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Stochastic Differential Equation Models and their Applications to Agriculture: An Overview

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Abstract

Nonlinear growth models play a very important role in almost all disciplines. The current status of applying them to data is to first assume an additive error in the model and then employing nonlinear estimation procedures. In this article, limitations of this methodology are highlighted. It is advocated that, for a more realistic modeling, a stochastic term should be added to the differential equation form of a growth model, thereby yielding a stochastic differential equation (SDE) growth model. A brief description is provided of the two types of stochastic calculi due respectively to Stratonovich and Ito. The methodologies for application of several univariate growth models, *viz*. Gompertz, Richards, von Bertalanffy and generalized logistic SDE models are described. Some work dealing with bivariate SDE growth models is also discussed. Finally, some future research problems in the area are outlined.

Key words: Bivariate growth models; Gompertz model; Richards model; Stochastic calculi; Stochastic differential equation; Von bertalanffy model.

1 Introduction

It is by now well-recognized that any type of statistical inquiry, in which principles from some body of knowledge are employed, would lead to a Nonlinear statistical model Seber and Wild (2003). Nonlinear growth models, such as Gompertz, Logistic, Richards and von Bertalanffy models are widely employed for describing growth of various species of plants, animals, etc. A heartening aspect of these models is that they are mechanistic in nature and so the underlying parameters have specific biological interpretations. Growth models are generally expressed in terms of nonlinear differential equations. An attractive feature of these models is that they can be converted to linear forms by means of some transformations, like Logarithmic and Reciprocal. Consequently, exact solutions of the underlying differential equations can be obtained, which are nonlinear in parameters. The usual practice for applying them to data is to add an additive term, with suitable assumptions, on the right hand side of the deterministic solutions and applying

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Nonlinear estimation procedures, such as Levenberg- Marquardt procedure Seber and Wild (2003) for estimation of parameters. A large number of articles dealing with this methodology have appeared in reputed research journals during the last two decades or so (See, *e.g.* Ross et al., 2010; Matis et al., 2011).

Although the above methodology has served many useful purposes in the past, it suffers from two main limitations. The first one is that it is applicable only when the data are available at equal time-intervals. The philosophy behind above models is that the growth rate is fast in the initial phase and then it slows down in the next phase, thereby leading to a sigmoid type of curve. Therefore, quite often, researchers record growth data in the initial phase at quick intervals and in the subsequent phase at wide intervals. This leads to generation of data at unequal intervals. Further, for culture fisheries, getting age-length data at equal intervals may not be difficult but for capture fisheries, these types of data are invariably at unequal intervals. Also, collection of growth data over time involves constraints of time, personnel, and budgets, etc. that do not always satisfy the requirement of obtaining data at equal intervals. Undoubtedly, the data that do exist in studies with missing data or data at unequal time-intervals are potentially informative, and precluding such data from analysis could affect conclusions adversely Dennis and Ponciano (2014). The other limitation is that, by simply adding an error term, a nonlinear statistical model is not capable of describing the underlying fluctuations of the system satisfactorily, particularly for longitudinal data. It may be highlighted that both the above limitations can be successfully tackled by employing the more general approach of 'Stochastic Differential Equation (SDE)' Oksendal (2003). These are generally obtained by adding a stochastic term on the right hand side of the differential equation form of deterministic formulation of a growth model. It may be noted that, in a physical situation, random environmental fluctuations due to variations in parameters, such as birth and death rates generally occur with great rapidity as compared to the time-scale of population growth. Therefore, the stochastic term is generally assumed to be a Gaussian white noise stochastic process. A heartening aspect of this prescription is that the resultant process becomes Markovian. However, the price to be paid is that the sample paths are very irregular and do not admit of derivatives in the conventional sense. To handle this situation, two types of stochastic calculi due respectively to, Stratonovich and Itô, have been developed in the literature. In the former, usual rules of calculus continue to apply whereas in the latter, these are suitably modified. However, for the present article, both these calculi yield identical results as we shall deal only models with additive noise, which is independent of state variable. Cohen and Elliott (2015).

