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Discrete Harris Extended Weibull Distribution and Applications

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Abstract

In this paper, we introduce a new family called Discrete Harris Extended (DHE) family of distributions and study its properties. It is shown that the new family is a generalization of discrete Marshall-Olkin family of distributions. In particular, we study the discrete version of Harris Extended Weibull distribution in detail. We give some selected special distributions from DHE family. We derive some basic distributional properties such as probability generating function, moments, hazard rate and quantiles of the DHEW distribution. Estimation of the parameters is done using maximum likelihood method and a simulation study is conducted to verify the performance. By using the method of maximum likelihood estimation we obtain the estimates of the proposed model parameters with respect to two discrete data sets.

Key words: Discrete Harris Extended Weibull distribution; Infinite divisibility; Marshall-Olkin family of distributions; Maximum likelihood.

AMS Subject Classifications: 60E05, 62E15, 62F10

1. Introduction

In the literature, there are several methods to obtain a discrete distribution from a continuous distribution: the discretization method based on the survival function (Nakagawa and Osaki, 1975), the discretization method based on an infinite series (Good, 1953; Kulasekera and Tonkyn, 1992; Kemp, 1997), the discretization method based on the hazard function (Stein, 1984), the compound two-phase method (Chakraborty, 2015), the discretization method based on reverse hazard function (Ghosh *et al.*, 2013), among many others.

The traditional discrete distributions (geometric, Poisson, etc.) have limited applicability as models for reliability, failure times, counts, etc. This has led to the development of new discrete distributions based on popular continuous models for reliability, failure times, etc. Of these, the most popular is the discrete Weibull distribution which was introduced by Nakagawa and Osaki (1975) and studied by Stein and Dattero (1984), and

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Khan *et al.*(1989). Gómez-Déniz (2010) developed a new generalization of the geometric distribution using Marshall-Olkin scheme. Discrete modified Weibull distribution proposed by Nooghabi *et al.* (2011) is the discrete analogue of the modified Weibull distribution in Lai *et al.* (2003). Nekoukhou and Bidram (2015) proposed exponentiated discrete Weibull distribution as a discrete analog of the exponentiated Weibull distribution of Mudholkar and Srivastava (1993). Sandhya and Prasanth (2012, 2013) have considered generalisations of geometric and discrete uniform distributions invoking the approach of Marshal and Olkin (1997), while Sandhya and Prasanth (2016) has developed another generalisation of the discrete uniform distribution by adding two parameters to it, generalizing the Marshal-Olkin scheme itself. Recently, Jayakumar and Sankaran (2018) have introduced a new discrete family of distributions using truncated discrete Mittag-Leffler distribution and studied its properties.

In this paper, we identify some members of the DHE family of distributions using the discretization method of Nakagawa and Osaki (1975). Our work mainly focuses on the DHEW distribution. This distribution is generated by discretizing the Harris extended Weibull (HEW) distribution of Batsidis and Lemonte (2014) with survival function (sf)

$$\bar{G}(x) = \left(\frac{\lambda e^{-k(\eta x)^\beta}}{1 - \bar{\lambda} e^{-k(\eta x)^\beta}} \right)^{1/k} \tag{1}$$

The HEW probabiity density function (pdf) is given by

$$g(x) = \frac{\lambda^{1/k} \beta \eta^\beta (x)^{\beta-1} e^{-(\eta x)^\beta}}{[1 - \bar{\lambda} e^{-k(\eta x)^\beta}]^{1+\frac{1}{k}}} \quad ; \quad x > 0, \quad (\lambda,\ \eta,\ k,\ \beta) > 0, \quad \bar{\lambda} = 1 - \lambda. \tag{2}$$

Here, $\lambda > 0$, $k > 0$, and $\beta > 0$ are shape parameters, $\eta > 0$ is the scale parameter.

The HEW distribution has many applications specially in quality control and reliability; see Jose *et al.* (2018). This distribution is a suitable competitor for gamma and Weibull distributions. But, sometimes, it is impossible or inconvenient to measure the life length of a device on a continuous scale. In practice, we come across situations where lifetimes are recorded on a discrete scale. For example, on/off switching devices, bulb of photocopier machine, to and fro motion of spring devices, *etc.*, are some obvious such situations. In the last two decades, standard discrete distributions like geometric and negative binomial have been employed to model lifetime data. However, there is a need to find more flexible discrete distributions to fit various types of data.

The rest of the paper is organized as follows. Discretization of continuous family of distributions is discussed in Section 2. In Section 3, we introduce the DHE family of distributions and study its properties. In Section 4, it is shown that the DHE family of distributions is a rich class and identify some members of this family. Section 5 is devoted to the study of various properties of the DHEW distribution. In Section 6, we discuss the method of maximum likelihood estimation of parameters of the distribution and a simulation study is conducted to verify the performance. Two real data sets are analyzed to illustrate the suitability of the proposed model and the results are presented in Section 7. Concluding remarks are given in the last section.

2. Discretization of continuous family of distributions

The general approach of discretizing a continuous variable is to introduce the greatest integer function of X namely, $[X]$ (the greatest integer less than or equal to X till it reaches the integer), in order to introduce grouping on a time axis.

Let the continuous failure time X has the sf, $\bar{Q}(x) = P(X > x)$ and $Y = [X]$; be the discrete random variable obtained by grouping the continuous failure time into unit intervals, then by Roy (2003) the probability mass function (pmf) of Y can be written as

$$\begin{aligned} P(Y = y) = P(y \leq X < y+1) &= P(X > y) - P(X > y+1) \\ &= \bar{Q}_x(y) - \bar{Q}_x(y+1), \qquad y = 0, 1, 2, ... \end{aligned} \tag{3}$$

where $\bar{Q}_x(y) = P(X > y)$.

Using (3) many researchers have developed discrete distributions corresponding to existing continuous distributions. For more details refer Nakagawa and Osaki (1975), Krishna and Pundir (2009), Chakraborty and Chakravarty (2012), Seethalekshmi *et al.* (2016), Gillariose *et al.* (2021).

