there exist atleast one root for $g_3(\theta;\lambda,\rho,y)=0$ for $\theta\in(0,\infty)$ and the solution is unique when

$$\sum_{j=1}^{n}\frac{y_j^2\lambda^2(1+\rho)\rho e^{-2\theta y_j}(A_1-A_2)-(1+y_j)^2\lambda^2e^{-2\theta(y_j+1)}((1+\rho)^2A_3-\rho^2A_4)-\lambda y_j^2e^{-\theta y_j}((1+\rho)A_2-\rho A_1)-\lambda(1+y_j)^2e^{-\theta(y_j+1)((1+\rho)A_3-\rho A_4)}}{A_1-A_2-A_3+A_4}<$$

$$\sum_{j=1}^{n}\frac{((y_j+1)\lambda e^{-\theta(y_j+1)}[(1+\rho)A_3-\rho A_4]-y_j\lambda e^{-\theta y_j}[(1+\rho)A_1-\rho A_2])(\lambda y_je^{-\theta y_j}[(1+\rho)A_2-\rho A_1]+\lambda(y_j+1)e^{-\theta(y_j+1)}[(1+\rho)A_3-\rho A_4])}{(A_1-A_2-A_3+A_4)^2}$$

where $A_1=e^{\lambda(\rho+1)e^{-\theta y_j}}$, $A_2=e^{\lambda\rho e^{-\theta y_j}}$, $A_3=e^{\lambda(\rho+1)e^{-\theta(y_j+1)}}$ and $A_4=e^{\lambda\rho e^{-\theta(y_j+1)}}$.

5.2. Ordinary least square estimation

This method is based on the observed sample $y_1,y_2,...,y_n$ from n ordered random sample of any distribution with CDF, where $F(.)$ denotes the CDF, we get

$$E(F(y_j))=\frac{j}{(n+1)}.$$

The OLS estimators are obtained by minimizing

$$OLS(\lambda,\rho,\theta)=\sum_{j=1}^{n}(F(y_j)-\frac{j}{n+1})^2. \tag{46}$$

Putting the CDF of DEIP in Equation (46) we get

$$OLS(\lambda,\rho,\theta)=\sum_{j=1}^{n}\left[1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{j}{n+1}\right]^2. \tag{47}$$

After differentiating Equation (47) with respect to the parameters λ,ρ and θ and equating to zero, the normal equations are as follows:

$$\begin{aligned}\frac{\partial OLS}{\partial\lambda}=&2\sum_{j=1}^{n}\left(1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{j}{n+1}\right)\frac{1}{(e^{\lambda}-1)^2}\\ &A_4e^{-\theta(y_j+1)-\lambda\rho}\left(\rho e^{\theta(y_j+1)}+(1-e^{\lambda})((1+\rho)e^{\lambda e^{-\theta(y_j+1)}}-\rho)\right)+\\ &e^{\theta(y_j+1)}(e^{\lambda}(1+\rho)(e^{\lambda e^{-\theta(y_j+1)}}-1)-\rho e^{\lambda e^{-\theta(y_j+1)}}).\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial OLS}{\partial\rho}=&2\sum_{j=1}^{n}\left(1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{j}{n+1}\right)\\ &\left(\frac{(A_4-A_3)(\lambda e^{\lambda\rho}(e^{-\theta(y_j+1)}-1))}{e^{\lambda}-1}\right).\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{\partial OLS}{\partial\theta}=&2\sum_{j=1}^{n}\left(1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{j}{n+1}\right)\\ &\frac{e^{-\lambda\rho}\lambda(1+y_j)e^{-\theta(y_j+1)}((1+\rho)A_3-\rho A_4)}{e^{\lambda}-1}\end{aligned} \tag{50}$$

where $A_3=e^{\lambda(\rho+1)e^{-\theta(y_j+1)}}$ and $A_4=e^{\lambda\rho e^{-\theta(y_j+1)}}$. The above non-linear equations cannot be solved analytically. So the OLS estimators of λ, ρ and θ can be obtained by using some iterative techniques likes Newton-Raphson method.

5.3. Weighted least square estimation

The WLS estimators can be obtained by minimizing

$$WLS(\lambda,\rho,\theta)=\sum_{j=1}^{n}w_j(F(y_i)-\frac{j}{n+1})^2 \tag{51}$$

with respect to the unknown parameters, where $w_j=\frac{1}{Var(F(Y_j))}=\frac{(n+1)^2(n+2)}{j(n-j+1)}$.

