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A New Family of Probability Density Functions With Deflated Tails

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Abstract

The objective of this study is to introduce a new family of probability density functions using a given probability density function and analyze some of its important theoretical properties involving quantiles and failure rate. As an offshoot of this new family of probability density functions, a two-component mixture density that behaves differently at the tail-ends viz., the two densities tend to be negligible to the left / right of two designated tail-end values, respectively, is proposed. This is important as mixture probability distributions have been extensively studied in the literature and their applications in real life are also well-known.

Key words: Mixture distribution; Failure rate; Quantiles; Stochastic ordering.

AMS Subject Classifications: 62E15, 60E15, 62N05

1. Introduction

The subject of mixture distributions is important from theoretical as well as practical points of view. Three major references that deal with this topic are Everittt and Hand (1981), Titterington, Smith and Makov (1985), and McLachlan and Basford (1988). Theoretical aspects are concerned with (i) obtaining parameter estimates within models and those of mixing distributions, (ii) identification of the number of components in a finite mixture, (iii) imputation of the missing indicators of component membership for mixture data, whereas practical aspects deal with areas such as fisheries research, economics, medicine, biology, psychology, palaeontology, geology, botany, agriculture, zoology, reliability and many other fields.

Mathematically, mixture distributions are typically formalized as follows.
Consider a pair $Y = (X, Z)$ of random variables with $g(y)$ as its joint probability density and suppose that

$$g(y) = g(x, z) = f(x|z)\pi(z)$$

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where f and π are, respectively, a conditional and marginal density. Then, in terms of this factorization, the marginal density for X, $p(x)$, is

$$p(x) = \int f(x|z)\pi(z)dz.$$

If the support of π is finite and concentrated on $c_1, c_2, \ldots, c_k$, say, then we get

$$p(x) = \sum_{i=1}^{k} \pi_i f(x_i)$$

where

$$f_i(x) = f(x|Z = c_i)$$

and

$$\pi_i = P(Z = c_i)$$

for $i = 1, 2, \ldots, k$.

In this case, X is said to have a finite mixture distribution, the f_i are called the component densities and the probabilities $\{\pi_i\}$ are called mixing weights.

In this paper, a new family of probability density functions using the given probability density function has been proposed and some of its important theoretical properties involving quantiles and failure rate have been analyzed. As an offshoot of this new family of probability density functions, a two-component mixture density that behaves differently at the tail-ends viz., the two densities tend to be negligible to the left / right of two designated tail-end values, respectively, is proposed. The applications for this version of mixture model are understood to arise in car industries.

The organization of this paper is as follows. Section 2 discusses new models and associated theoretical results. Section 3 contains simulation results, while Section 4 presents conclusions.

2. General Model

This section presents a new class of probability density functions and the same is given in Definition 1 below.

Definition 1: Let X be an absolutely continuous random variable having $f(x)$ as its probability density function and $F(x)$ as its cumulative density function. Then for every pair of real numbers s and t, $s < t$, we may define an induced random variable $X^*(s,t)$ having probability density function $f^*(x|(s,t))$ given by

$$f^*(x|(s,t)) = \begin{cases} \lambda f(x) & x < s \\ (\lambda + \delta) f(x) & s \le x < t \\ \delta f(x) & x \ge t \end{cases}$$

with $p \in (0,1)$, $\lambda = \dfrac{(1-p)}{D_1}$, $\delta = \dfrac{p}{D_1}$, and $D_1 = (1-p)F(t) + p(1-F(s))$.

The corresponding Cumulative Distribution Function (CDF) is given by

$$F^*(x|(s,t)) = \begin{cases} \lambda F(x) & x < s \\ (\lambda+\delta)F(x) - \delta F(s) & s \le x < t \\ \lambda F(t) - \delta F(s) + \delta F(x) & x \ge t. \end{cases}$$

Remark 1: It may be noted that the probability density function $f^*(x|(s,t))$ defined above in Definition 1 is discontinuous at s and t where s, t, and p are assumed to be fixed. However, further, it may be noted that the corresponding CDF $F^*(x|(s,t))$ is continuous at s and t.

Remark 2: It may be noted that $\{X^*(s,t), s < t\}$ is a family of random variables induced by the given absolutely continuous random variable X and it is such that it is deflated in both the tails and inflated in the middle.