2 The Stratonovich and Ito Calculi

Consider a deterministic model described by the differential equation:

$$dX/dt = f(X) + g(X)Z, \tag{1}$$

where f and g are deterministic functions of $X(t)$ and Z reflects the state of the environment. To incorporate environmental variation, Z is replaced by some stochastic process $Z(t)$. If $Z(t)$ is a "well-behaved" stochastic process in the sense that its sample paths are smooth functions, then the solution $X(t)$ becomes non-Markovian. Considering the fact that compared to the theory of Markov processes, the theory of non-Markov processes has not yet been developed to such a point that it can be applied to concrete physical or biological problems, it would be very desirable for the mathematical treatment if the solution $X(t)$ is a Markov process. For that to be so, the sample paths of $Z(t)$ should be deprived of memory, i.e. the random variables $Z(t_1)$ and $Z(t_2)$ should be uncorrelated, for all $t_2 \neq t_1$, a situation clearly in contrast with the smoothness property of sample paths. It is well-known that $Z(t)$ is Markov only when $Z(t)$ is a Gaussian white noise process. The price that is paid for this convenience is that the sample paths of $X(t)$ are very irregular and do not admit of derivatives in the conventional sense. To overcome this difficulty, two new calculi have been proposed. These are now discussed briefly.

Consider the stochastic differential equation (SDE):

$$dX/dt= f(X) + g(X)Z(t), \tag{2}$$

where $f(X)$ and $g(X)$ are deterministic functions of $X(t)$, and $Z(t)$ is a Gaussian white noise process. In order to avoid the mathematical pathology associated with white noise, eq. (2) is expressed in terms of the Weiner process $W(t)$ as

$$dX = f(X)dt + g(X)dW(t). \tag{3}$$

This SDE is actually shorthand for the stochastic integral equation:

$$X(t) = X(t_o) + \int_{t_o}^{t} f\left(X(s)\right)ds + \int_{t_o}^{t} g\left(X(s)\right)dW(s), \tag{4}$$

The ambiguity arises from the fact that the Weiner process is of unbounded variation, i.e. almost every realization has infinite length on a finite time-interval. Therefore, the last integral of (4), viz.

$$\int_{t_o}^{t} g\left(X(s)\right)dW(s) \tag{5}$$

cannot, in general, be interpreted as an ordinary Riemann-Stieltjes integral. In other words, the limit of the approximating sums

$$S_n = \Sigma\, g\big(X(\tau_i)\big)\{W(t_{i+1}) - W(t_i)\} \tag{6}$$

is dependent on the particular choice of the evaluation points. If $t_0<t_1<\ldots<t_n = t$ is a partition of the interval of integration, then the stochastic integral (5) is defined as the limit of sums of the form

$$\sum_{i=o}^{n-1} g[\lambda X(t_i) + (1-\lambda)X(t_{i+1})]\,\{W(t_{i+1}) - W(t_i)\}. \tag{7}$$

Thus, there are many possible definitions of (5) each of which gives rise to a different result. Only two of these definitions, those attributed to Ito (obtained on taking λ=1) and Stratonovich (obtained on taking λ=0.5) yield solutions which are likely to be useful as models. Only the Ito's definition has the appealing mathematical property that the process $X(t)$ is a martingale, i.e. for $t \geq s$, with probability one

$$E[X(t)|X(u), u \leq s] = X(s).$$

Also $E[X(t)] = 0$. However, it is disconcerting that ordinary rules of calculus do not apply for Ito calculus. For example

$$\int_{t_o}^{t} W(s)\, dW(s) = \frac{1}{2}\{W^2(t) - W^2(t_o)\} - \frac{1}{2}(t - t_o),$$

and

$$d\big(W^n(t)\big) = nW^{n-1}(t)dW(t) + \frac{1}{2}n(n-1)W^{n-2}(t)dt.$$

The Stratonovich definition has the advantage that here the ordinary rules of calculus apply. Mathematically, Ito calculus is more fundamental and more general than the Stratonovich calculus. It may be mentioned that the two prescriptions are equivalent in the sense that it is possible to pass from the results obtained under one interpretation to the results for the other interpretation via a transformation formula. A problem arises as to which of these calculi should one use in a given physical or biological situation. If the SDE is obtained as the white noise limit of a real noise problem, then the Stratonovich interpretation is appropriate. Further, if the process of interest and the noise it is subject to are inherently continuous, then the Stratonovich calculus is better; if the process is discrete with uncorrelated noise, then the Ito calculus is appropriate. Unfortunately, neither of these results addresses the situation which is most common in population biology, namely the case in which "real model" is a difference equation subject to autocorrelated noise. A good description of various aspects of stochastic calculus is available in Oksendal (2003) and Cohen and Elliott (2015).