3. Discrete Harris extended family of distributions

Let $F(x)$ be the baseline cumulative distribution function (cdf) of a random variable X and let $\overline{F}(x)$ be the survival function (sf) of a distribution. Then the Harris family has the survival probabilities

$$\bar{Q}(x) = \left[\frac{\lambda \bar{F}(x)^k}{1 - \bar{\lambda}\bar{F}(x)^k}\right]^{1/k} \tag{4}$$

Now the probability mass function (pmf) of the new family is

$$\begin{aligned} p_Y(x) &= \bar{Q}(x) - \bar{Q}(x+1) \\ &= \lambda^{1/k}\left\{\frac{\bar{F}(x)}{[1-\bar{\lambda}\bar{F}(x)^k]^{1/k}} - \frac{\bar{F}(x+1)}{[1-\bar{\lambda}\bar{F}(x+1)^k]^{1/k}}\right\}, \quad x = 0, 1, 2, ... \end{aligned} \tag{5}$$

where, $\lambda, k > 0$, $\bar{\lambda} = 1 - \lambda$. We denote this family of distribution by DHE(λ, k) family. Note that , when $k = 1$, the distribution with pmf (5) reduces to discrete Marshall-Olkin distribution discussed in Supanekar and Shirke (2015). Let $R(x)$ be the hazard rate function (hrf) of DHE family of the discrete random variable X, then

$$\begin{aligned} R(x) &= \frac{p_Y(x)}{\bar{Q}(x)} \\ &= 1 - \frac{\bar{F}(x)[1-\bar{\lambda}\bar{F}(x+1)^k]^{1/k}}{\bar{F}(x)[1-\bar{\lambda}\bar{F}(x+1)^k]^{1/k}} \end{aligned} \tag{6}$$

3.1. Probability generating function, moments and quantiles

The probability generating function (pgf) of (5) is given by

$$P_Y(s) = 1 + \lambda^{1/k}(s-1)\sum_{x=1}^{\infty} s^{x-1}\frac{\bar{F}(x)}{\left[1-\bar{\lambda}\bar{F}(x)^k\right]^{1/k}} \tag{7}$$

Mean and Variance of the random variable X is given by

$$E(X) = \lambda^{1/k}\sum_{x=1}^{\infty}\frac{\bar{F}(x)}{\left[1-\bar{\lambda}\bar{F}(x)^k\right]^{1/k}} \tag{8}$$

$$V(X) = \lambda^{1/k}\sum_{x=1}^{\infty}(2x-1)\frac{\bar{F}(x)}{\left[1-\bar{\lambda}\bar{F}(x)^k\right]^{1/k}} - \left\{\lambda^{1/k}\sum_{x=1}^{\infty}\frac{\bar{F}(x)}{\left[1-\bar{\lambda}\bar{F}(x)^k\right]^{1/k}}\right\}^2 \tag{9}$$

Quantiles q_m and Median of DHE family are

$$q_m = \left[F^{-1}\left(1-(1-m)\left(\lambda+\bar{\lambda}(1-m)^k\right)^{-1/k}\right)-1\right] \tag{10}$$

Median is given by

$$Median = \left[F^{-1}\left(1-\left(2^k\lambda+\bar{\lambda}\right)^{-1/k}\right)-1\right] \tag{11}$$

where[.] denote the integer part.

4. Some members of DHE family of distributions

In this section, we give some selected special distributions from DHE family. The selected models are DHE exponential, DHE Uniform, DHE Fréchet, DHE Burr type XII, DHE Lomax and DHE Lindley.

4.1. DHE exponential(DHEE) distribution

Consider the sf of exponential distribution with parameter θ is given by $\bar{F}(x) = e^{-\theta x}$. Let $p = e^{-\theta}$, $0 < p < 1$. Then the probability mass function (pmf), survival function (sf), hazard rate function (hrf) of the DHEE distribution using equation (5) are respectively given by

$$p_x = \frac{\lambda^{1/k}p^x}{\left[1-\bar{\lambda}p^{kx}\right]^{1/k}} - \frac{\lambda^{1/k}p^{x+1}}{\left[1-\bar{\lambda}p^{x+1}\right]^{1/k}}, \quad x = 0, 1, 2, ...$$

$$\bar{Q}(x) = \frac{\lambda^{1/k}p^x}{\left[1-\bar{\lambda}p^{kx}\right]^{1/k}}$$

$$R(x) = 1 - \frac{\left[1-\bar{\lambda}p^{kx}\right]^{1/k}}{\left[1-\bar{\lambda}p^{k(x+1)}\right]^{1/k}}\ p$$

For $k = 1$, the distribution reduces to generalized geometric distribution obtained by discretizing the generalized exponential distribution of Marshall-Olkin (1997).

4.2. DHE Uniform (DHEU) distribution

Let X $\sim$ U$(0, a)$ follows Uniform distribution with parameter a. Then the sf of X is given by $\bar{F}(x) = 1 - \frac{x}{a}$. Then the pmf, sf, hrf of the DHEU distribution using (5) are respectively given by

$$p_x = \frac{\lambda^{1/k}(a-x)}{\left[a^k - \bar{\lambda}(a-x)^k\right]^{1/k}} - \frac{\lambda^{1/k}(a-x-1)}{\left[a^k - \bar{\lambda}(a-x-1)^k\right]^{1/k}}, \quad x = 1, 2, ..., a$$

$$\bar{Q}(x) = \frac{\lambda^{1/k}(a-x)}{\left[a^k - \bar{\lambda}(a-x)^k\right]^{1/k}}$$

$$R(x) = 1 - \frac{(a-x-1)\left[a^k - \bar{\lambda}(a-x)^k\right]^{1/k}}{(a-x)\left[a^k - \bar{\lambda}(a-x-1)^k\right]^{1/k}}$$

This distribution is obtained and studied by Prasanth and Sandhya (2016).

4.3. DHE Fr$\acute{e}$chet (DHEF) distribution

Consider the survival function of Fréchet distribution with parameter α and β is given by $\bar{F}(x) = 1 - e^{-(\frac{\alpha}{x})^\beta}$. Let $p = e^{-\alpha^\beta}$, $0 < p < 1$. Then the pmf, sf, hrf of the DHEF distribution using equation (5) are respectively given by

$$p_x = \frac{\lambda^{1/k}(1 - p^{-(\frac{1}{x})^\beta})}{\left[1 - \bar{\lambda}1 - p^{-(\frac{1}{x})^\beta}\right]^{1/k}} - \frac{\lambda^{1/k}(1 - p^{-(\frac{1}{x+1})^\beta})}{\left[1 - \bar{\lambda}(1 - p^{-(\frac{1}{x+1})^\beta})\right]^{1/k}}, \quad x = 0, 1, 2, ...$$

$$\bar{Q}(x) = \frac{\lambda^{1/k}(1 - p^{-(\frac{1}{x})^\beta})}{\left[1 - \bar{\lambda}1 - p^{-(\frac{1}{x})^\beta}\right]^{1/k}}, \quad x = 0, 1, 2, ...$$

$$R(x) = 1 - \frac{(1 - p^{-(\frac{1}{x+1})^\beta})\left[1 - \bar{\lambda}(1 - p^{-(\frac{1}{x})^\beta})\right]^{1/k}}{(1 - p^{-(\frac{1}{x})^\beta})\left[1 - \bar{\lambda}(1 - p^{-(\frac{1}{x+1})^\beta})\right]^{1/k}}$$

