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Statistical Inference for Fréchet Distribution Based on Dual Generalized Order Statistics

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Abstract

This article discusses the idea of an ordered random variable and its basic structure. Under the umbrella of dual generalized order statistics, the problem of Bayesian estimation of Fréchet distribution with parameters α and λ is addressed. Both symmetric (squared error) and asymmetric (linear exponential and general entropy) loss functions are taken into account to enable flexibility in the outcomes. For the aim of estimation, two approximation methods (Lindley and Markov Chain Monte Carlo) have been employed and presented. Simulation tools have been used to elaborate the findings clearly.

Key words: Fréchet distribution; Dual generalized order statistics; Bayesian methods; Markov chain Monte Carlo.

AMS Subject Classifications: 62C10, 62F10, 62F15, 62G30

1. Introduction

While dealing with data analysis using statistical tools and techniques, extreme value theory is inevitable. Jenkinson (1955) described how Generalized Extreme Value (GEV) is the most preferred distribution in this regard. The cumulative density function is given by

$$F(y \mid \sigma, \mu, \xi) = \begin{cases} \exp\left(-[1+\xi(y-\mu)/\sigma]_{+}^{-1/y}\right\}, & \text{for } \xi \neq 0 \\ \exp(-\exp[-(y-\mu)/\sigma]], & \text{for } \xi = 0 \end{cases}$$

where $\sigma > 0, \mu, \xi \in \mathbb{R}$. The considered distribution in this manuscript is Fréchet which is a special cases of GEV distribution. Its name spawned from Maurice René Fréchet, a French mathematician, who developed this distribution in 1920 as a maximum value distribution. It is also known as the extreme value distribution of type II.

The probability density function (PDF), cumulative density function (CDF) and reliability function of the random variable y following Fréchet distribution are given as

$$f(\mathrm{y} \mid \lambda, \alpha) = \lambda\alpha y^{-(\alpha+1)} e^{-\lambda y^{-\alpha}}, \tag{1}$$

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$$F(\mathrm{y} \mid \lambda, \alpha) = e^{-\lambda y^{-\alpha}}, \tag{2}$$

$$R(\mathrm{t} \mid \lambda, \alpha) = 1 - e^{-\lambda t^{-\alpha}}. \tag{3}$$

where $y > 0$, $t > 0$, $\alpha > 0$ is shape parameter and $\lambda > 0$ is the scale parameter. Depending on the form of parameters, the PDF might be unimodal or declining, although the hazard function is always unimodal. This is the only CDF that can be established on non-negative real numbers and is also a limiting CDF for the maxima of random variables. For a range of engineering applications, this feature is crucial for simulating the issues associated with investigating the statistical behavior of material properties.

It was explained by Kotz and Nadarajah (2000) that how Fréchet distribution can be used in a variety of contexts, including accelerated life testing, natural disasters, horse racing, rainfall, grocery store lines, sea currents, wind speeds, track race records and so on. Harlow (2002) demonstrated that the Fréchet distribution is the best option for simulating the case where high values are crucial. The literature on Fréchet distribution is extensive. Maximum likelihood estimation has been performed by Calabria and Pulcini (1989), and the features of its estimator (MLE) have been studied. Maximum likelihood estimation was carried out by Ramos *et al.* (2017) in the presence of the cure fraction, and Loganathan and Uma (2017) compared the MLE, the LSE, the weighted LSE, and the method of moment estimation for the Fréchet distribution. In order statistics, the Fréchet distribution was investigated by Salman and AMER (2003), while generalised order statistics was researched by Maswadah (2003). Many scholars have also addressed the issue of Bayesian estimate for the Frechet distribution. For instance, Calabria and Pulcini (1994) and Kundu and Howlader (2010) have performed Bayesian estimation using Gamma or other informative or arbitrary priors. Fréchet distribution was examined using Jeffreys and reference priors in Abbas and Tang (2015).

After carefully searching the literature, we were unable to locate any articles addressing its application to order statistics or lower record data. Therefore, utilizing the setup of Dual Generalised Order Statistics (*dgos*), we have addressed the Bayesian estimation of the Fr'echet distribution. The manuscript is arranged as follows: Mathematical formulation of *dgos* is thoroughly discussed in Section 2. Also, in this section, Bayesian framework for estimation using different loss functions is given. Bayes estimators are obtained using the Lindley approximation, a method for approximation that is detailed in Section 3. Bayes estimators are obtained in Section 4 using Markov chain Monte Carlo approach. Simulation analysis for *dgos* submodels such as order statistics and lower record values is provided in Section 5 along with conclusions regarding the obtained results.

2. Formulation of Bayesian framework

Let us take independent and identically distributed sequence containing $X_1, X_2, \ldots$ random variables having absolutely continuous distribution function $F(\cdot)$ and the probability density function $f(\cdot)$. Let $n \in \mathbb{N}$, $(n \geq 2)$, $k \geq 1$ and m be the parameters such that $\gamma_r = k + (n-r)(m+1) > 0$, for all $r \in \{1, 2, \ldots, n-1\}$ and $Y(1, n, m, k), Y(2, n, m, k), \ldots, Y(n, n, m, k)$ be the n *dgos*. Then the joint density function of $Y_1, Y_2, \ldots, Y_n$ is of the form

$$k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} (F(y_i))^m f(y_i) \right) (F(y_n))^{k-1} f(y_n), \tag{4}$$

where $F^{-1}(1) > y_1 \geq y_2 \geq \cdots \geq y_n > F^{-1}(0)$, $Y_i = Y(i,n,m,k)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ is the realization of $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$.