3 Gompertz SDE growth model

The differential equation form of Gompertz growth model Seber and Wild (2003) is given by

$$dy_t/dt = ry_t \log_e(K/y_t). \tag{8}$$

The transformed Gompertz statistical growth model can be written as Prajneshu and Ghosh, (2017):

$$\log_e y_t = [\log_e K + \log_e(y_0/K)exp(-rt)] + \varepsilon_t^*. \tag{9}$$

Following along similar lines as Filipe et al. (2013), analogous Gompertz SDE model with constant diffusion coefficient is given by

$$dZ_t = r(\alpha - Z_t)dt + \sigma dW_t, \tag{10}$$

where $\alpha = \log_e K$, $Z_t = \log_e y_t$, and W_t is a Wiener process with variance parameter unity. After integration of both sides and applying Ito calculus, solution Z_t of the SDE model given $F_{t_k} = \left\{Z_{t_j} : j \le k\right\}$ is

$$Z_t = \alpha + \left(Z_{t_k} - \alpha\right)e^{-r(t-t_k)} + \sigma exp(-rt)\int_{t_k}^{t} exp(rs)\, dW_s. \tag{11}$$

Note that solution of the above Gompertz SDE model given by eq. (11) is a Gaussian process with conditional mean and variance given respectively by

$$\mu_{Z:t|t_k} = E\{Z_t|F_{t_k}\} = \alpha + \left(Z_{t_k} - \alpha\right)e^{-r(t-t_k)},\ \sigma^2_{Z:t|t_k} = V\{y_t|F_{t_k}\} = \frac{\sigma^2\left(1-e^{-2r(t-t_k)}\right)}{2r}. \tag{12}$$

The transition probability density function of the process $\{Z_t, t \geq 0\}$ is asymptotically stationary with mean and variance given respectively by α and $\sigma^2/(2r)$. however, it may be noted that the mean-value function of $\{Z_t\}$, i.e. $E[Z_t] = \alpha + \left(Z_{t_0} - \alpha\right)e^{-r(t-t_k)}$ is a sigmoid curve. It may be pointed out that the solution of Gompertz SDE model given by eq. (11) is capable to model growth under dependent error processes $\left\{\sigma exp(-rt) \int_{t_0}^{t} exp(rs)\, dW_s, t \geq 0\right\}$. Finally, predicted value of y_t may be obtained by evaluating conditional mean of y_t given past values of the process $\left\{y_{t_j}: j \leq k\right\}$, and the same is given by

$$\mu_{y:t|t_k} = exp\left(\mu_{Z:t|t_k} + 0.5\sigma^2_{Z:t|t_k}\right), \tag{13}$$

The parameters of Gompertz SDE model in eq. (11) may be estimated by the Method of maximum likelihood, which is carried out by maximizing joint likelihood of the transformed process$\{Z_t\}$. To this end, joint likelihood is expressed in terms of product of conditional likelihoods at time-epoch t given F_{t_k}, which are Gaussian with conditional means and variances given by eq. (12). Relevant computer code in SAS software package for fitting the model to data is available in Prajneshu and Ghosh (2017).

4 Gompertz SDE growth model with time-dependent diffusion

One limitation of the model discussed in Section 3 is that the underlying diffusion term is assumed to be time-independent and, therefore, its solution is a homogenous Markov chain. In other words, it implies that the conditional probability distribution evolves only with respect to time elapsed between past and present but not on the time over which the system is recorded. However, it is quite reasonable that the growth rate of living organisms would be changing as age increases. Therefore, for a more realistic modelling, diffusion term should be time-dependent Ghosh and Prajneshu (2017a).

Ghosh and Prajneshu (2017a) recently considered time-varying diffusion coefficient in eq. (10) as

$$dZ_t = r(\alpha - Z_t)dt + \sigma V_t dW_t. \tag{14}$$

Taking $V_t = a + bt$ as a first-order approximation, solution of eq.(14) is $Z_t = \alpha + (y_0 - \alpha)e^{-r(t-t_0)} + \sigma e^{-rt} \int_{t_0}^{t} exp(rs) V_s dW_s$. Thus, the conditional distribution of Z_t given F_{t_k} is normal with mean $\alpha + \left(Z_{t_k} - \alpha\right)e^{-r(t-t_k)}$ and variance

$$\sigma^2_{L:Z} = \sigma^2\left[a^2\left(1 - e^{-2r(t-t_k)}\right) + 2b\left\{\left(t - t_k e^{-2r(t-t_k)}\right) - (2r)^{-1}\left(1 - e^{-2r(t-t_k)}\right)\right\} + b^2\left\{\left(t^2 - t_k^2\, e^{-2r(t-t_k)}\right) - q\right\}\right]/(2r), \tag{15}$$

where

$$q = r^{-1}\left(\left[t - t_k e^{-2r(t-t_k)}\right] - (2r)^{-1}\left[1 - e^{-2r(t-t_k)}\right]\right).$$