4.4. DHE Burr type XII(DHEBXII) and Lomax (DHELX) distributions

Consider the survival function of Burr type III distribution with parameter c and b is given by $\bar{F}(x) = (1 + x^c)^{-b}$. Let $p = e^{-b}$, $0 < p < 1$. Then the pmf, sf, hrf of the DHEBXII distribution using equation (5) are respectively given by

$$p_x = \frac{\lambda^{1/k} p^{log(1+x^c)}}{\left[1 - \bar{\lambda}p^{klog(1+x^c)}\right]^{1/k}} - \frac{\lambda^{1/k} p^{log(1+(x+1)^c)}}{\left[1 - \bar{\lambda}p^{klog(1+(x+1)^c)}\right]^{1/k}}, \quad x = 0, 1, 2, ...$$

$$\bar{Q}(x) = \frac{\lambda^{1/k} p^{log(1+x^c)}}{\left[1 - \bar{\lambda}p^{klog(1+x^c)}\right]^{1/k}}$$

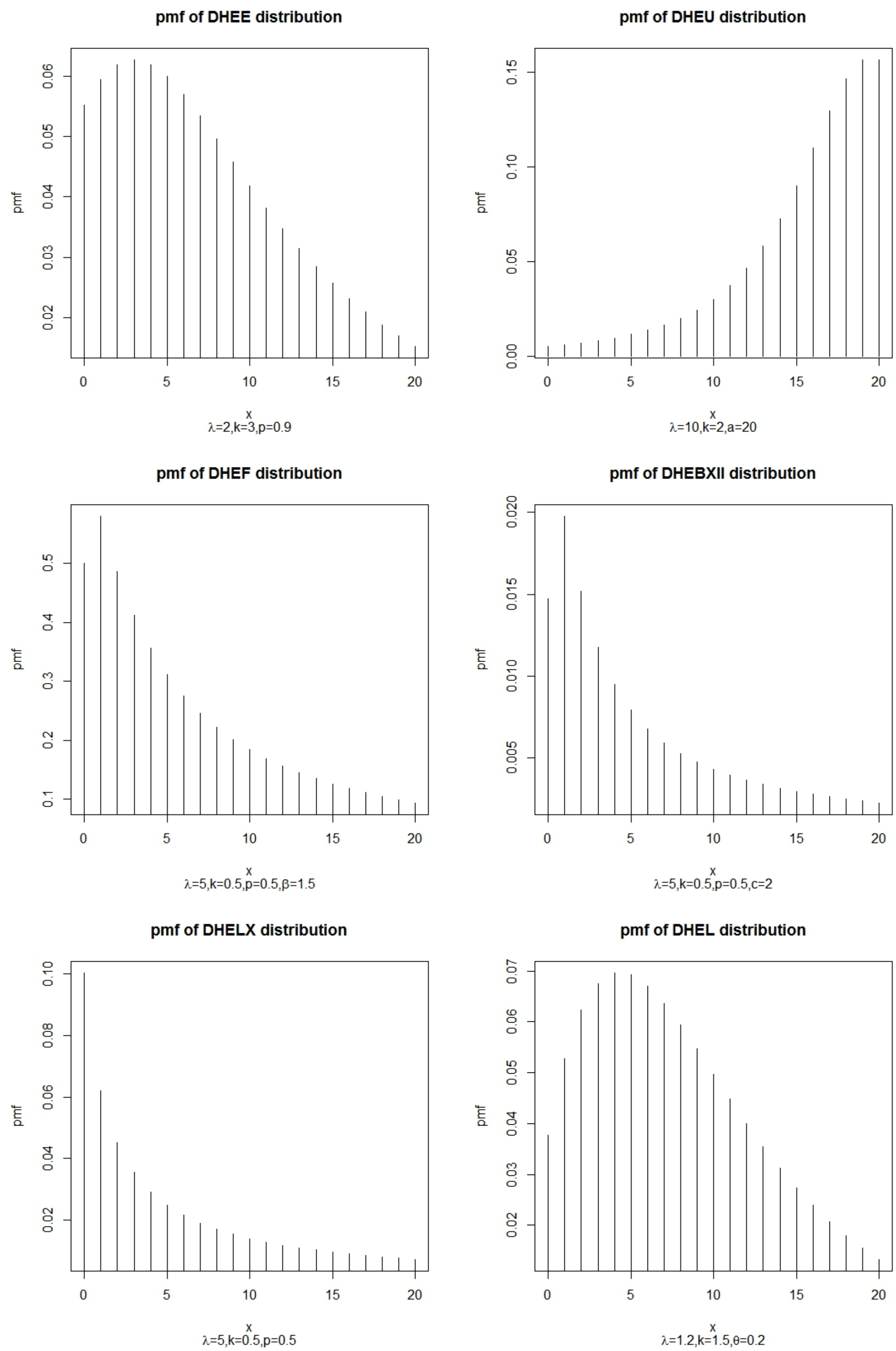

Figure 1: pmf of discrete HE family of distributions

$$R(x) = \frac{p^{log(1+(x+1)^c)}\left[1-\bar{\lambda}p^{klog(1+x^c)}\right]^{1/k}}{p^{log(1+x^c)}\left[1-\bar{\lambda}p^{klog(1+(x+1)^c)}\right]^{1/k}}$$

When $c = 1$, the DHEBXII distribution becomes DHELX distribution.