Remark 3: A special case of interest is when s and t, $s < t$, are chosen such that $F(s) = \overline{F}(t)$, where $\overline{F}(t)$ is the survival function of the original absolutely continuous random variable X which gives rise to a new random variable $X^*(s,t)$. In this case, $\overline{F}(s) = F(t) = D_1$ and for any $0 < p < 1$,

$$f^*(x) = \begin{cases} \frac{(1-p)}{\overline{F}(s)} f(x) & x < s \\ \frac{1}{\overline{F}(s)} f(x) & s \le x < t \\ \frac{p}{\overline{F}(s)} f(x) & x \ge t. \end{cases}$$

Further to this, we may also note the following:
(a) For $p = 1$,

$$f^*(x) = \begin{cases} 0 & x < s \\ \frac{f(x)}{\overline{F}(s)} & x \ge s. \end{cases}$$

Note that $f^*(x)$ is a probability density function of a random variable $X^*(s,.)$ which is truncated to the left of s.
(b) For $p = 0$,

$$f^*(x) = \begin{cases} \frac{f(x)}{F(t)} & x < t \\ 0 & x \ge t. \end{cases}$$

Note that $f^*(x)$ is a probability density function of a random variable $X^*(.,t)$ which is truncated to the right of t.

Remark 4: It is easy to verify for selected values of s, t that

$$\lambda \le 1 \Leftrightarrow p \ge \frac{\overline{F}(t)}{\overline{F}(t)+\overline{F}(s)}$$

and

$$\delta \le 1 \Leftrightarrow p \le \frac{F(t)}{F(t)+F(s)}.$$

Here is an interesting result one that makes connection between percentile points of the original probability density function f and those of the new probability density function f^*. We state it in the form of properties of f^* in relation to f.

Let $F^*(x_q^*) = q$ and $F(x_q) = q$, and η_q be such that $F^*(x_q) = q + \eta_q$ where the sign of η_q is positive (negative) when f^* is positively skewed (negatively skewed). Then

Property A: For $x < s$, $x_q^* = F^{-1}\Big(\dfrac{qD_1}{(1-p)}\Big)$.

Property B: For $s \le x < t$, $x_q^* = F^{-1}\Big(q + D_1\eta_q\Big)$.

Property C: For $x \ge t$, $x_q^* = F^{-1}\Big(q + \dfrac{D_1}{p}\eta_q\Big)$.

We provide the proofs of these properties in the Appendix A.

A question regarding stochastic comparison between X^* and X is in order. The following result is geared towards that. The concept of a random variable being stochastically larger than another random variable and the concept of failure rate can be found in Barlow and Proschan (1975).

Theorem 1: The random variable X^* having f^* as its pdf is stochastically larger than the random variable X with pdf f if and only if $p \ge \frac{F(t)}{F(t)+F(s)}$.

Proof: Not to obscure the essential steps of reasoning, we will go through the following Lemmas.

Lemma 1: For $x < s$, the random variable X^* having f^* as its pdf is stochastically larger than the random variable X with pdf f if and only if $p \ge \frac{\overline{F}(t)}{\overline{F}(t)+\overline{F}(s)}$.

Proof: For $x < s$,

$$F^*(x) = \lambda F(x) < F(x) \Leftrightarrow \lambda = \frac{(1-p)}{D_1} \le 1 \Leftrightarrow p \ge \frac{\overline{F}(t)}{\overline{F}(s)+\overline{F}(t)}.$$

Lemma 2: For $s \le x < t$, the random variable X^* having f^* as its pdf is stochastically larger than the random variable X with pdf f if and only if $p \ge \frac{F(x)\overline{F}(t)}{\overline{F}(x)F(s)+F(x)\overline{F}(t)}$.