It may be noted that the transition probability distribution of the process $\{Z_t, t \geq 0\}$ is non-homogeneous and the process is asymptotically non-stationary with variance given by $\sigma_t^2 = \sigma^2\left\{a^2 + b\left(t - \frac{1}{2r}\right)\left(2 - \frac{b}{r}\right) + b^2t^2\right\}/(2r)$, while asymptotic mean is α. Since the transformation $g(.)$ is monotonically non-decreasing, therefore, approximate conditional mean-value function of untransformed process $\{Y_t\}$, viz. $E\left(y_t|\mathcal{F}_{y:t_k}\right) = g^{-1}\left(E\left(Z_t|\mathcal{F}_{t_k}\right)\right)$ is also monotonically non-decreasing and tends to $g^{-1}(\alpha)$ as $t \to \infty$. For dynamic computation of predicted values of $\{Y_t\}$, above approach may be used. Therefore, using inverse transformation, viz. $g^{-1}(Z_t) = exp(Z_t)$, evaluation of conditional expectation of y_t reduces to computation of moment generating function of Gaussian random variable Z_t. Therefore, predicted value of y_t is given by conditional mean of y_t as

$$\mu_{y:t|t_k} = exp\left(\mu_{Z:t|t_k} + 0.5\sigma^2_{Z:t|t_k}\right), \tag{16}$$

and prediction error variance, viz. conditional variance of y_t, is given by

$$\sigma^2_{L,y:t|t_k} = \left\{exp\left(2\left[\mu_{Z:t|t_k} + \sigma^2_{L,Z:t|t_k}\right]\right)\right\}\left\{1 - exp\left(-\sigma^2_{L,Z:t|t_k}\right)\right\}. \tag{17}$$

To carry out the above analysis to a dataset, salient codes in SAS software package, Ver. 9.4 are available in Ghosh and Prajneshu (2017a).

5 Richards SDE growth model

Richards four-parameter nonlinear growth model, which is a generalization of the well-known logistic and Gompertz models, is a very versatile model for describing many growth processes. It may be pointed out that the corresponding SDE growth model becomes very cumbersome. Therefore, the methodology is developed for a particular value of m, say $m = -1/2$. as discussed by Ghosh and Prajneshu (2017b), the Richards SDE growth model can be written as

$$dy_t/dt = \left\{-2ry_t\left(K^{-1/2} - y_t^{-1/2}\right)/K^{-1/2}\right\}\left\{1 + \left\{\eta_t/r\left(K^{1/2} - y_t^{1/2}\right)\right\}\right\}. \tag{18}$$

Using the variance stabilization transformation $Z_t = g(y_t) = y_t^{1/2}$ and chain rule of differentiation, linear SDE in transformed variable Z_t may be obtained. Hence, advantage of the nonlinear SDE model given in eq. (18) is that it is capable of yielding closed form solution by getting solution of SDE in transformed variable. Thus, after necessary simplification, eq. (18) is reduced to Linearized Richards SDE (LRSDE) model, given by

$$dZ_t = r(\alpha - Z_t)dt + \sigma dW_t, \tag{19}$$

where $\alpha = K^{1/2}$ and W_t is the Brownian or Wiener process with variance parameter unity. Given $\mathcal{F}_{t_k} = \{Z_t : t \leq t_k\}$, solution of the LRSDE model, obtained by using Ito calculus, is given by Filipe et al. (2013)

$$Z_t = K^{1/2} + \left(Z_{t_k} - K^{1/2}\right)e^{-r(t-t_k)} + \sigma exp(-rt)\int_{t_k}^{t} exp(rs)\, dW_s\,. \tag{20}$$

Note that solution of the above LRSDE model is Markovian and follows Gaussian process with conditional mean and variance given by $\mu_{Z:t|t_k} = E\{Z_t|F_{t_k}\} = \alpha + (Z_{t_k} - \alpha)e^{-r(t-t_k)}$ and $\sigma^2_{Z:t|t_k} = V\{y_t|F_{t_k}\} = \sigma^2(1 - e^{-2r(t-t_k)})/(2r)$ respectively. The mean-value function of $\{Z_t\}$, i.e. $E[Z_t] = \alpha + (Z_{t_0} - \alpha)e^{-r(t-t_k)}$ is a sigmoid curve, whereas the transition probability is homogeneous and homoscedastic. It may be noted that variance-function of $\{Z_t\}$ depends on time, which allows the variance of $\{y_t\}$ to change over time. The process is also asymptotically stationary with mean and variance given by α and $\sigma^2/(2r)$.