4.5. DHE Lindley (DHEL) distribution

Consider the survival function of Lindley distribution with parameter θ is given by $\bar{F}(x) = \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}$. Then the pmf, sf, hrf of the DHEL distribution using equation (5) are respectively given by

$$\begin{aligned} p_x &= \frac{\lambda^{1/k}(1+\theta+\theta x)e^{-\theta x}}{[(1+\theta)^k - \bar{\lambda}(1+\theta+\theta x)^k e^{-k\theta x}]^{1/k}} \\ &- \frac{\lambda^{1/k}(1+\theta+\theta(x+1))e^{-\theta(x+1)}}{[(1+\theta)^k - \bar{\lambda}(1+\theta+\theta(x+1))^k e^{-\theta k(x+1)}]^{1/k}}, \quad x = 0, 1, 2, ... \end{aligned}$$

where, $(\lambda, k, \theta) > 0$

$$\bar{Q}(x) = \frac{\lambda^{1/k}(1+\theta+\theta x)e^{-\theta x}}{[(1+\theta)^k - \bar{\lambda}(1+\theta+\theta x)^k e^{-k\theta x}]^{1/k}}$$

$$R(x) = 1 - \frac{[(1+\theta)^k - \bar{\lambda}(1+\theta+\theta x)^k e^{k\theta x}]^{1/k}[1+\theta+\theta(x+1)]}{[(1+\theta)^k - \bar{\lambda}(1+\theta+\theta(x+1))^k e^{-k\theta(x+1)}]^{1/k}[1+\theta+\theta x]}e^{-\theta} \tag{12}$$

We can obtain discrete half-logistic, discrete half-normal and discrete Rayleigh distribution as members of new family of distributions, defined in (5), by substituting respective distribution function. In the next section, we study discrete HEW distribution in detail. Figure 1 displays possible shapes of the selected discrete Harris extended models.

5. Discrete Harris extended Weibull(DHEW) distribution

The sf of Weibull distribution with scale parameter η and shape parameter β is given by

$$\bar{F}(x) = e^{-(\eta x)^\beta}; \quad x > 0, \quad \eta > 0, \quad \beta > 0$$

Let $e^{-\eta^\beta} = p; 0 < p < 1$. Hence the sf of the resulting discrete distribution is given by

$$\bar{Q}(x) = \frac{\lambda^{1/k}p^{x^\beta}}{[1-\bar{\lambda}p^{kx^\beta}]^{1/k}}; x = 0, 1, 2, ... \tag{13}$$

$$p_x = \frac{\lambda^{1/k}p^{x^\beta}}{\left[1-\bar{\lambda}p^{kx^\beta}\right]^{1/k}} - \frac{\lambda^{1/k}p^{(x+1)^\beta}}{\left[1-\bar{\lambda}p^{(x+1)^\beta}\right]^{1/k}}, \quad x = 0, 1, 2, ...$$

We call the random variable X, with sf (13), as DHEW distribution with parameters $\lambda > 0$,$k > 0$,$0 < p < 1$, $\beta > 0$ and denote it by DHEW (λ, k, p, β). Many properties of the continuous HEW distribution also hold for DHEW (λ, k, p, β). Figure 2 displays possible

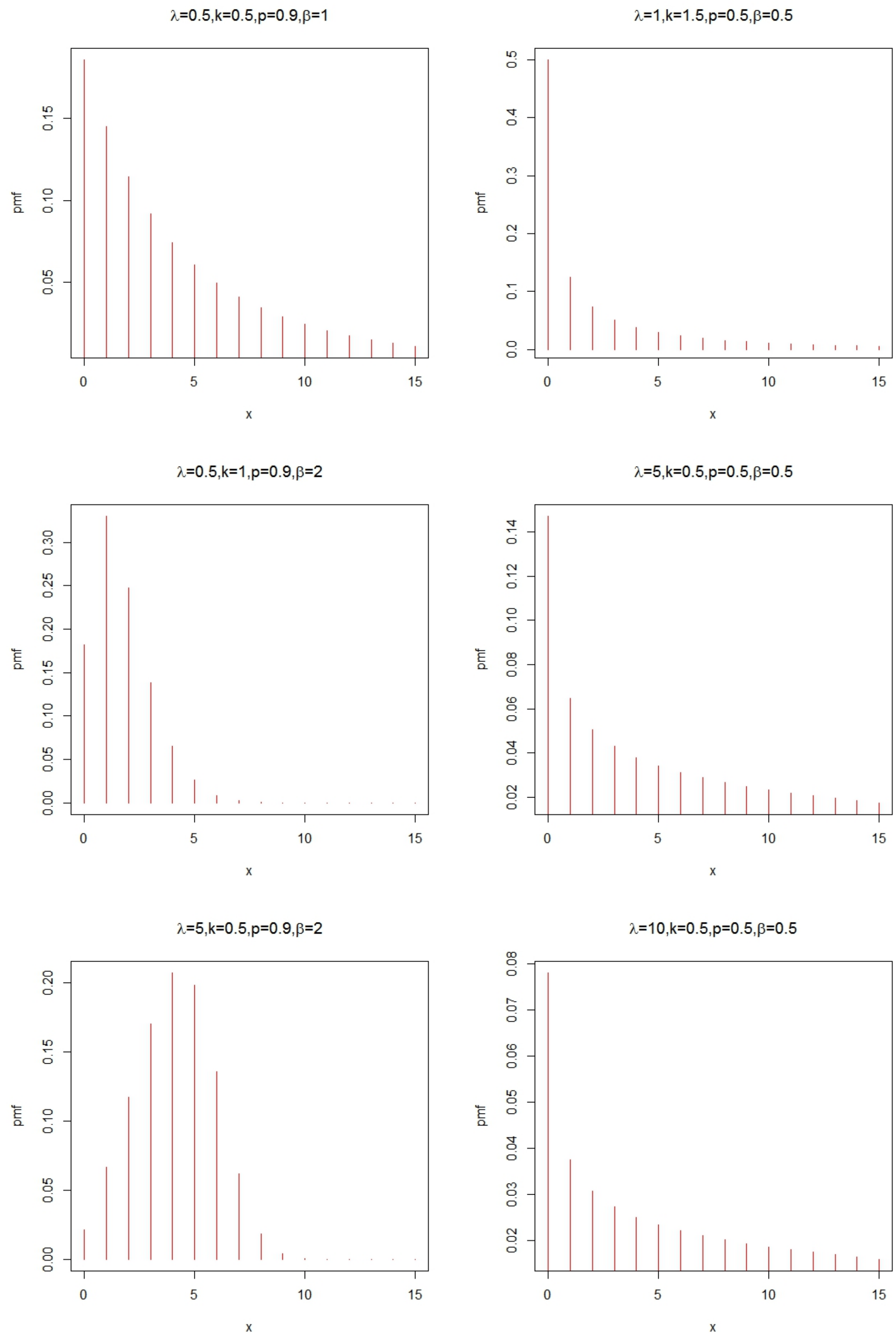

Figure 2: pmf of DHEW distribution for various values of parameters

shapes of the pmf of the DHEW distribution. The pmf can be increasing, decreasing and upside-down bathtub shaped. The hazard rate is given by

$$R(x) = 1 - \frac{p^{(x+1)^\beta}\left[1-\bar{\lambda}p^{kx^\beta}\right]^{1/k}}{p^{x^\beta}\left[1-\bar{\lambda}p^{k(x+1)^\beta}\right]^{1/k}} \tag{14}$$

Figure 3 displays possible shapes of the hrf of DHEW distribution for selected values of the parameters λ,$k > 0$,p and $\beta > 0$ respectively. Obviously, from figure it is clear that the hrf can be increasing, decreasing, bathtub and upside-down bathtub shaped.