The *dgos* is a combination of many ordered random models, and we may create different models by accounting for different *dgos* model characteristics. For instance, when $m = 0$ and $k = 1$ are used, the *dgos* model reduces to reverse order statistics; when $m = -1$ is used, the *dgos* model reduces to the k^{th} lower record values; and when $m = -1$ and $k = 1$ are used, the *dgos* model reduces to standard lower record values; *etc.* The following books and articles are suggested for readers who want to learn more about ordered statistics and record data: Ahsanullah (2004), Arnold *et al.* (2008), Devi *et al.* (2017), Arshad and Jamal (2019a,b), Sharma *et al.* (2019) Arshad and Baklizi (2019), Tripathi *et al.* (2019), Gupta and Jamal (2019), Anwar *et al.* (2020) and Azhad *et al.* (2021, 2022, 2023).

Now, let $Y_1, Y_2, \ldots, Y_n$ be the n *dgos* drawn from Fréchet(α, λ), then by using equation (4), equation (1) and equation (2), the likelihood function is given as

$$L(\alpha, \lambda|\boldsymbol{y}) = k(\alpha\lambda)^n \left(\prod_{j=1}^{n-1} \gamma_j\right) \left(\prod_{i=1}^{n} y_i^{-(\alpha+1)} e^{-\lambda y_i^{-\alpha}}\right) \prod_{i=1}^{n-1} \left(e^{-\lambda y_i^{-\alpha}}\right)^m \left(e^{-\lambda y_n^{-\alpha}}\right)^{k-1}. \tag{5}$$

Assuming that informative priors are independent and have a two-parameter gamma distribution with the following set of hyperparameters, we now investigate informative priors for each parameter.

$$\left.\begin{aligned} \pi(\alpha) &= \frac{b_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1-1} e^{-b_1\alpha}, \quad a_1,\ b_1,\ \alpha > 0, \\ \pi(\lambda) &= \frac{b_2^{a_2}}{\Gamma(a_2)} \lambda^{a_2-1} e^{-b_2\lambda}, \quad a_2,\ b_2,\ \lambda > 0. \end{aligned}\right\} \tag{6}$$

We take into account symmetric and asymmetric loss functions to demonstrate the adaptability of our findings and to provide a wide range of applicability for diverse real-life scenarios. The symmetric loss function is taken into consideration since it equally penalises underestimation and overestimation, which are typically highly helpful. The majority of the time, nevertheless, we observe that positive losses can sometimes be more severe than negative losses, and vice versa. Asymmetric loss functions are necessary in these circumstances. Here, we have taken into account one symmetric loss function, the squared error loss function (SELF), as well as two asymmetric loss functions, the linear exponential (LINEX) and general entropy (GE). For more details about these loss function, one may refer to Jaheen (2003), Dey (2009), Ali (2015), Zhang and Gui (2020), Nagamani *et al.* (2020). The SELF is defined as

$$L_1(\delta, \beta) = (\delta - \beta)^2, \quad \beta > 0. \tag{7}$$

The Bayes estimator under SELF is posterior mean (δ_{SEL}). The LINEX loss function is defined as

$$L_2(\delta, \beta) = e^{c(\delta-\beta)} - c(\delta - \beta) - 1, \qquad c \neq 0 \tag{8}$$

with corresponding Bayes estimator as

$$\delta_{LINEX} = -\frac{1}{c} \ln\left(E(e^{-c\beta})\right).$$

The GE loss function is given as

$$L_3(\delta, \beta) \propto \left(\frac{\delta}{\beta}\right)^c - c \ln\left(\frac{\delta}{\beta}\right) - 1, \qquad c \neq 0 \tag{9}$$

with corresponding Bayes estimator as

$$\delta_{GE} = \left[E(\beta)^{-c}\right]^{-1/c}.$$

Now, the joint posterior density of α and λ is obtained by using equation (5) and equation (6), and is given as

$$\pi(\alpha, \lambda|\boldsymbol{y}) \propto \alpha^{n+a_1-1}\lambda^{n+a_2-1}\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n} y_i^{-(\alpha+1)}e^{-\lambda y_i^{-\alpha}}\right)\prod_{i=1}^{n-1}\left(e^{-\lambda y_i^{-\alpha}}\right)^m\left(e^{-\lambda y_n^{-\alpha}}\right)^{k-1} \tag{10}$$
$$e^{(-b_1\alpha - b_2\lambda)}; \alpha > 0, \lambda > 0.$$

Joint posterior density has a complex structure, making it difficult to construct exact Bayes estimators. Lindley approximation and the Markov chain Monte Carlo approach are two extensively used approximation techniques that are used to address this scenario.