Proof: For $s \le x < t$,

$$F^*(x) = (\lambda+\delta)F(x) - \delta F(s) < F(x) \Leftrightarrow \frac{F(x)}{D_1} - F(x) < \delta F(s) \quad as \quad \lambda + \delta = \frac{1}{D_1}$$

$$\Leftrightarrow \frac{(1-D_1)}{D_1}F(x) < \frac{p}{D_1}F(s) \Leftrightarrow p \ge \frac{F(x)\overline{F}(t)}{\overline{F}(x)F(s)+F(x)\overline{F}(t)}.$$

Lemma 3: For $x > t$, the random variable X^* having f^* as its pdf is stochastically larger than the random variable X with pdf f if and only if $p \geq \frac{F(t)}{F(t)+F(s)}$.
Proof: For $x > t$,

$$F^*(x) = \lambda F(t) + \delta F(x) - \delta F(s) \leq F(x) \Leftrightarrow \frac{(1-p)}{D_1}F(t) + \frac{p}{D_1}F(x) - \frac{p}{D_1}F(s) \leq F(x)$$

$$\Leftrightarrow p \geq \frac{F(t)\overline{F}(x)}{[F(t)\overline{F}(x) + F(s)\overline{F}(x)]} = \frac{F(t)}{F(t)+F(s)}.$$

Proof of Theorem 1: The property of F^* being stochastically larger than F implies that simultaneously last inequalities involving p in the proofs of Lemma 1, Lemma 2, and Lemma 3 have to necessarily hold good. This amounts to saying that

$$p \geq \max\Big\{\frac{\overline{F}(t)}{\overline{F}(t)+\overline{F}(s)}, \sup_{x\in[s,t)} \frac{F(x)\overline{F}(t)}{\overline{F}(x)F(s)+F(x)\overline{F}(t)}, \frac{F(t)}{F(t)+F(s)}\Big\}.$$

We establish that this is equivalent to the condition stipulated in the statement of this Theorem.

To this end, we make two claims and prove them.
Claim 1:

$$\frac{\overline{F}(t)}{\overline{F}(t)+\overline{F}(s)} \leq \frac{F(t)}{F(t)+F(s)}$$

$$\Leftrightarrow \overline{F}(t)(F(t)+F(s)) \leq F(t)(\overline{F}(s)+\overline{F}(t))$$

$$\Leftrightarrow \overline{F}(t)F(t) + \overline{F}(t)F(s) \leq \overline{F}(s)F(t) + \overline{F}(t)F(t)$$

$$\Leftrightarrow F(s) \leq F(t) \quad \textit{always holds as } s < t.$$

Thus, Claim 1 proved.
Claim 2:

$$\sup_{x\in[s,t)} \frac{F(x)\overline{F}(t)}{\overline{F}(x)F(s)+\overline{F}(t)F(x)} \leq \frac{F(t)}{F(t)+F(s)}.$$

Consider

$$\frac{F(x)\overline{F}(t)}{\overline{F}(x)F(s)+\overline{F}(t)F(x)} \leq \frac{F(t)}{F(t)+F(s)}$$

$$\Leftrightarrow F(x)\overline{F}(t)F(t) + F(x)\overline{F}(t)F(s) \leq F(t)\overline{F}(x)F(s) + F(t)\overline{F}(t)F(x)$$

$$\Leftrightarrow F(x)\overline{F}(t) \leq F(t)\overline{F}(x)$$

$$\Leftrightarrow F(x) \leq F(t) \quad \textit{which} \quad \textit{is valid for } x \in [s,t).$$

Thus, Claim 2 is proved and the proof is complete.

The following Theorem attempts to make connection between the failure rate of f^* and that of f.

Theorem 2: The probability density function f^* corresponding to the random variable X^* has increasing (decreasing) failure rate η^* over the set C iff for $x \in C$

$$\max\left[(\eta(x) - \frac{d(\ln \eta(x))}{dx}), 0\right] < (>) \min\left[\frac{\delta f(x)}{[1 - \lambda F(t) - \delta F(x) + \delta F(s)]}, \frac{\delta f(x)}{[1 - \lambda F(x)]}\right]$$

where $\eta(x)$ is the failure rate of the random variable X with f as its probability density function and the set C is such that $C \subseteq \mathcal{R}^+$, and, further, for $x \in C$ $x \neq s, t$.

Proof: The proof of this Theorem is rather long and a bit complicated. We provide details of this proof in the Appendix B.

It is interesting to note that the family of probability distribution functions given in Definition 1 results from the following tweaked version of a mixture of two probability distribution functions having an appealing feature that this mixture density behaves differently at the tail-ends *viz.*, the two densities tend to be negligible to the left / right of two designated tail-end values, respectively.