Ghosh and Prajneshu (2017b) also derived the optimal predictor for untransformed data along with prediction error variance. Relevant computer programs are included in the above article. Finally, as an illustration, pig growth data are considered.

6 Von Bertalanffy (VB) SDE Growth Model

The well-known VB growth model plays a very important role in estimating the age-length relationship in fisheries. The deterministic VB age-length growth model can be written

$$l_t = \alpha[1 - exp\{-\beta(t - t_0)\}] \tag{21}$$

where α, β, t_0 represent respectively the ultimate fish length, curvature parameter and initial time-epoch at which fish length is zero. As discussed by Prajneshu et al. (2017), the analogous VBSDE model can be written as

$$dexp(rt)l_t = r\alpha exp(rt)dt + \sigma exp(rt)dW_t, \quad l_{t_0} = 0. \tag{22}$$

where W_t is a Wiener process with variance parameter unity. Integrating both sides and applying Ito calculus, solution l_t of the VBSDE model, given $F_{t_k} = \{l_{t_j}: j \leq k\}$ is

$$l_t = \alpha + (l_{t_k} - \alpha)e^{-r(t-t_k)} + \sigma exp(-rt)\int_{t_k}^{t} exp(rs)\, dW_s. \tag{23}$$

Since the age-length data is observed in controlled environment and the length data is obtained from age at time-epoch $t = 0$, therefore it is of interest to estimate t_0 in addition to r, α, σ^2. Note that the processes $\{l_t: t \geq t_0\}$ is Markovian and stationary with conditional mean $\mu_{l:t|t_k}$ and variance $\sigma^2_{l:t|t_k}$ given by

$$\mu_{l:t|t_k} = E\{l_t|\, l_s: s \leq t_k\} = \alpha + (l_{t_k} - \alpha)e^{-r(t-t_k)}, \tag{24a}$$

$$\sigma^2_{l:t|t_k} = V\{l_t|\, l_s: s \leq t_k\} = \frac{\sigma^2(1-e^{-2r(t-t_k)})}{2r} \tag{24b}$$

Method of maximum likelihood is applied to obtain estimates of parameters. To this end, joint likelihood is expressed in terms of product of conditional likelihoods at time-epoch t given $\mathcal{F}_{t_k} = \{\, l_s: s \leq t_k\}$, which are Gaussian with conditional means and variances respectively given by eqs. (24a) and (24b). It may be highlighted that only the VBSDE model is capable of predicting future length at any time-epoch continuously. The optimal (exact) predictor of l_t given $\{\, l_s: s \leq t_k\}$ is given by $\mu_{l:t|t_k} = E\{l|\, l_s: s \leq t_k\} = \alpha + (l_{t_k} - \alpha)e^{-r(t-t_k)}$. One may also use naïve approach for prediction of l_t by considering the predicted value of $l_{t'}$ at some intermediate time-epoch t', where $t_k < t' < t$. Relevant computer code for fitting the model to data in SAS software package are available in Prajneshu et al. (2017).

7 Generalized Logistic SDE Growth Model

Rahman et al. (2009) considered the generalized logistic differential equation model given by

$$dN/dt = aN^{\beta} - bN^{\gamma}. \tag{25}$$

This equation was then perturbed by extrinsic Gaussian white noise through its growth coefficient a. The drift and diffusion parameters of the stochastic logistic model were estimated using Levenberg-Marquardt optimization method of nonlinear least squares. The solution of the deterministic model was approximated using Fourth Order Runge-Kutta method. As the SDE model does not have an exact analytical solution, numerical methods are required to solve them approximately. The usual approximation is the Euler-Maruyama and Milstein numerical schemes Picchini et al. (2006). Rahman et al. (2009) employed the latter scheme as it is more precise than the former. Superiority of the logistic SDE growth model over the corresponding deterministic model was shown for data pertaining to cell growth of Clostridium Acetobutylicum.