3. Lindley approximation

Using the Taylor series expansion, Lindley (1980) estimated the ratio of the two integrals. The expectation of posterior densities can be calculated using this method to a reasonable extent. Typically, a Bayes estimator takes the following form for any loss function of the β parameter:

$$E(z(\beta)|\boldsymbol{y}) = \frac{\int z(\beta)e^{\mathbb{L}(\beta)+\rho(\beta)}d\beta}{\int e^{\mathbb{L}(\beta)+\rho(\beta)}d\beta}, \tag{11}$$

where $\mathbb{L}$ denotes the logarithm of likelihood function, logarithm of the prior distribution of β is denoted by ρ. In present case $\beta = (\alpha, \lambda)$, we can transform equation (11) to

$$E(z(\alpha, \lambda)|\boldsymbol{y}) = \frac{\int\int z(\alpha, \lambda)e^{\mathbb{L}(\alpha,\lambda)+\rho(\alpha,\lambda)}d\alpha d\lambda}{\int\int e^{\mathbb{L}(\alpha,\lambda)+\rho(\alpha,\lambda)}d\alpha d\lambda}, \tag{12}$$

The values of the quantities in above equation are $\mathbb{L}(\alpha, \lambda) = \ln L(\alpha, \lambda|\boldsymbol{y})$ and $\rho(\alpha, \lambda) = \ln \pi(\alpha) + \ln \pi(\lambda)$. Utilizing the method by, we get (see Lindley (1980))

$$E(z(\alpha, \lambda)|\boldsymbol{y}) \approx z(\alpha, \lambda) + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} z_{ij}\sigma_{ij} + \sum_{i=1}^{2}\rho_i Q_i \tag{13}$$
$$\frac{1}{2}\sum_{i=1}^{2}\mathbb{L}_{iii}\sigma_{ii}Q_i + \frac{1}{2}\left[\mathbb{L}_{112}(2\sigma_{12}Q_1 + \sigma_{11}Q_2) + \mathbb{L}_{122}(\sigma_{22}Q_1 + 2\sigma_{12}Q_2)\right],$$

where,

$$\left.\begin{aligned}
&z_1=\frac{\partial z(\alpha,\lambda)}{\partial\alpha},\ z_2=\frac{\partial z(\alpha,\lambda)}{\partial\lambda},\ z_{11}=\frac{\partial^2 z(\alpha,\lambda)}{\partial\alpha^2},\ z_{22}=\frac{\partial^2 z(\alpha,\lambda)}{\partial\lambda^2},\ z_{12}=\frac{\partial^2 z(\alpha,\lambda)}{\partial\alpha\partial\lambda}=z_{21},\\
&\mathbb{L}_{11}=\frac{\partial^2\ln L(\alpha,\lambda|\boldsymbol{y})}{\partial\alpha^2},\ \mathbb{L}_{22}=\frac{\partial^2\ln L(\alpha,\lambda|\boldsymbol{y})}{\partial\lambda^2},\ \mathbb{L}_{112}=\frac{\partial^3\ln L(\alpha,\lambda|\boldsymbol{y})}{\partial\alpha^2\partial\lambda},\ \mathbb{L}_{111}=\frac{\partial^3\ln L(\alpha,\lambda|\boldsymbol{y})}{\partial\alpha^3},\\
&\mathbb{L}_{222}=\frac{\partial^3\ln L(\alpha,\lambda|\boldsymbol{y})}{\partial\lambda^3},\ \rho_1=\frac{\partial\rho(\alpha,\lambda)}{\partial\alpha},\ \rho_2=\frac{\partial\rho(\alpha,\lambda)}{\partial\lambda},\ Q_r=\textstyle\sum_{j=1}^2 z_j\sigma_{rj}
\end{aligned}\right\} \tag{14}$$

and σ_{rj} denotes $(r,j)^{th}$ element of the inverse of matrix $[-\mathbb{L}_{ij}]$. For obtaining Bayes estimator, we have to calculate all the unknown values in equation (13) by using the MLES of α and λ.

We have deduced the unknown quantities in equation (14) as per our problem. These are :

$$\left.\begin{aligned}
\mathbb{L}_{11}=&\ -\frac{n}{\alpha^2}-(k-1)\lambda(\ln y_n)^2y_n^{-\alpha}-m\lambda\textstyle\sum_{i=0}^{n-1}(\ln y_i)^2y_i^{-\alpha}-\sum_{i=0}^{n}\lambda(\ln y_i)^2y_i^{-\alpha}\\
\mathbb{L}_{12}=&\ -(k-1)(\ln y_n)^2y_n^{-\alpha}-m\textstyle\sum_{i=0}^{n-1}(\ln y_i)^2y_i^{-\alpha}-\sum_{i=0}^{n}(\ln y_i)^2y_i^{-\alpha}\\
\mathbb{L}_{22}=&\ -\frac{n}{\lambda^2},\ \mathbb{L}_{222}=\frac{2n}{\lambda^3},\ \mathbb{L}_{122}=0,\ \rho_1=\frac{a_1-1}{\alpha}-b_1,\ \rho_2=\frac{a_2-1}{\lambda}-b_2\\
\mathbb{L}_{111}=&\ \frac{2n}{\alpha^3}+(k-1)\lambda(\ln y_n)^3y_n^{-\alpha}-m\lambda\textstyle\sum_{i=0}^{n-1}-(\ln y_i)^3y_i^{-\alpha}-\sum_{i=0}^{n}-\lambda(\ln y_i)^3y_i^{-\alpha}
\end{aligned}\right\} \tag{15}$$

According to the defined loss functions, we have derived the quantities required. It is evident that except $z(\alpha,\lambda)$ and its derivatives, all the other quantities are same.