Definition 2: Let X_i be an absolutely continuous random variable with $f_i(x)$ as its probability density function and $F_i(x)$ as its cumulative density function for $i = 1, 2$. For every $p \in (0, 1)$ and every pair of real numbers s, t $(s < t)$,

$$h_2(u|(s,t)) = \begin{cases} \frac{(1-p)}{D_2} f_1(u) & u < s \\ \frac{1}{D_2}((1-p)f_1(u) + pf_2(u)) & s \leq u < t \\ \frac{p}{D_2} f_2(u) & u \geq t \end{cases}$$

defines a probability density function of a random variable $U(s, t)$, say, for fixed values of s and t. Here $D_2 = (1 - p)F_1(t) + p\overline{F}_2(s)$.

The corresponding Cumulative Distribution Function (CDF) is given by

$$H_2(u|(s,t)) = \begin{cases} \frac{(1-p)}{D_2} F_1(u) & u < s \\ \frac{1}{D_2}((1-p)F_1(u) + p(F_2(u) - F_2(s)) & s \leq u < t \\ \frac{1}{D_2}(1-p)F_1(t) + p(F_2(u) - F_2(s)) & u \geq t. \end{cases}$$

Remark 5: It may be noted that Definition 2 reduces to Definition 1 if X_1 and X_2 are identically distributed random variables.

Remark 6: The applications for this version of mixture model are understood to arise in car industries.

Conclusions: In this paper, a new family of probability density functions from a given probability density function is generated, and the relationships of the former to the latter in terms quantiles and failure rate are studied. This family of probability density function results from the tweaked version of a mixture of two probability density functions.

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APPENDIX A

Proof of Property A: For $x < s$, $F^*(x) = \lambda F(x)$.
Let x_q^* be such that $F^*(x_q^*) = q$ and x_q be such that $F(x_q) = q$.
Thus, we have, $F^*(x_q^*) = \lambda F(x_q^*) = q$.
This implies that $F(x_q^*) = \frac{q}{\lambda}$ with $\lambda = \frac{(1-p)}{D_1}$ and $D_1 = (1-p)F(t) + p\overline{F}(s)$.
i.e. $x_q^* = F^{-1}(\frac{q}{\lambda}) = F^{-1}\left(\dfrac{qD_1}{(1-p)}\right)$.
Proof of Property B: For $s \leq x < t$,

$$\begin{aligned} F^*(x) &= \lambda F(x) + \delta F(x) - \delta F(s) \\ &= (\lambda + \delta)F(x) - \delta F(s) \\ &= \frac{1}{D_1}F(x) - \delta F(s). \end{aligned}$$

With x_q^* and x_q as defined above, we have

$$\begin{aligned} F^*(x_q^*) &= \frac{1}{D_1}F(x_q^*) - \delta F(s) \\ &= q \end{aligned} \tag{1}$$

$$\begin{aligned} F^*(x_q) &= \frac{1}{D}F(x_q) - \delta F(s) \\ &= \frac{1}{D_1}q - \delta F(s) \\ &= q + |\eta_q| \end{aligned} \tag{2}$$

where $|\eta_q|$ is such that (i) it will be positive if F^* is positively skewed and (ii) it will be negative if F^* is negatively skewed. From (1) and (2) we have

$$\begin{aligned} |F^*(x_q^*) - F^*(x_q)| &= \Big|\frac{1}{D_1}F(x_q^*) - \delta F(s) - \frac{1}{D_1}F(x_q) - \delta F(s)\Big| \\ &= \Big|\frac{1}{D_1}F(x_q^*) - \frac{1}{D_1}F(x_q)\Big| \\ &= |\eta_q|. \end{aligned}$$

This implies that $\frac{1}{D_1}F(x_q^*) - \frac{1}{D_1}F(x_q) = \eta_q$ i.e. $\frac{1}{D_1}F(x_q^*) = \frac{1}{D_1}F(x_q) + \eta_q$ This, in turn, implies that $x_q^* = F^{-1}\Big(q + D_1\eta_q\Big)$.