8 Bivariate SDE Growth Models

Gutierrez et al. (2008) proposed a bivariate stochastic Gompertz diffusion model as the solution for a system of two Ito SDE that are similar as regards the drift and diffusion coefficients to those considered in the univariate Gompertz diffusion model. Probabilistic characteristics of this model, such as bivariate transition density, bidimensional moment functions, conditioned trend functions and in particular, correlation function between each of the components of the model were established. Maximum likelihood estimation of the bidimensional drift and the diffusion matrix of the diffusion were carried out and a computational statistical methodology for this purpose based on discrete observations over time was proposed. Thus, a method for trend analysis was established. It was applied to the real case of two dependent variables, Gross Domestic Product (GDP) and CO_2 emission in Spain on the basis of annual observations of the variables over the period 1986–2003. The application is a new methodology in environmental and climate change studies, and provides an alternative to other approaches of a more econometric nature, or those corresponding to the methodology of secular trends in time-series.

Rupsys and Petrauskas (2010) presented a new method for describing the bivariate diameter and height distribution of trees growing in a pure, uneven-aged forest by using a SDE framework to derive a bivariate age-dependent probability density function of tree diameter and height when the tree diameter and height follow a bivariate stochastic Gompertz shape growth process. The bivariate stochastic Gompertz model is fit to diameter and height observations for pine trees in the Dubrava district of Lithuania. A considerable advantage of the bivariate stochastic Gompertz growth model is that the model parameters are easily interpretable. All results are implemented in the symbolic algebra system MAPLE.

9 Some Future Research Problems

These are outlined below:

(i) In Section 4 of the present article, the methodology is discussed for application of Gompertz model in random environment when diffusion term is assumed to be a linear function of time. However, methodology valid for more complicated time-dependent diffusion terms needs to be developed.

(ii) In Section 5 of the present article, the methodology is discussed for application of Richards model in random environment for one particular value of the parameter m. This work may be extended for any value of m.

(iii) In this article, the noise term is assumed to be 'white noise'. However, for a more realistic modeling, it should be 'coloured' (Behera and O'Rourke 2008). But then the process becomes non-Markovian, and development of efficient estimation procedures may be a challenging task.

(iv) All the models discussed so far assumed that that the rate of change of the variable depends only on their current values. However, in the real world, growth rate of a population often does not respond immediately to changes, but rather would do so after some time lag. Allen (2014) derived stochastic versions of several discrete delay and continuous delay differential equation models. This type of work may be extended to other growth models and efficient estimation procedures for fitting those models to data needs to be explored.

(v) For a nonlinear SD, Yamamura and Shoji (2008) developed multi-step ahead forecasts for discrete sampling through a local linear model for estimating the drift term nonparametrically using Gaussian kernel. Effort needs to be directed towards applying efficient technique of Wavelet analysis for estimation of drift term. It may be noted that the procedure of Yamamura and Shoji (2008) yields point estimates for multi-step ahead forecasts. However, it is widely recognized that this is meaningful only when the corresponding standard errors due to forecasts are also computed. Therefore, to this end, attempt should be made to estimate the theoretical conditional variance of the diffusion process.

10 Concluding Remarks

Nonlinear growth models have a long history of applications in several disciplines, such as Agriculture, medicine, and industry. An attractive feature of these models is they can be converted to linear forms by means of suitable transformations. Before 1963, only 'intrinsically nonlinear growth models' were used, as efficient nonlinear estimation procedure, viz. Levenberg-Marquardt algorithm was not discovered. It may be pointed out that this methodology was not mathematically correct and so has also led to erroneous conclusions. But there was no choice. Even when Levenberg-Marquardt algorithm was discovered in 1963, it was very difficult to apply and so researchers continue to apply the 'intrinsically nonlinear models' until statistical software packages, such as SAS and SPSS were developed and reached researchers sometime in 1980's.

So, during the last three decades or so, nonlinear growth models have been widely employed through nonlinear estimation procedures. These models also suffer from some limitations as described in the present article. So, there is a need to abandon these in favour of Stochastic differential equation growth models. Although a large number of research papers dealing with the theoretical aspects of Stochastic differential equations have been published, it is disconcerting to note that very few of them have so far been applied to real data (See, e.g. Prajneshu 1980 and Skiadas 2010). Thus, it is high time to reduce this wide gap and at the same time build a synergistic interaction between the theoretical and applied aspects in this area.

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