We know that the posterior mean is the Bayes estimator in SELF. So, Bayes estimator of α, is obtained using

$$z(\alpha,\lambda)=\alpha,\ z_1=1,\ z_2=0=z_{11}=z_{12}=z_{22}=z_{21}.$$

Similarly, the quantities

$$z(\alpha,\lambda)=\lambda,\ z_2=1,\ z_1=0=z_{11}=z_{12}=z_{22}=z_{21}.$$

are used for Bayes estimator of λ,
and,

$$\begin{aligned}
&z(\alpha,\lambda)=1-e^{-\alpha(t^{-\lambda}-1)},\ z_1=-e^{-t^{-\alpha}\lambda}t^{-\alpha}\lambda\ln t,\ z_2=e^{-t^{-\alpha}\lambda}t^{-\alpha},\\
&z_{11}=e^{-t^{-\alpha}\lambda}t^{-2\alpha}\left(t^{\alpha}-\lambda\right)\lambda(\ln t)^2,\ z_{22}=-e^{-t^{-\alpha}\lambda}t^{-2\alpha},\\
&z_{12}=-e^{-t^{-\alpha}\lambda}t^{-2\alpha}\left(t^{\alpha}-\lambda\right)\ln t=z_{21}.
\end{aligned}$$

are used for Bayes estimator of $R(t)$.

Under LINEX loss function, following quantities are used for the Bayes estimator of α and λ, respectively

$$z(\alpha,\lambda)=e^{-c\alpha},\ z_1=-ce^{-c\alpha},\ z_{11}=c^2e^{-c\alpha},\ z_2=0=z_{12}=z_{21}=z_{22}.$$

$$z(\alpha,\lambda)=e^{-c\lambda},\ z_2=-ce^{-c\lambda},\ z_{22}=c^2e^{-c\lambda},\ z_1=0=z_{12}=z_{21}=z_{11}.$$

and for $R(t)$, the used quantities are:

$$z(\alpha,\lambda)=e^{-c\left(1-e^{-\lambda t^{-\alpha}}\right)},\ z_1=ce^{-c\left(1-e^{-t^{-\alpha}\lambda}\right)-t^{-\alpha}\lambda}t^{-\alpha}\lambda\ln t$$

$$z_{11}=-ce^{-c\left(1-e^{-t^{-\alpha}\lambda}\right)-t^{-\alpha}\lambda}t^{-\alpha}\lambda(\ln t)^2+ce^{-c\left(1-e^{-t^{-\alpha}\lambda}\right)-t^{-\alpha}\lambda}t^{-\alpha}\lambda\ln t \times\left(t^{-\alpha}\lambda\ln t+ce^{-t^{-\alpha}\lambda}t^{-\alpha}\lambda\ln t\right)$$

$$z_2=-ce^{-c\left(1-e^{-t^{-\alpha}\lambda}\right)-t^{-\alpha}\lambda}t^{-\alpha}$$

$$z_{22}=-ce^{-c\left(1-e^{-t^{-\alpha}\lambda}\right)-t^{-\alpha}\lambda}t^{-\alpha}\left(-t^{-\alpha}-ce^{-t^{-\alpha}\lambda}t^{-\alpha}\right)$$

$$z_{12}=ce^{-c\left(1-e^{-t^{-\alpha}\lambda}\right)-t^{-\alpha}\lambda}t^{-\alpha}\ln t+ce^{-c\left(1-e^{-t^{-\alpha}\lambda}\right)-t^{-\alpha}\lambda}t^{-\alpha}\left(-t^{-\alpha}-ce^{-t^{-\alpha}\lambda}t^{-\alpha}\right)\lambda\ln t=z_{21}.$$

Similarly, in case of GE loss function, Bayes estimator of α can be obtained by the following quantities

$$z(\alpha,\lambda)=\alpha^{-c},\ z_1=-c\alpha^{-c-1},\ z_{11}=c(c+1)\alpha^{-c-2},\ z_2=0=z_{12}=z_{21}=z_{22}.$$

For Bayes estimator of λ, we have,

$$z(\alpha,\lambda)=\lambda^{-c},\ z_2=-c\lambda^{-c-1},\ z_{22}=c(c+1)\lambda^{-c-2},\ z_1=0=z_{12}=z_{21}=z_{11}.$$

and for Bayes estimator of $R(t)$, following quantities are used

$$z(\alpha,\lambda)=\left(1-e^{-\lambda t^{-\alpha}}\right)^{-c},\ z_1=ce^{-t^{-\alpha}\lambda}\left(1-e^{-t^{-\alpha}\lambda}\right)^{-1-c}t^{-\alpha}\lambda\ln t$$

$$z_2=-ce^{-t^{-\alpha}\lambda}\left(1-e^{-t^{-\alpha}\lambda}\right)^{-1-c}t^{-\alpha}$$