Proof of Property C: For $x \geq t$, $F^*(x) = \lambda F(t) - \delta F(s) + \delta F(x)$. Thus, for x_q^* and x_q as defined above, we have

$$\begin{aligned} F^*(x_q^*) &= \lambda F(t) - \delta F(s) + \delta F(x_q^*) \\ &= q \end{aligned} \tag{3}$$

$$\begin{aligned} F^*(x_q) &= \lambda F(t) - \delta F(s) + \delta F(x_q) \\ &= q + |\eta_q|. \end{aligned} \tag{4}$$

From (3) and (4) we have

$$\begin{aligned} |F^*(x_q^*) - F^*(x_q)| &= |\delta F(x_q^*) - \delta F(x_q)| \\ &= |\delta F(x_q^*) - \delta q| \\ &= |\eta_q|. \end{aligned}$$

This yields $x_q^* = F^{-1}\Big(q + \dfrac{D_1}{p}\eta_q\Big)$.

APPENDIX B

From the expressions of $f^*(x)$ and $F^*(x)$, it follows that the corresponding failure rate η^* is given by

$$\eta^*(x) = \begin{cases} \dfrac{\lambda\overline{F}(x)\eta(x)}{1 - \lambda F(x)} & x < s \\ \dfrac{(\lambda+\delta)\overline{F}(x)\eta(x)}{1 - (\lambda+\delta)F(x) + \delta F(s)} & s \leq x < t \\ \dfrac{\delta\overline{F}(x)\eta(x)}{1 - \lambda F(t) - \delta F(x) + \delta F(s)} & x \geq t \end{cases}$$

where $\eta(x)$ denotes the hazard rate of $f(x)$, while $\eta^*(x)$ denotes the hazard rate of $f^*(x)$.

First, let $x < s$. Then $\eta^*(x)$ is increasing in x if and only if its derivative $\frac{d\eta^*(x)}{dx}$ is greater than 0. Note that

$$\begin{aligned}\frac{d\eta^*(x)}{dx} &= \frac{d[\frac{\lambda\overline{F}(x)(\eta(x))}{1-\lambda F(x)}]}{dx}\\ &= \frac{(1-\lambda F(x))(-\lambda f(x)\eta(x)+\lambda\overline{F}(x)\frac{d\eta(x)}{dx})}{(1-\lambda F(x))^2}+\frac{(\lambda)^2\overline{F}(x)f(x)\eta(x)}{(1-\lambda F(x))^2}.\end{aligned}$$

Thus $\frac{d\eta^*(x)}{dx} > 0$ iff $(1-\lambda F(x))(-\lambda f(x)\eta(x)+\lambda\overline{F}(x)\frac{d\eta(x)}{dx})+(\lambda)^2\overline{F}(x)f(x)\eta(x) > 0$
i.e., $\frac{d\eta^*(x)}{dx} > 0$ iff $(\lambda)^2\overline{F}(x)f(x)\eta(x) > \lambda\overline{F}(x)[1-\lambda F(x)][(\eta(x))^2-\frac{d\eta(x)}{dx}]$
i.e., $\frac{d\eta^*(x)}{dx} > 0$ iff $(\lambda)f(x)\eta(x) > [1-\lambda F(x)][(\eta(x))^2-\frac{d\eta(x)}{dx}]$.
Next, whenever $s \le x < t$, we have

$$\eta^*(x) = \frac{(\lambda+\delta)\overline{F}(x)\eta(x)}{1-(\lambda+\delta)F(x)+\delta F(s)}.$$

Then $\eta^*(x)$ is increasing in x iff its derivative $\frac{d\eta^*(x)}{dx}$ is greater than 0.
Note that

$$\begin{aligned}\frac{d\eta^*(x)}{dx} &= \frac{d[\frac{(\lambda+\delta)\overline{F}(x)\eta(x)}{1-(\lambda+\delta)F(x)+\delta F(s)}]}{dx}\\ &= \frac{[1-(\lambda+\delta)F(x)+\delta F(s)][-(\lambda+\delta)f(x)\eta(x)+(\lambda+\delta)\overline{F}(x)\frac{d\eta(x)}{dx}]}{[1-(\lambda+\delta)F(x)+\delta F(s)]^2}\\ &\quad+\frac{(\lambda+\delta)^2\overline{F}(x)f(x)\eta(x)}{[1-(\lambda+\delta)F(x)+\delta F(s)]^2}.\end{aligned}$$