5.1. Special sub-models

Some discrete distributions that are special cases of DHEW distribution are:

(1) When $k = 1$, we obtain

$$p_x = \frac{\lambda[p^{x^\beta} - p^{(x+1)^\beta}]}{[\lambda + (1-\lambda)(1-p^{x^\beta})][\lambda + (1-\lambda)(1-p^{(x+1)^\beta})]}$$

which is considered as the discrete version of Marshall-Olkin Weibull distribution.

(2) When $\lambda = 1$,$k = 1$, we obtain discrete Weibull distribution of Nakagawa and Osaki(1975).In addition $\beta = 1$ geometric distribution is achieved.

(3) If $\beta = 2$, then the pmf reduce to

$$P(X = x) = p_x = \frac{\lambda^{1/k}p^{x^2}}{\left[1-\bar{\lambda}p^{kx^2}\right]^{1/k}} - \frac{\lambda^{1/k}p^{(x+1)^2}}{\left[1-\bar{\lambda}p^{(x+1)^2}\right]^{1/k}}, \quad x = 0, 1, 2, ...$$

which defines discrete version of Harris Extended Rayleigh distribution.

(4) If $\beta = 2$ and $\lambda = 1$,

$$p_x = \frac{\lambda[p^{x^2} - p^{(x+1)^2}]}{[\lambda + (1-\lambda)(1-p^{x^2})][\lambda + (1-\lambda)(1-p^{(x+1)^2})]}$$

which is the discrete version of Marshall-Olkin Rayleigh distribution. Moreover with $k = 1$, we get discrete Rayleigh distribution of Roy (2014).

5.2. Probability generating function, quantiles, mean and variance

The pgf of DHEW(λ, k, p, β) is given by

$$P_X(s) = 1 + \lambda^{1/k}(s-1)\sum_{x=1}^{\infty} s^{x-1}\frac{p^{x^\beta}}{\left[1-\bar{\lambda}p^{kx^\beta}\right]^{1/k}}$$

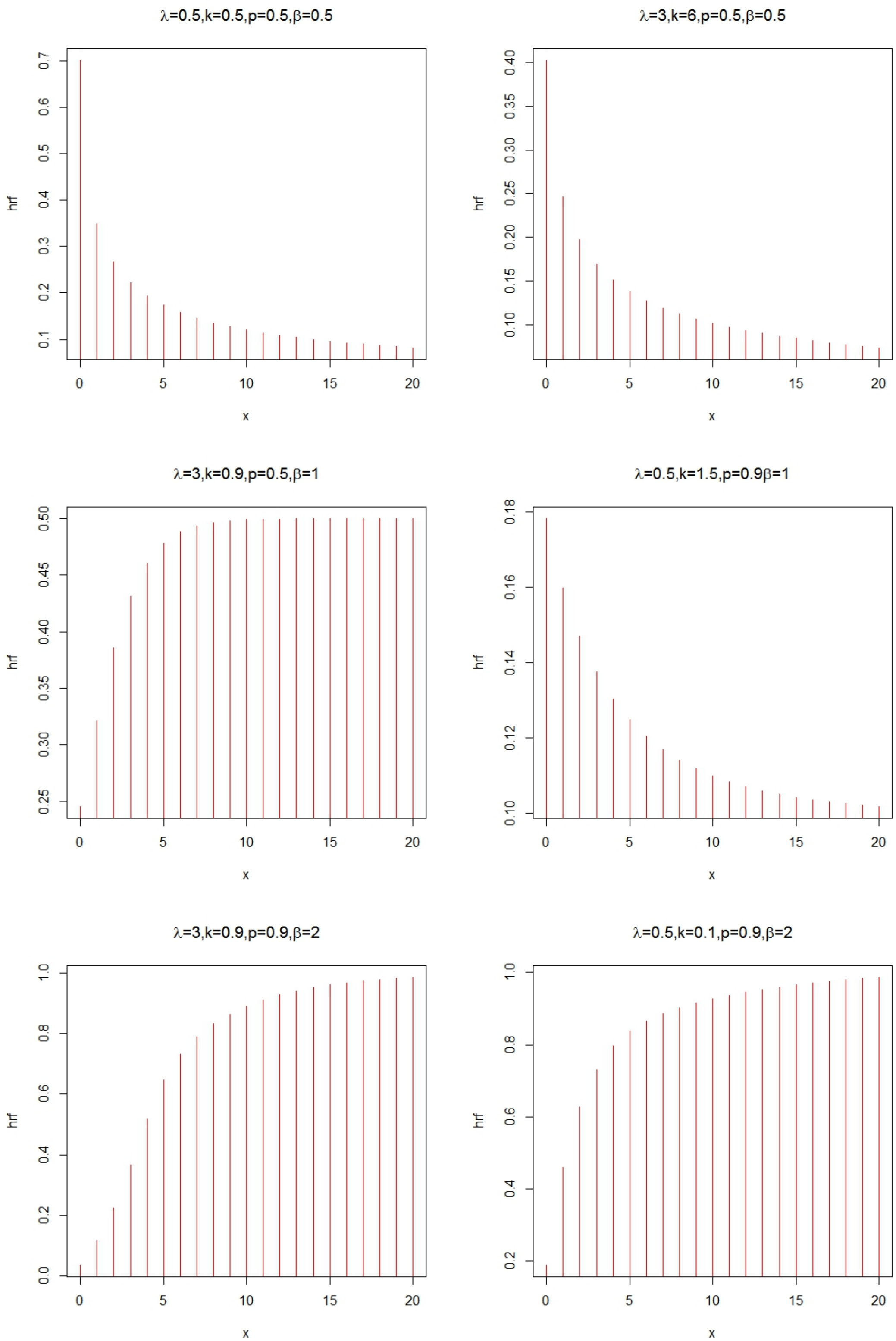

Figure 3: hrf of DHEW distribution for various values parameters

The m^{th} quantile of DHEW distribution is denoted by q_m and is given by

$$q_m = \left\{ \frac{\log p \ \log[\bar{\lambda} + \lambda(1-m)^{-k}]}{k} \right\}^{1/\beta} - 1$$

