Statistics and Applications {ISSN 2454-7395 (online)}
Volume 19, No. 2, 2021(New Series), pp 115–125

On Association in Time of Markov Process With Application to Reliability and Survival Analysis

M. Manoharan[1] **and V.M. Chacko**[2]

[1] *Department of Statistics, University of Calicut, Kerala.*

[2] *Department of Statistics, St.Thomas College(Autonomous), Thrissur, Kerala.*

Received: 17 June 2020; Revised: 08 October 2020; Accepted: 14 October 2020

Abstract

In this paper, the association in time of a Markov process is considered. A measure based on transition probability function is proposed to obtain and compare the degree of association in time of two processes. A real data is analyzed.

Key words: Association; Degree of association; Transition probability function; Reliability; Markov process; Multistate system; Right tail increasing

AMS Subject Classifications: 60J25, 60J28

1. Introduction

For a multistate system it is difficult to calculate system reliability. The calculation of system reliability can become even more difficult, perhaps impossible, if the components of the system are maintained, or are interdependent. Hence, reliability bounds for reliability of multistate systems (MSSs) are useful. In reliability and survival analysis, the lifetime random variables are not independent but are associated. In structures in which the components share load, the failure of one component results in increased load on each of the remaining components. Associated random variables and time associated stochastic processes are useful for obtaining the reliability bounds for MSSs. Association of random variables is mathematically described by Esary *et al.* (1967). Esary and Proschan (1970) obtained a minimal cut lower bound for a non-maintained system, if the joint performance process of the components is associated in time. A repairable system modeled by semi-Markov process is considered by Dharmadhikari and Kuber (2006) and derived a sufficient condition for the association in time of the process governing the system. Hjort *et al.* (1985) introduced a sufficient condition for association when the marginal processes are Markovian. Bound for reliability of maintained systems without imposing conditions of association in time of marginal process of components is given by Natvig (1993). Minimal path structures of a coherent system having components in common behave in a similar manner, so that failure of a component will adversely affect the performance of all the minimal path structures.

A sufficient condition for association in time of the Markov performance process of a binary system, in terms of its transition probability functions, is given by Esary and Proschan (1970). Reliability analysis of MSSs can be seen in Barlow and Wu (1978).

Corresponding Author: V. M. Chacko
Email: chackovm@gmail.com

Relative degree (or strength) of association for two sets of random variables is described by Karlin (1983). In order to assess the degree of association of a Markov process or of comparing the relative strength of association of two Markov processes, we need measures in terms of transition probability functions. In this paper, in order to find degree of association of the Markov process or to compare the relative degree of association of two Markov processes, we consider a measure of association based on transition probability function. In MSS modeling, the information regarding exact state visited by components before the present state may not be available. At the same time, information regarding either the state is above or below a certain state of performance may be available. The conditions for association in time of the stochastic process which governs the MSS is still worthwhile. A weaker sufficient condition for association in time of the finite Markov process is given.

In section 2, we discuss the measure of degree of association in time of the Markov process in terms of the transition probability function. We examine the correlation in terms of transition probability functions to asses the relative degree (or strength) of association when comparing two Markov processes. In section 3, we present the weaker sufficient condition in terms of transition functions and intensities for the Markov process to be associated in time. An illustrative example is provided in section 4.

2. Measure of Degree of Association

An approach for assessing the level and form of dependence for multivariate observations is provided by Karlin (1983). It provides a fine tuning in evaluating relationships of pair of random variable by transforming the data in natural manifold ways and then computing the associated correlations whose totality reflects on the nature of dependence between array of transformed variables. The degree of dependence between two random variables X and Y can be computated by a single statistics.

The following definition gives the measure for ordering bivariate distributions by the strength of their association.