$$z_{11}=\frac{c\left(1-e^{-t^{-\alpha}\lambda}\right)^{-c}t^{-2\alpha}\lambda\left(-\left(e^{t^{-\alpha}\lambda}-1\right)t^{\alpha}+\left(c+e^{t^{-\alpha}\lambda}\right)\lambda\right)(\ln t)^2}{\left(e^{t^{-\alpha}\lambda}-1\right)^2}$$

$$z_{22}=\frac{c\left(1-e^{-t^{-\alpha}\lambda}\right)^{-c}\left(c+e^{t^{-\alpha}\lambda}\right)t^{-2\alpha}}{\left(e^{t^{-\alpha}\lambda}-1\right)^2}$$

$$z_{12}=\frac{c\left(1-e^{-t^{-\alpha}\lambda}\right)^{-c}t^{-2\alpha}\left(-\left(e^{t^{-\alpha}\lambda}-1\right)t^{\alpha}+\left(c+e^{t^{-\alpha}\lambda}\right)\lambda\right)\ln t}{\left(e^{t^{-\alpha}\lambda}-1\right)^2}=z_{21}.$$

4. Markov chain Monte Carlo

From equation (10), we see that posterior density is complex in nature and exact Bayes estimates of parameters are not easy to compute. To tackle this situation, one of the most popular tools known as Markov chain Monte Carlo (MCMC) is applied here. MCMC is is a powerful computational method used for generating samples from complex probability distributions and obtaining approximate Bayes estimates of the unknown parameters. This tools has significant popularity in various scientific fields, including statistics, machine learning, physics, and computational biology. To derive the approximate Bayes estimator of α, λ and $R(t)$, we use the MCMC technique in this part. With the use of posterior densities, the MCMC method is utilised to generate a random samples of unknown quantities. The Bayes estimator for the loss functions is then obtained using the generated samples. For this we first derived the conditional posterior densities of α and λ, from equation (10) as,

$$\left.\begin{array}{ll}\pi(\alpha|\lambda,\boldsymbol{y})\propto & \alpha^{n+a_1-1}\left(\prod_{i=1}^{n} y_i^{-(\alpha+1)}e^{-\lambda y_i^{-\alpha}}\right)\prod_{i=1}^{n-1}\left(e^{-\lambda y_i^{-\alpha}}\right)^m\left(e^{-\lambda y_n^{-\alpha}}\right)^{k-1}e^{-b_1\alpha}\\ \pi(\lambda|\alpha,\boldsymbol{y})\propto & \lambda^{n+a_2-1}\left(\prod_{i=1}^{n} e^{-\lambda y_i^{-\alpha}}\right)\prod_{i=1}^{n-1}\left(e^{-\lambda y_i^{-\alpha}}\right)^m\left(e^{-\lambda y_n^{-\alpha}}\right)^{k-1}e^{-b_2\lambda}\end{array}\right\}. \quad (16)$$

From equation (16), we observe that the marginal posterior densities of α and λ do not have known form of any probability distribution. So, we adopt the technique of Metropolis Hasting (MH) algorithm with normal distribution (see Gelman *et al.* (2013)) as the proposal density to generate samples. The algorithm and steps are followed from Arshad *et al.* (2021).

5. Simulation study

This section comprises of studying the behavior of the derived estimators on the simulated model. Various configurations of the parameters, sample sizes and priors have been tested and reported in this section. Since *dgos* is an umbrella term containing many models having different configurations for random variables of ordered nature, we have confined ourselves to study the lower record data and order statistics. To assess the credibility of Bayes estimators, risk function is taken to be the measure. The first thing is to generate the random samples from the *dgos* setup. For this purpose the algorithm discussed by Azhad *et al.* (2021) is considered here. Using the generated samples, for 1000 replications, all the estimators are obtained along with the risks in their estimation. For assessing the different possibilities, we have considered two set of priors i.e., Prior I : $(a_i, b_i) = (2, 2), i = 1, 2$ and Prior II : $(a_i, b_i) = (0.05, 0.05), i = 1, 2$, and different configurations of shape and scale parameters. The calculation is performed using R software (R Core Team (2022)). In addition to this, the convergence behaviour of generated Markov chain is tested with the aid of Gelman Rubin (GR) diagnostic (See Gelman *et al.* (2013)). With GR diagnostic we find that as we increase the number of iterations, the value of shrink reduction factor is getting close to 1. Hence, we conclude that convergence is achieved. The risks of various estimators are reported in Table [1-4] (see Appendix). From these tables, the following observations are made.

(i) The Table [1] (see Appendix) reports risks of Bayes estimates obtained using Lindley Approximation method for lower record values. From the table, it is observed that risks based on asymmetric loss functions (LINEX and GELF) are much smaller than symmetric loss function.

(ii) The Table [2] (see Appendix) reports risks of Bayes estimates obtained using MCMC method for lower record values. From the table, it is observed that risks based on asymmetric loss functions (LINEX and GELF) are much smaller than symmetric loss function. It is also observed that mostly risks of estimators based on MCMC method smaller than risk of estimators based on Lindley method.

(iv) The Table [3 - 4] (see Appendix) report risks of Bayes estimates obtained using Lindley and MCMC method for order statistics, respectively. Similar observations are seen for risks of all estimators for order statistics as these were for lower record values.

(v) From all the Tables, it is observed that the risks of all estimators are decreasing as we increase the sample size irrespective of ordered random models. Also, on average, Prior I seems to have showm lesser risk that Prior II.