Putting the CDF of DEIP distribution in Equation (51), we get

$$WLS(\lambda,\rho,\theta)=\sum_{j=1}^{n}w_j\left[1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{j}{n+1}\right]^2. \tag{52}$$

The Equation(52) is differentiated with respect to the parameters λ, ρ and θ and then equating to zero, the normal equations are as follows:

$$\begin{aligned}\frac{\partial WLS}{\partial \lambda} =&2\sum_{j=1}^{n}\left(1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{j}{n+1}\right)\frac{1}{(e^{\lambda}-1)^2}\\ &A_4 e^{-\theta(y_j+1)-\lambda\rho}w_j(\rho e^{\theta(y_j+1)}+(1-e^{\lambda})((1+\rho)e^{\lambda e^{-\theta(y_j+1)}}-\rho)+\\ &e^{\theta(y_j+1)}(e^{\lambda}(1+\rho)(e^{\lambda e^{-\theta(y_j+1)}}-1)-\rho e^{\lambda e^{-\theta(y_j+1)}})).\end{aligned} \tag{53}$$

$$\begin{aligned}\frac{\partial WLS}{\partial \rho} =&2\sum_{j=1}^{n}\left(1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{j}{n+1}\right)\\ &w_j\left(\frac{(A_4-A_3)(\lambda e^{\lambda\rho}(e^{-\theta(y_j+1)}-1))}{e^{\lambda}-1}\right).\end{aligned} \tag{54}$$

$$\begin{aligned}\frac{\partial WLS}{\partial \theta} =&2\sum_{j=1}^{n}\left(1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{j}{n+1}\right)\\ &w_j\frac{e^{-\lambda\rho}\lambda(1+y_j)e^{-\theta(y_j+1)}((1+\rho)A_3-\rho A_4)}{e^{\lambda}-1}.\end{aligned} \tag{55}$$

where $A_3 = e^{\lambda(\rho+1)e^{-\theta(y_j+1)}}$ and $A_4 = e^{\lambda\rho e^{-\theta(y_j+1)}}$. These above nonlinear equations cannot be solved analytically. Therefore the WLS estimates can be obtained by using any iterative procedure techniques such as Newton-Raphson type algorithms.

5.4. Cramer-von Mises estimation

The CVM estimates of the parameter λ, ρ and θ are obtained by minimizing the following expression with respect to the parameters λ, ρ and θ respectively.

$$CVM_{\lambda,\rho,\theta} = \frac{1}{12n} + \sum_{j=1}^{n}(F(y_j) - \frac{-1+2j}{2n})^2. \tag{56}$$

For in the case of DEIP distribution, put CDF of DEIP in Equation(56).

$$CVM_{\lambda,\rho,\theta} = \frac{1}{12n} + \sum_{j=1}^{n}\left[1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{-1+2j}{2n}\right)^2. \tag{57}$$

By differentiating Equation(57) with respect to the parameters λ, ρ and θ and equating to zero, we get the normal equations as follows:

$$\begin{aligned}\frac{\partial CVM}{\partial \lambda} =&2\sum_{j=1}^{n}(\left(1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{-1+2j}{2n}\right)\\ &\frac{1}{(e^{\lambda}-1)^2}A_4 e^{-\theta(y_j+1)-\lambda\rho})(\rho e^{\theta(y_j+1)}+(1-e^{\lambda})((1+\rho)e^{\lambda e^{-\theta(y_j+1)}}-\rho)+\\ &e^{\theta(y_j+1)}(e^{\lambda}(1+\rho)(e^{\lambda e^{-\theta(y_j+1)}}-1)-\rho e^{\lambda e^{-\theta(y_j+1)}})).\end{aligned} \tag{58}$$

$$\frac{\partial CVM}{\partial \rho} = 2\sum_{j=1}^{n}\left(1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{-1+2j}{2n}\right)\left(\frac{(A_4-A_3)(\lambda e^{\lambda\rho}(e^{-\theta(y_j+1)}-1))}{e^{\lambda}-1}\right). \tag{59}$$

$$\frac{\partial CVM}{\partial \theta} = 2\sum_{j=1}^{n}\left(1-\left[\frac{e^{\lambda(1+\rho)e^{-\theta(y_j+1)}}-e^{\lambda\rho e^{-\theta(y_j+1)}}}{e^{\lambda\rho}(e^{\lambda}-1)}\right]-\frac{-1+2j}{2n}\right)\frac{e^{-\lambda\rho}\lambda(1+y_j)e^{-\theta(y_j+1)}((1+\rho)A_3-\rho A_4)}{e^{\lambda}-1} \tag{60}$$

where $A_3 = e^{\lambda(\rho+1)e^{-\theta(y_j+1)}}$ and $A_4 = e^{\lambda\rho e^{-\theta(y_j+1)}}$.
These Equations (58–60) cannot be solved analytically. The estimates of λ, ρ and θ can be obtained by setting the normal equations equations equal to zero and solving simultaneously with the help of statistical packages like *optim* or *nlm* in **R** programming.