Thus, $\frac{d\eta^*(x)}{dx} > 0$ iff

$$[1-(\lambda+\delta)F(x)+\delta F(s)][-(\lambda+\delta)f(x)\eta(x)+(\lambda+\delta)\overline{F}(x)\frac{d\eta(x)}{dx}]+(\lambda+\delta)^2\overline{F}(x)f(x)\eta(x) > 0$$

$\frac{d\eta^*(x)}{dx} > 0$ iff $(\lambda+\delta)^2\overline{F}(x)f(x)\eta(x) > (\lambda+\delta)\overline{F}(x)[1-(\lambda+\delta)F(x)+\delta F(s)][(\eta(x))^2-\frac{d\eta(x)}{dx}]$
$\frac{d(\eta^*(x))}{dx} > 0$ iff $(\lambda+\delta)f(x)\eta(x) > [1-(\lambda+\delta)F(x)+\delta F(s)][(\eta(x))^2-\frac{d\eta(x)}{dx}]$.
Lastly, for $x \ge t$ we have

$$\eta^*(x) = \frac{\delta\overline{F}(x)\eta(x)}{1-\lambda F(t)-\delta F(x)+\delta F(s)}.$$

Then $\eta^*(x)$ is increasing in x iff its derivative $\frac{d\eta^*(x)}{dx}$ is greater than 0.
Here

$$\begin{aligned}\frac{d\eta^*(x)}{dx} &= \frac{d[\frac{\delta\overline{F}(x)\eta(x)}{1-\lambda F(t)-\delta F(x)+\delta F(s)}]}{dx}\\ &= \frac{[1-\lambda F(t)-\delta F(x)+\delta F(s)][-\delta f(x)\eta(x)+\delta\overline{F}(x)\frac{d\eta(x)}{dx}]}{[1-\lambda F(t)-\delta F(x)+\delta F(s)]^2}\\ &\quad+\frac{(\delta)^2 f(x)\overline{F}(x)\eta(x)}{[1-\lambda F(t)-\delta F(x)+\delta F(s)]^2}\end{aligned}$$

$\frac{d\eta^*(x)}{dx} > 0$ iff $[1-\lambda F(t)-\delta F(x)+\delta F(s)][-\delta f(x)\eta(x)+\delta\overline{F}(x)\frac{d(\eta(x))}{dx}]+(\delta)^2 f(x)\overline{F}(x)\eta(x) > 0$ i.e., $\frac{d\eta^*(x)}{dx} > 0$ iff $(\delta)^2 f(x)\overline{F}(x)\eta(x) > \delta\overline{F}(x)[1-\lambda F(t)-\delta F(x)+\delta F(s)][(\eta(x))^2-\frac{d(\eta(x))}{dx}]$ i.e., $\frac{d\eta^*(x)}{dx} > 0$ iff $(\delta)f(x)\eta(x) > [1-\lambda F(t)-\delta F(x)+\delta F(s)][(\eta(x))^2-\frac{d\eta(x)}{dx}]$.
Thus $\eta(x)$ is increasing in x in the respective intervals iff

$$\begin{aligned}(\lambda f(x)\eta(x) &> [1-\lambda F(x)][(\eta(x))^2-\frac{d\eta(x)}{dx}]\\ (\lambda+\delta)f(x)\eta(x) &> [1-(\lambda+\delta)F(x)+\delta F(s)][(\eta(x))^2-\frac{d\eta(x)}{dx}]\\ (\delta)f(x)\eta(x) &> [1-\lambda F(t)-\delta F(x)+\delta F(s)][(\eta(x))^2-\frac{d\eta(x)}{dx}].\end{aligned}$$

Note that in the above, if the first and the third inequalities hold then the second inequality automatically holds.

In view of this, $\eta(x)$ is increasing (decreasing) in respective intervals iff

$$\max[((\eta(x))^2-\frac{d\eta(x)}{dx}),0] < (>) \min[\frac{\delta f(x)\eta(x)}{1-\lambda F(t)-\delta F(x)+\delta F(s)},\frac{\lambda f(x)\eta(x)}{1-\lambda F(x)}]$$

that is,

$$\max[(\eta(x)-(\frac{d\ln\eta(x)}{dx})),0] < (>) \min[\frac{\delta f(x)}{1-\lambda F(t)-\delta F(x)+\delta F(s)},\frac{\lambda f(x)}{1-\lambda F(x)}].$$

The proof is complete.