In particular, the Median is

$$Median = \left[\frac{\log p \ \log(\bar{\lambda} + 2^k \lambda)}{k} \right]^{1/\beta} - 1$$

The expression for mean and variance of DHEW(λ, k, p, β) is given by

$$E(X) = \lambda^{1/k} \sum_{x=1}^{\infty} \frac{p^{x^\beta}}{\left[1 - \bar{\lambda} p^{kx^\beta}\right]^{1/k}} \tag{15}$$

and

$$V(X) = \lambda^{1/k} \sum_{x=1}^{\infty} (2x-1) \frac{p^{x^\beta}}{\left[1 - \bar{\lambda} p^{kx^\beta}\right]^{1/k}} - \left[\lambda^{1/k} \sum_{x=1}^{\infty} \frac{p^{x^\beta}}{\left[1 - \bar{\lambda} p^{kx^\beta}\right]^{1/k}} \right]^2 \tag{16}$$

The mean and variance of a DHEW(λ, k, p, β) distribution for different values of parameters are calculated numerically in Table 1 using the expression (15) and (16). From the Table 1, we can see that depending on the values of parameters, the mean of the distribution can be equal, smaller or greater than the variance. Hence DHEW models are appropriate for modelling both over and under dispersed data.

5.3. Infinite divisibility

According to Steutel and van Harn (2004, pp. 56) if p_x, $x \in N_0$ is infinitely divisible, then $p_x < e^{-1}$ for all $x \in N$. However, *e.g.*, in a DHEW(0.25, 0.15, 0.9, 2) distribution, we see that $p_1 = 0.4493 > e^{-1} = 0.367$. Therefore, in general, DHEW(λ, k, p, β) distribution is not infinitely divisible. In addition, since the class of self decomposable and stable distributions, in their discrete concept, are subclass of infinitely divisible distributions, we can conclude that DHEW distribution can be neither self decomposable nor stable, in general.

6. Estimation

To apply the method of maximum likelihood for estimating λ, k, p and β assume that X_1, X_2, ..., X_n is a random sample of size n from DHEW distribution. The log-likelihood function is

$$L = \frac{n}{k} log\lambda + \sum_{i=1}^{n} log \left[\frac{p^{x_i^\beta}}{[1 - \bar{\lambda} p^{kx_i^\beta}]^{1/k}} - \frac{p^{(x_i+1)^\beta}}{[1 - \bar{\lambda} p^{k(x_i+1)^\beta}]^{1/k}} \right] \tag{17}$$

Hence, the likelihood equations are,

$$\frac{\partial L}{\partial \lambda} = \frac{n}{k\lambda} + \sum_{i=1}^{n} \frac{[V_{\lambda,k,\beta}(x_i) - V_{\lambda,k,\beta}(x_i+1)]}{km_{\lambda,k,\beta}(x_i)} \tag{18}$$

Table 1: The mean(standard deviation) of DHEW for different parameters

	$p \longrightarrow$ $\beta \downarrow$	0.25	0.5	0.75
$k = 0.5$	0.50	1.2787(1.2759)	2.555(5.3605)	11.3245(31.4465)
$\lambda = 0.5$	0.75	1.16958(0.5932)	1.70562(1.6561)	3.9703(5.5288)
	3.50	1.1111(0.3141)	1.2992(0.4580)	1.5951(0.5143)
$k = 0.5$	0.50	2.2751(3.0416)	7.0255(12.4002)	37.9023(72.2195)
$\lambda = 1.5$	0.75	1.36(1.2370)	3.2516(3.2333)	9.2204(10.542)
	3.50	1.1644(0.48)	1.61491(0.4883)	1.8937(0.4887)
$k = 1$	0.50	1.4074(1.6471)	3.1233(6.8603)	14.6716(40.145)
$\lambda = 0.5$	0.75	1.2357(0.7304)	1.8981(2.0028)	4.6081(6.6489)
	3.50	1.1428(0.3498)	1.3335(0.4762)	1.6196(0.5244)
$k = 1$	0.50	2.0718(2.6700)	6.1911(10.918)	33.0475(6.3592)
$\lambda = 1.5$	0.75	1.5943(1.1210)	3.0204(2.9410)	8.4713(9.5965)
	3.50	1.333(0.4713)	1.6005(0.4909)	1.8749(0.4721)
$k = 3$	1.00	1.3808 (0.7000)	2.1324(1.4620)	4.3543(3.5536)
$\lambda = 1.5$	2.00	1.2899(0.4634)	1.6347(0.6195)	2.2640(0.8981)
	3.50	1.2854(0.4515)	1.5613(0.4970)	1.8496(0.4648)

$$\frac{\partial L}{\partial k} = \frac{-n}{k^2\lambda} + \sum_{i=1}^{n} \frac{\bar{\lambda}[W_{\lambda,k,\beta}(x_i+1) - W_{\lambda,k,\beta}(x_i)]}{m_{\lambda,k,\beta}(x_i)} \tag{19}$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} \frac{\bar{\lambda} logp[U_{\lambda,k,\beta}(x_i+1) - U_{\lambda,k,\beta}(x_i)]}{m_{\lambda,k,\beta}(x_i)} \tag{20}$$

$$\frac{\partial L}{\partial p} = \sum_{i=1}^{n} \frac{\bar{\lambda}[(x_i+1)^{\beta} V_{\lambda,k,\beta}(x_i+1) - (x_i)^{\beta} V_{\lambda,k,\beta}(x_i)]}{m_{\lambda,k,\beta}(x_i)} \tag{21}$$

where,

$$m_{\lambda,k,\beta}(x) = \frac{p^{x^\beta}}{[1-\bar{\lambda}p^{kx^\beta}]^{1/k}} - \frac{p^{(x+1)^\beta}}{[1-\bar{\lambda}p^{k(x+1)^\beta}]^{1/k}}$$

$$V_{\lambda,k,\beta}(x) = p^{x^\beta}\left(\frac{1}{1-\bar{\lambda}p^{kx^\beta}}\right)^{\frac{1}{k}-1} p^{kx^\beta}$$

$$W_{\lambda,k,\beta}(x) = p^{x^\beta}\left(\frac{1}{1-\bar{\lambda}p^{kx^\beta}}\right)^{\frac{1}{\beta}} log\left(\frac{1}{1-\bar{\lambda}p^{kx^\beta}}\right) p^{kx^\beta} log(p^{x^\beta})$$

$$U_{\lambda,k,\beta}(x) = p^{x^\beta}\left(\frac{1}{1-\bar{\lambda}p^{kx^\beta}}\right)^{\frac{1}{k}-1} p^{kx^\beta} x^\beta logx + \left(\frac{1}{1-\bar{\lambda}p^{kx^\beta}}\right)^{\frac{1}{k}} p^{x^\beta} x^\beta logx$$

The solutions of likelihood equations (18)-(21) provide the maximum likelihood estimators (MLEs) of $\theta = (\lambda, k, p, \beta)^T$, say $\hat{\theta} = (\hat{\lambda}, \hat{k}, \hat{p}, \hat{\beta})^T$, which can be obtained by a numerical method such as the four variable Newton -Raphson type procedure.