(vi) From these observations it is evident that Bayes estimators based on asymmetric loss functions (LINEX and GELF) are performing better based on their risks. So, In practical scenarios where the underlying assumptions considered in this study are satisfied, it is recommended to use asymmetric loss functions as it provides more flexibility to the model. Also, estimators based on MCMC method are performing better than Lindley estimators.

6. Discussion and conclusions

In the present manuscript Fréchet distribution is considered and Bayesian perspective on estimation is explored under the *dgos* configuration. The considered distribution has many applications like it is used in hydrology to describe severe occurrences like annual maximum one-day rainfall and river discharges, used to depict a falling pattern in time series data of oil or gas production rate over time for a well, employed to simulate the idiosyncratic element of people's preferences for various goods, places , or businesses *etc.* The reliability function and Bayes estimators of unknown quantities are thoroughly addressed. For Bayesian methods, it makes sense to take distinct loss functions into account. In addition, a discussion of the findings for order statistics under *dgos's* setup and lower record values under *dgos's* setup is given. After careful examination of the simulation results, we come to the conclusion that, MCMC is a better choice than Lindley approximation for estimation of parameters α, λ, and $R(t)$ in both the cases of lower record values and order statistics for the considered distribution.

For future studies scaled squared error loss function, precautionary loss function, K-loss function, regression loss function, *etc.*, may be used. This research may possibly be expanded by assuming additional estimating techniques and applying them on censored data.

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APPENDIX

Table 1: Risk of Lindley Bayes estimates based on lower record values for $(c, t) = (0.5, 0.5)$

(α, λ)	n	SELF			Linex			General Entropy		
		$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$\hat{R(t)}_{risk}$	$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$\hat{R(t)}_{risk}$	$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$\hat{R(t)}_{risk}$
$(a_1, a_2, b_1, b_2) = (2, 2, 2, 2)$										
(1,1)	5	0.1493	0.2405	0.2423	0.0200	0.0340	0.0304	0.0124	0.0362	0.0467
	10	0.0896	0.1096	0.2209	0.0079	0.0125	0.0263	0.0033	0.0200	0.0361
	15	0.0306	0.0690	0.2208	0.0017	0.0047	0.0260	0.0011	0.0103	0.0330
(1.5,1)	5	0.3810	0.3609	0.2753	0.0611	0.0358	0.0330	0.0408	0.0476	0.0562
	10	0.2700	0.1910	0.2628	0.0181	0.0331	0.0330	0.0107	0.0275	0.0445
	15	0.1532	0.1246	0.2595	0.0059	0.0146	0.0302	0.0041	0.0194	0.0393
(1,1.5)	5	0.1600	0.3802	0.3791	0.0211	0.0348	0.0452	0.0109	0.0633	0.0628
	10	0.0912	0.3383	0.3664	0.0083	0.0252	0.0423	0.0064	0.0409	0.0537
	15	0.0398	0.3117	0.3620	0.0040	0.0237	0.0416	0.0041	0.0315	0.0498
(1.5,1.5)	5	0.2758	0.3346	0.3793	0.0349	0.0271	0.0451	0.0231	0.0640	0.0640
	10	0.1552	0.2322	0.3720	0.0083	0.0166	0.0432	0.0056	0.0316	0.0559
	15	0.0812	0.2025	0.3599	0.0041	0.0119	0.0411	0.0035	0.0213	0.0504
$(a_1, a_2, b_1, b_2) = (0.05, 0.05, 0.05, 0.05)$										
(1,1)	5	0.1853	0.3233	0.2036	0.0239	0.0337	0.0254	0.0221	0.0501	0.0453
	10	0.1394	0.2609	0.2030	0.0167	0.0284	0.0248	0.0154	0.0415	0.0381
	15	0.0849	0.2423	0.1996	0.0107	0.0272	0.0240	0.0100	0.0347	0.0342
(1.5,1)	5	0.1254	0.3322	0.2742	0.0155	0.0347	0.0335	0.0188	0.0492	0.0560
	10	0.0864	0.2873	0.2716	0.0107	0.0305	0.0310	0.0119	0.0410	0.0455
	15	0.0648	0.2483	0.2568	0.0082	0.0279	0.0322	0.0086	0.0366	0.0440
(1,1.5)	5	0.1919	0.5153	0.2899	0.0233	0.0537	0.0353	0.0211	0.0878	0.0575
	10	0.1363	0.4065	0.2890	0.0175	0.0434	0.0345	0.0162	0.0700	0.0496
	15	0.0987	0.3688	0.2732	0.0128	0.0400	0.0323	0.0120	0.0620	0.0437
(1.5,1.5)	5	0.1092	0.4841	0.3193	0.0138	0.0507	0.0387	0.0169	0.0851	0.0620
	10	0.0834	0.4338	0.3143	0.0106	0.0458	0.0374	0.0115	0.0734	0.0531
	15	0.0736	0.3466	0.3122	0.0096	0.0382	0.0367	0.0100	0.0601	0.0488

Table 2: Risk of MCMC Bayes estimates based on lower record values for $(c,t) = (0.5, 0.5)$