5.5. Simulation

Here we examine the performance of the estimates of DEIP parameters using simulation study with 1000 replications. We calculate the estimates and mean square errors(MSE) using the **R** package. We used "*nlm*" function in **R** program for ML estimation and "*optim*" function is used for the estimation of OLS, WLS and CVM. The simulation procedure is given below.

1. Generate $N = 1000$ samples of sizes $n = 50, 100, 300$ from DEIP(0.1, 0.1, 0.1) and DEIP(0.5, 0.9, 0.1).

   Here, the random variable X possesses a continuous Exponential Intervened Poisson (EIP) distribution with parameters λ, ρ and θ. Then $Y = \lfloor X \rfloor$ follows the DEIP distribution with parameters λ, ρ and θ. To generate data from the DEIP distribution, first we have to generate data from EIP. Then take the integer values of each generated observation to get the simulated data set. The procedure of generating random samples from EIP distribution is explained in Jayakumar and Sankaran (2019). Initial values are chosen to compute the estimates in such way that the optimization function having minimum bias.

2. Compute the estimates for the 1000 samples, say $\hat{\beta}$ for j = 1, 2, ..., 1000.

3. Compute MSE by using the below quantity.

$$MSE(\hat{\beta}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\beta}-\beta)^2. \tag{61}$$

4. Compute the coverage probabilities [CP] of the estimates.

The empirical result from the Table 3 is when the sample size increases the MSEs of the parameter decreases. This shows the consistency of the estimators. Also, CVM estimates perform better when compared to other estimates.

Table 3: Estimates of of λ, ρ and θ

	True values→	$\lambda=0.1,\ \rho=0.1,\ \theta=0.1$				$\lambda=0.5,\ \rho=0.9,\ \theta=0.1$			
Sample size(n)	Parameter↓	ML	OLS	WLS	CVM	ML	OLS	WLS	CVM
50	$\hat{\lambda}$	0.1383	0.1761	0.1596	0.0961	0.5670	0.6458	0.6351	0.5293
	(MSE)	(0.2121)	(0.3562)	(0.4970)	(0.1831)	(0.4031)	(0.6234)	(0.6037)	(0.3960)
	[CP]	[0.841]	[0.617]	[0.700]	[0.863]	[0.796]	[0.635]	[0.641]	[0.801]
	$\hat{\rho}$	0.1312	0.1796	0.1623	0.1071	0.9606	0.8764	0.8823	0.9724
	(MSE)	(0.3210)	(0.4013)	(0.4433)	(0.0961)	(0.2683)	(0.4801)	(0.4573)	(0.2541)
	[CP]	[0.800]	[0.672]	[0.727]	[0.854]	[0.801]	[0.765]	[0.768]	[0.834]
	$\hat{\theta}$	0.1402	0.1634	0.1706	0.1324	0.1451	0.1937	0.1969	0.1433
	(MSE)	(0.2150)	(0.4146)	(0.5231)	(0.1612)	(0.1732)	(0.4176)	(0.5154)	(0.1365)
	[CP]	[0.829]	[0.631]	[0.786]	[0.847]	[0.896]	[0.719]	[0.703]	[0.902]
100	$\hat{\lambda}$	0.1238	0.1571	0.1431	0.0989	0.5312	0.6032	0.5993	0.5240
	(MSE)	(0.1972)	(0.3237)	(0.4130)	(0.1645)	(0.3632)	(0.5710)	(0.5651)	(0.3649)
	[CP]	[0.850]	[0.651]	[0.711]	[0.867]	[0.804]	[0.691]	[0.699]	[0.820]
	$\hat{\rho}$	0.1300	0.1586	0.1604	0.1009	0.9510	0.8923	0.8967	0.9813
	(MSE)	(0.2291)	(0.3913)	(0.4312)	(0.0801)	(0.2402)	(0.4154)	(0.4403)	(0.2130)
	[CP]	[0.838]	[0.699]	[0.732]	[0.861]	[0.864]	[0.791]	[0.793]	[0.856]
	$\hat{\theta}$	0.1352	0.1546	0.1695	0.1308	0.1363	0.1891	0.1893	0.1382
	(MSE)	(0.1676)	(0.3001)	(0.4961)	(0.1532)	(0.1565)	(0.4073)	(0.5035)	(0.1325)
	[CP]	[0.851]	[0.657]	[0.789]	[0.857]	[0.916]	[0.763]	[0.746]	[0.917]
300	$\hat{\lambda}$	0.1211	0.1503	0.1364	0.1061	0.5021	0.5638	0.5712	0.5009
	(MSE)	(0.1681)	(0.3146)	(0.3291)	(0.0261)	(0.1641)	(0.3173)	(0.3630)	(0.1512)
	[CP]	[0.891]	[0.672]	[0.780]	[0.893]	[0.899]	[0.747]	[0.731]	[0.902]
	$\hat{\rho}$	0.1281	0.1492	0.1470	0.1006	0.9371	0.8974	0.8982	0.9503
	(MSE)	(0.1441)	(0.3530)	(0.3021)	(0.0541)	(0.2121)	(0.3903)	(0.4102)	(0.2053)
	[CP]	[0.886]	[0.724]	[0.786]	[0.899]	[0.881]	[0.803]	[0.800]	[0.874]
	$\hat{\theta}$	0.1206	0.1488	0.1501	0.1201	0.1235	0.1709	0.1742	0.1292
	(MSE)	(0.1121)	(0.2943)	(0.3213)	(0.1036)	(0.2751)	(0.4156)	(0.5019)	(0.2601)
	[CP]	[0.894]	[0.786]	[0.779]	[0.864]	[0.930]	[0.796]	[0.758]	[0.929]