6.1. Simulation study

Here we study the performance of the MLEs of the model parameters of DHEW distribution using Monte Carlo simulation for various sample sizes and for selected parameter values. We have taken the parameter values as λ= 1,β= 0.5, k= 0.2and p= 0.8 and generated random samples of size $n = 30$, 50 and 60 respectively. The MLEs of λ,β k and p are determined by maximizing the log-likelihood function using the nlm package of R software based on each generated samples. This simulation is repeated 1000 times and the average estimates of bias and MSE are computed and presented in Table 2. It can be seen that, as the sample size increases, the bias tends to zero and MSE decreases.

Table 2: Simulation results related to the paramters of the DHEW distribution

Sample size	Estimates	Average bias	MSE
	0.6079	-0.3920	0.9131
	0.2916	-0.2083	0.1060
30	0.1193	-0.0806	0.0238
	0.4630	-0.3369	0.2700
	0.9081	-0.0918	0.0917
	0.4548	-0.0451	0.0228
50	0.1828	-0.0171	0.0048
	0.7273	-0.0726	0.0582
	0.9990	-0.0009	0.0009
	0.5004	0.0005	0.0002
60	0.2021	0.0021	0. 0043
	0.8002	0.0002	3.8699e-05

7. Application

In this section, we illustrate the flexibility of the proposed distribution using two real data sets. Maximum likelihood estimation is used to obtain the parameter estimates of the models(using R software). We compare the fit of the DHEW distribution with the following discrete life time distributions.

(a) Exponentiated discrete Weibull (EDW) distribution (Nekoukhou and Bidram 2015) having pmf

$$P(X = x) = (1 - p^{(x+1)^\alpha})^\beta - (1 - p^{x^\alpha})^\beta;\quad 0 < p < 1,\ \alpha > 0,\ \beta > 0,\ x = 0, 1, 2, ...$$

(b) The pmf of the discrete Gamma (DG) distribution, which has been used first by Yang (1994) and recently considered by Chakraborty and Chakravarty (2012), is given by

$$P(X = x) = \frac{\gamma(\alpha, \beta(x + 1)) - \gamma(\alpha, \beta x)}{\Gamma(\alpha)},\ \alpha > 0,\ \beta > 0$$

where,$\gamma(\alpha, x) = \int_0^x t^{\alpha-1}e^{-t}dt$ denotes the incomplete gamma function.

Table 3: Aarset data

Time of failure	0	1	2	3	6	7	11	12	18	21	32	36	40	45	46
No. of failures	2	5	1	1	1	1	1	1	5	1	1	1	1	1	1
Time of failure	47	50	55	60	63	67	72	75	79	82	83	84	85	86	
No. of failures	1	1	1	1	2	4	1	1	1	2	1	3	5	2	

Table 4: Fitted estimates for Aarset data

Distribution	MLEs	AIC	K-S
DHEW	$(\hat{\lambda}, \hat{k}, \hat{\beta}, \hat{p}) = (2.9, 0.30, 0.1248, 0.0403)$	**484.777**	**0.1739**
EDW	$(\hat{\alpha}, \hat{\beta}, \hat{p}) = (13.0059, 0.2517, 0.2675)$	509.864	0.2194
DW	$(\hat{\beta}, \hat{p}) = (1.0228, 0.9805)$	487.2202	0.1867

(c) A generalization of discrete Rayleigh (GDR) distribution of Roy (2004) having pmf

$$P(X = x) = (1 - p^{(x+1)^2})^\gamma - (1 - p^{x^2})^\gamma; \quad 0 < p < 1, \ \alpha > 0, \ x = 0, 1, 2, ...$$

(d) Discrete Weibull(DW) distribution (Nakagawa and Osaki 1975) having pmf

$$P(X = x) = p^{x^\alpha} - p^{(x+1)^\alpha}; \quad 0 < p < 1, \ \alpha > 0, \ x = 0, 1, 2, ...$$

The values of the K-S (Kolmogrov- Smirnov) statistic and AIC (Akaike Information Criterion with correction) are calculated for the four distributions in order to verify which distribution fits better to the data. The better distribution corresponds to smaller values of -log L, K-S statistic and AIC as well as larger p-value. Here, $AIC = -2LogL + 2k$, where, L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size.

7.1. Discrete Aarset data

Aarset (1987) data consist of the failure times (in weeks) of 50 devices put on a life test. The TTT (Total Time on Test) plot for this data shows that the hazard rate has a bathtub-shape. The data set is given in Table 3.

The MLE of parameters of the models and the measures AIC and K-S statistic are given in Table 4. From Table4, we can see that AIC, K-S statistic are smallest for DHEW with AIC=484.77 and K-S statistic value=0.1739. Hence DHEW model gives a better fit to the data.

7.2. Discrete Karlis and Xekalaki data

In this section, the DHEW model will be examined for a real data set which is given by Karlis and Xekalaki (2001) on the numbers of fires in Greece for the period from 1 July 1998 to 31 August of the same year. This data set consists of 123 observations and are presented

Table 5: Numbers of fires in Greece

Numbers	0	1	2	3	4	5	6	7	8	9	10	11	12	15	16	20	43
Frequency	16	13	14	9	11	13	8	4	9	6	3	4	6	4	1	1	1

Table 6: Fitted estimates for discrete Karlis and Xekalaki data

Distribution	Estimated Parameters	AIC	K-S
DHEW	$(\hat{\lambda}, \hat{k}, \hat{\beta}, \hat{p}) = (8.5913, 0.9718, 0.6705, 0.4393)$	**693.5843**	**0.128**
EDW	$(\hat{\alpha}, \hat{\beta}, \hat{p}) = (1.1573, 1.0511, 0.8449)$	694.1897	0.1285
GDR	$(\hat{\alpha}, \hat{p}) = (0.3934, 0.9924)$	694.6178	0.1467
DG	$(\hat{\alpha}, \hat{\beta}) = (0.7525, 0.1543)$	749.7162	0.2683

in Table 5. Only fires in forest districts are considered. Bakouch et al. (2014) considered these data to indicate the potentiality of discrete Lindley (DL) distribution in data modeling and compared it with Poisson, geometric and discrete gamma (DG) distributions.