(α,λ)	n	SELF			Linex			General Entropy		
		$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$R\hat{(t)}_{risk}$	$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$R\hat{(t)}_{risk}$	$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$R\hat{(t)}_{risk}$
$(a_1,a_2,b_1,b_2) = (2,2,2,2)$										
(1,1)	5	0.0626	0.0314	0.1837	0.0079	0.0040	0.0214	0.0057	0.0033	0.0380
	10	0.0601	0.0302	0.1832	0.0076	0.0038	0.0214	0.0056	0.0032	0.0379
	15	0.0578	0.0287	0.1816	0.0073	0.0036	0.0212	0.0054	0.0030	0.0375
(1.5,1)	5	0.0646	0.2124	0.1874	0.0082	0.0247	0.0218	0.0059	0.0179	0.0390
	10	0.0589	0.2060	0.1833	0.0075	0.0239	0.0214	0.0055	0.0159	0.0379
	15	0.0557	0.1928	0.1801	0.0070	0.0225	0.0210	0.0052	0.0174	0.0371
(1,1.5)	5	0.0875	0.0329	0.1848	0.0106	0.0041	0.0215	0.0064	0.0035	0.0380
	10	0.0839	0.0319	0.1839	0.0102	0.0040	0.0214	0.0061	0.0033	0.0383
	15	0.0837	0.0289	0.1836	0.0101	0.0036	0.0214	0.0061	0.0030	0.0380
(1.5,1.5)	5	0.0873	0.2180	0.1844	0.0106	0.0253	0.0215	0.0064	0.0184	0.0382
	10	0.0840	0.2139	0.1808	0.0101	0.0248	0.0211	0.0061	0.0181	0.0373
	15	0.0835	0.2022	0.1816	0.0102	0.0235	0.0212	0.0061	0.0168	0.0375
$(a_1,a_2,b_1,b_2) = (0.05,0.05,0.05,0.05)$										
(1,1)	5	0.0627	0.0319	0.1851	0.0079	0.0040	0.0216	0.0058	0.0035	0.0384
	10	0.0591	0.0308	0.1834	0.0075	0.0039	0.0211	0.0055	0.0032	0.0372
	15	0.0560	0.0306	0.1804	0.0071	0.0039	0.0214	0.0052	0.0032	0.0379
(1.5,1)	5	0.0649	0.2036	0.1839	0.0082	0.0240	0.0215	0.0060	0.0173	0.0381
	10	0.0605	0.2031	0.1821	0.0077	0.0237	0.0213	0.0056	0.0169	0.0376
	15	0.0585	0.2068	0.1804	0.0074	0.0237	0.0211	0.0054	0.0168	0.0371
(1,1.5)	5	0.0863	0.0328	0.1835	0.0105	0.0041	0.0214	0.0063	0.0034	0.0380
	10	0.0845	0.0299	0.1833	0.0102	0.0038	0.0214	0.0062	0.0032	0.0379
	15	0.0832	0.0296	0.1833	0.0101	0.0037	0.0214	0.0061	0.0031	0.0379
(1.5,1.5)	5	0.0880	0.2113	0.1848	0.0106	0.0245	0.0215	0.0064	0.0176	0.0383
	10	0.0841	0.2087	0.1820	0.0102	0.0243	0.0212	0.0061	0.0174	0.0376
	15	0.0833	0.2053	0.1819	0.0101	0.0239	0.0212	0.0061	0.0172	0.0375

Table 3: Risk of Lindley Bayes estimates based on order statistics for $(c, t) = (0.5, 0.5)$

(α, λ)	n	SELF			Linex			General Entropy		
		$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$\hat{R(t)}_{risk}$	$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$\hat{R(t)}_{risk}$	$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$\hat{R(t)}_{risk}$
$(a_1, a_2, b_1, b_2) = (2, 2, 2, 2)$										
(1,1)	5	0.0711	0.0632	0.6544	0.0086	0.0049	0.0718	0.0102	0.0074	0.0744
	10	0.0643	0.0599	0.6168	0.0078	0.0079	0.0680	0.0078	0.0087	0.0719
	15	0.0481	0.0369	0.4837	0.0059	0.0073	0.0540	0.0057	0.0074	0.0631
(1.5,1)	5	0.1965	0.0571	0.7713	0.0154	0.0045	0.0837	0.0178	0.0060	0.0864
	10	0.0608	0.0562	0.7173	0.0074	0.0069	0.0782	0.0083	0.0077	0.0818
	15	0.0503	0.0380	0.5412	0.0063	0.0069	0.0597	0.0067	0.0071	0.0684
(1,1.5)	5	0.0882	0.3452	0.8049	0.0112	0.0412	0.0871	0.0153	0.0420	0.0898
	10	0.0640	0.0974	0.7617	0.0079	0.0106	0.0828	0.0074	0.0131	0.0869
	15	0.0471	0.0711	0.6257	0.0058	0.0092	0.0690	0.0056	0.0103	0.0799
(1.5,1.5)	5	0.2417	0.2860	0.8531	0.0249	0.0417	0.0920	0.0286	0.0403	0.0947
	10	0.0799	0.0996	0.7967	0.0096	0.0095	0.0863	0.0094	0.0119	0.0903
	15	0.0500	0.0698	0.6197	0.0064	0.0089	0.0682	0.0067	0.0102	0.0780
$(a_1, a_2, b_1, b_2) = (0.05, 0.05, 0.05, 0.05)$										
(1,1)	5	0.1562	0.1643	0.6613	0.0203	0.0209	0.0725	0.0194	0.0239	0.0750
	10	0.1004	0.1094	0.6263	0.0130	0.0131	0.0690	0.0121	0.0136	0.0731
	15	0.0653	0.0870	0.5379	0.0082	0.0108	0.0599	0.0077	0.0113	0.0694
(1.5,1)	5	0.1124	0.1681	0.7867	0.0135	0.0196	0.0853	0.0154	0.0224	0.0880
	10	0.0713	0.1163	0.7492	0.0091	0.0142	0.0815	0.0097	0.0145	0.0860
	15	0.0583	0.0836	0.6535	0.0071	0.0102	0.0720	0.0073	0.0100	0.0821
(1,1.5)	5	0.1566	0.2145	0.8007	0.0193	0.0250	0.0867	0.0177	0.0307	0.0895
	10	0.0920	0.1135	0.7594	0.0111	0.0144	0.0826	0.0101	0.0164	0.0872
	15	0.0589	0.0814	0.6712	0.0076	0.0103	0.0738	0.0073	0.0111	0.0840
(1.5,1.5)	5	0.0961	0.1915	0.8692	0.0122	0.0225	0.0936	0.0147	0.0289	0.0964
	10	0.0702	0.1213	0.8295	0.0091	0.0146	0.0897	0.0096	0.0164	0.0944
	15	0.0612	0.0841	0.7239	0.0076	0.0104	0.0792	0.0077	0.0112	0.0899