6. Application

In this section, we illustrate the flexibility of the proposed distribution using a real data set.

The fit of the proposed distribution is compared with the following distributions:

- Poisson (P) distribution having pmf

$$\mathrm{P(Y = y)} = \frac{e^{-\lambda}\lambda^y}{y!};\ \lambda \geq 0,\ \ y = 0, 1, 2, ...$$

- Discrete Burr (Krishna and Pundir (2009)) (DB) distribution having pmf

$$P(Y = y) = \theta^{log(1+y^{\alpha})} - \theta^{log(1+(1+y^{\alpha}))};\ 0 < \theta < 1,\ \alpha > 0\ ,\ y = 0, 1, 2, ...$$

- Discrete Gumbel (Chakraborty and Chakravarty (2014)) (DG) distribution having pmf

$$P(Y = y) = exp(-\alpha p^{y+1}) - exp(-\alpha p^{y}); \alpha > 0, 0 < p < 1,\ \ y = 0, 1, 2, ...$$

- A new three-parameter Poisson-Lindley (NTPPL) distribution (Das *et al.* (2018)) having pmf

$$P(Y=y)=\frac{\theta^2}{(\theta+1)^{x+2}}\left(1+\frac{\alpha+\beta x}{\theta\alpha+\beta}\right);\theta>0,\beta>0,\theta\alpha+\beta>0,y=0,1,2,...$$

The data set is taken from Efron (1988), a study of 51 patients with head and neck cancer conducted by the Northern California Oncology Group. We compare the fits of the DEIP distribution with the competitive models DG, DB, NTPPL and Poisson. The data is given below:

{0, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 5, 6, 7, 7, 7, 8,8, 9, 9, 9, 10, 13, 13, 13, 14, 17, 17, 19, 19, 36, 36, 37, 40, 44, 46, 46}

Table 4: Goodness of fit for various models fitted for the dataset.

Model	P	DG	DB	NTPPL	DEIP
Estimates	$\hat{\lambda}=11.314$	$\hat{\alpha}=3.207$ $\hat{p}=0.668$	$\hat{\alpha}=2.551$ $\hat{\theta}=0.730$	$\hat{\alpha}=1.506$ $\hat{\beta}=1.282\times10^6$ $\hat{\theta}=0.088$	$\hat{\lambda}=2.212$ $\hat{\rho}=4.404$ $\hat{\theta}=0.007$
K-S	0.4787	0.9821	0.2955	6.665	0.1656
p-value	<0.000	<0.000	<0.000	0.000	0.1217

The K-S statistic given in Table 4 is smallest for the DEIP distribution with the value of 0.1656 and p- value is 0.1217, which is higher when compared to other models. That is, Table 4 gives that the DEIP distribution leads to a better fit for the data set compared to the other four models.

7. Conclusion

In the present article, we have introduced a new family of discrete distributions called DIPc family. One special model of the proposed family are studied in detail. Further we have noticed DIPc family can be used for modelling variety of failure data because its hazard rate can take different shapes. The methods of ML, OLS, WLS and CVM estimations have been utilized to estimate the unknown parametres of the models. Some characterizations of the proposed distribution have been also studied. An extensive simulation is carried out to evaluate the behaviour of the above stated estimation methods. The flexibility of the family has also been elucidated using a real data set. The new distribution can serve as a better alternative for modelling count data in various fields including reliability, insurance, medicine, engineering *etc.*

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