The MLEs of parameters of the models and the measures AIC and K-S statistic are given in Table 6. The MLEs and K-S test statistic values of the DG distribution, given in this table, are directly reported from Table 7 of Bakouch et al. (2014). From Table 6, we can see that AIC, K-S statistic are smallest for DHEW with AIC=693.5843 and K-S statistic value=0.128. Hence DHEW model gives a better fit to this data.

8. Conclusion

In this paper, we have introduced a new family of discrete Harris extended distributions. This family is a generalization of discrete Marshall-Olkin family of distributions. We obtained generalizations of discrete exponential, discrete uniform, discrete Weibull and many other discrete distributions using this family. As an illustration, we have studied discrete Harris extended Weibull distribution in detail. From the results presented here, it can be seen that the generalized discrete Harris extended Weibull distribution introduced in this paper appears to be more suitable for modeling many real data sets and is a better alternative to some existing distributions.

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References

Abouammoh, A. M., Alshangiti, A. M. and Ragab, I. E. (2015). A new generalized Lindley distribution. *Journal of Statistical Computation and Simulation*, **85**, 3662-3678.

Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*. **36**, 106-108.

Batsidis, A. and Lemonte, A. J. (2014). On the Harris extended family of distributions. *Statistics: A Journal of Theoretical and Applied Statistics*, **49**, 1400-1421.

Bakouch, H. S., Jazi, M. A. and Nadarajah, S. (2014). A new discrete distribution. *Statistics*, **48**, 200-240.

Chakraborty, S. (2015). Generating discrete analogues of continuous probability distributions-a survey of methods and constructions. *Journal of Statistical Distributions and Applications*, **2**, 1-30.

Chakraborty, S. and Chakravarty, D. (2012). Discrete gamma distributions: Properties and parameter estimations. *Communications in Statistics-Theory and Methods*, **41**, 3301-3324.

Ghosh, T., Roy, D. and Chandra, N. K. (2013). Reliability approximation through the discretization of random variables using reversed hazard rate function. *International Journal of Mathematical, Computational, Statistical, Natural and Physical Engineering, International Science Index*, **7**, 96 -100.

Gillariose, J., Balogun, O. S., Almetwally , E. M., Sherwani, R. A. K., Jamal, F. and Joseph, J. (2021). On the Discrete Weibull Marshall Olkin family of distributions: Properties, characterizations, and applications. *Axioms*, **10**, 287.

Good, I. J. (1953). The population frequencies of species and the estimation of population parameters. *Biometrika*, **40**, 237-264.

Gómez-Déniz E. (2010). Another generalization of the geometric distribution. *Test*, **19**, 399-415.

Jayakumar, K. and Sankaran, K. K. (2018). A generalization of discrete Weibull distribution. *Communications in Statistics - Theory and Methods*, **47**, 6064-6078

Jose, K. K., Tomy L. and Thomas, S. P. (2018). On a generalization of the Weibull istribution and its application in quality control. *Stochastics and Quality Control*, **33** , 113-124.

Kemp, A. W.(1997). Characterizations of a discrete normal distribution. *Journal of Statistical Planning and Inference*, **63**, 223- 229.

Karlis, D. and Xekalaki, E. (2001). *On Some Discrete Valued Time Series Models Based on Mixtures and thinning*, in Proceedings of the Fifth Hellenic-European Conference on Computer Mathematics and Its Applications, E. A. Lipitakis Ed., 872-877.

Krishna, H. and Pundir, P. S., (2009). Discrete burr and discrete Pareto distributions. *Statistical Methodology*, **6**, 177-188.

Khan, M. S. A., Khalique, A. and Abouammoh, A. M. (1989). On estimating parameters in a discrete Weibull distribution. *IEEE Transactions on Reliability*, **38** , 348-350.

Kulasekera, K. and Tonkyn D. W. (1992). A new discrete distribution, with applications to survival, dispersal and dispersion. *Communications in Statistics - Simulation and Computation*, **21**, 499-518.

Lai, C. D., Xie M. and Murthy D. N. P. (2003). A modified Weibull distribution. *IEEE Transactions on Reliability*, **52**, 33-37.

Marshall A. W. and Olkin I. (1997). A new method for adding a parameter to a family of distributions with Application to the Exponential and Weibull Families. *Biometrika*, **84**, 641-652.

Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure rate data. *IEEE Transactions on Reliability*, **42**, 299-302.

Nakagawa, T. and Osaki, S. (1975). Discrete Weibull distribution. *IEEE Transactions on Reliability*, **24**, 300-301.

Nekoukhou, V. and Bidram, H. (2015). The exponentiated discrete Weibull distribution. *SORT*, **39**, 127-146.

Stein, W. E. and Dattero, R. (1984). A new discrete Weibull distribution. *IEEE Transactions on Reliability*, **33**, 196-197.

Roy, D. (2004). Discrete Rayleigh distribution. *IEEE Transactions on Reliability*, **53**, 255-260.

Sandhya, E. and Prasanth, C. B. (2012). *A Generalized Geometric Distribution.* In Proceedings Of International Conference on Frontiers of Statistics and Application and 32^{nd} Annual Conference of Indian Society for Probability and Statistics, Department of Statistics, Podichery University, Dec-2012, Bonfring publication, 261-269, ISBN 978-93-82338-78-9

Sandhya, E. and Prasanth, C. B. (2013). Marshall-Olkin discrete uniform distribution. *Journal of Probability*, **2014**, 1-10.

Sandhya, E. and Prasanth, C. B. (2016). A generalized discrete uniform distribution. *Journal of Statistics Applications and Probability*, **1**, 109-121.

Seethalekshmi, V., Sebastian, S. and Joseph J. (2016). Frechet distribution on integers: Properties and estimation. *International Journal of Computer and Mathematical Sciences*, **5**, 2347-8527.

Stein, W. E. and Dattero, R. (1984). A new discrete Weibull distribution. *IEEE Transactions on Reliability*, **33**, 196-197.

Supanekar, S. R. and Shirke, D. T. (2015). A new discrete family of distributions. *ProbStat Forum*, **8**, 83-94.

Satheesh, S., Sandhya, E. and Sherly, S. (2006). A generalization of stationary AR(1) schemes. *Statistical Methods*, **8**, 213-225.

Steutel, F. W. and Van Harn, K. (2004). *Infinite Divisibility of Probability Distributions on the Real Line*. New York: Marcel Dekker.