Table 4: Risk of MCMC Bayes estimates based on order statistics for $(c,t) = (0.5, 0.5)$

(α,λ)	n	SELF			Linex			General Entropy		
		$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$\hat{R(t)}_{risk}$	$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$\hat{R(t)}_{risk}$	$\hat{\alpha}_{risk}$	$\hat{\lambda}_{risk}$	$\hat{R(t)}_{risk}$
$(a_1,a_2,b_1,b_2) = (2,2,2,2)$										
(1,1)	5	0.0428	0.2054	0.2548	0.0052	0.0249	0.0294	0.0077	0.0670	0.0603
	10	0.0401	0.1966	0.2512	0.0049	0.0238	0.0290	0.0072	0.0632	0.0591
	15	0.0379	0.1961	0.2473	0.0047	0.0237	0.0285	0.0068	0.0629	0.0579
(1.5,1)	5	0.0457	0.8683	0.2593	0.0056	0.0952	0.0299	0.0083	0.1353	0.0618
	10	0.0442	0.8594	0.2554	0.0054	0.0942	0.0294	0.0081	0.1318	0.0608
	15	0.0422	0.8490	0.2549	0.0052	0.0932	0.0294	0.0077	0.1288	0.0605
(1,1.5)	5	0.4668	0.2074	0.2550	0.0525	0.0251	0.0294	0.0452	0.0671	0.0602
	10	0.4553	0.2028	0.2500	0.0513	0.0245	0.0289	0.0439	0.0656	0.0588
	15	0.4471	0.1914	0.2465	0.0504	0.0232	0.0285	0.0427	0.0602	0.0575
(1.5,1.5)	5	0.4587	0.8904	0.2509	0.0516	0.0975	0.0290	0.0441	0.1410	0.0573
	10	0.4562	0.8883	0.2500	0.0514	0.0973	0.0288	0.0439	0.1392	0.0587
	15	0.4417	0.8818	0.2454	0.0498	0.0966	0.0283	0.0422	0.1352	0.0589
$(a_1,a_2,b_1,b_2) = (0.05,0.05,0.05,0.05)$										
(1,1)	5	0.0402	0.2067	0.2564	0.0054	0.0250	0.0289	0.0080	0.0673	0.0591
	10	0.0403	0.1973	0.2509	0.0049	0.0239	0.0288	0.0073	0.0635	0.0585
	15	0.0438	0.1884	0.2493	0.0049	0.0229	0.0295	0.0072	0.0596	0.0609
(1.5,1)	5	0.0432	0.8842	0.2544	0.0053	0.0968	0.0293	0.0078	0.1371	0.0603
	10	0.0405	0.8697	0.2527	0.0050	0.0954	0.0291	0.0073	0.1357	0.0596
	15	0.0378	0.8654	0.2477	0.0046	0.0949	0.0286	0.0067	0.1345	0.0579
(1,1.5)	5	0.4582	0.2015	0.2515	0.0516	0.0244	0.0290	0.0443	0.0651	0.0591
	10	0.4582	0.2014	0.2510	0.0516	0.0244	0.0290	0.0442	0.0641	0.0590
	15	0.4562	0.1980	0.2505	0.0514	0.0240	0.0289	0.0442	0.0634	0.0592
(1.5,1.5)	5	0.4581	0.8825	0.2517	0.0516	0.0967	0.0290	0.0442	0.1380	0.0593
	10	0.4557	0.8772	0.2507	0.0511	0.0961	0.0288	0.0438	0.1368	0.0589
	15	0.4535	0.8746	0.2496	0.0513	0.0958	0.0289	0.0436	0.1353	0.0585