Definition 1: For two bivariate distributions corresponding to the random variables (X, Y) and (Z, W) we say that dependence of (X, Y) is stronger than the dependence of (Z, W) with respect to classes of non-decreasing functions F and G if $\rho[h(X), g(Y)] \geq \rho[h(Z), g(W)]$ for all $h \in \mathrm{F} and \mathrm{g} \in \mathrm{G}$.

The comparisons are made with respect to the same transformations on the variables (X, Y) and (Z, W) for all functions $h \in \mathrm{F} and \mathrm{g} \in \mathrm{G}$.

A measure which can be used to measure the degree of association of the Markov process is proposed below. We first discuss the measure of degree of association in discrete time stochastic process $\{X_k, k \geq 0\}$ with state space $E = \{1, 2, ..., M\}$. We have, $\mathrm{Cov}(\mathrm{X}_k, X_{k-1}) = E(X_k.X_{k-1}) - E(X_k)E(X_{k-1})$
$= \sum_{i,j \in E} P(X_k \geq j, X_{k-1} \geq i) - \sum_{j \in E} P(X_k \geq j) \sum_{i \in E} P(X_{k-1} \geq i)$.

But, X_k and X_{k-1}, associated if, $Cov(X_k, X_{k-1}) \geq 0$.
$\Rightarrow \left(\sum_{i,j\in E} P(X_k \geq j, X_{k-1} \geq i) - \sum_{i,j\in E} P(X_k \geq j)P(X_{k-1} \geq i)\right) \geq 0$
$\Rightarrow \sum_{i,j\in E} [P(X_k \geq j, X_{k-1} \geq i) - P(X_k \geq j)P(X_{k-1} \geq i)] \geq 0$
or

$$\sum_{i,j\in E} [P(X_k \geq j|X_{k-1} \geq i) - P(X_k \geq j)]P(X_{k-1} \geq i) \geq 0. \quad (1)$$

Using one step transition probability, we get,

$$\sum_{i,j\in E} \sum_{i,j\in E} [P(X_k = j|X_{k-1} = i) - P(X_k = j)]P(X_{k-1} = i) \geq 0 \quad (2)$$

We can use the measure, $\text{Cov}(\text{X}_k, X_{k-1}) = \sum_{i,j\in E} \sum_{i,j\in E}[P(X_k = j|X_{k-1} = i) - P(X_k = j)]P(X_{k-1} = i)$, for assessing the association of the discrete time stochastic process.

Standardization of the covariance may be desired to achieve scale invariance and enable meaningful comparisons between different data sets. The condition of association,

$$Cov(h(X), g(Y)) \geq 0$$

for all functions $h \in F$ and $g \in G$, of two random variables with respect to the classes F and G is replaced by an equivalent requirement $\rho(X,Y) = \frac{Cov(h(X),g(Y))}{\sqrt{Var(h(X)).Var(g(Y))}} \geq 0$. For two stochastic processes $\{X_k, k \geq 0\}$ and $\{Y_k, k \geq 0\}$, the following measure of association is used for comparing two processes in terms of their strength of association. $\rho_{(X_k,X_{k-1})} = \frac{Cov(X_k,X_{k-1})}{\sqrt{Var(X_k).Var(X_{k-1})}}$ where $Var(X_k) = \sum_{i,j\in E} P(X_k \geq max(i,j)) - P(X_k \geq j)P(X_k \geq i)$.

If

$$\rho_{(X_k,X_{k-1})} \geq \rho_{(Y_k,Y_{k-1})} \quad (3)$$

the association between X_k and X_{k-1} is larger than association between Y_k and Y_{k-1}. If (3) is true for every k, then the stochastic process $\{X_k, k \geq 0\}$ is more associated than $\{Y_k, k \geq 0\}$.

Here we also consider a continuous time Markov process $\{X(t), t \geq 0\}$. Consider the random variables $X(t)$, $X(s)$, $s < t$ in the Markov process. It is clear that if $X(t)$ and $X(s)$, $s < t$ are associated if

$$Cov(X(t), X(s)) = \int_R \int_R P(X(t) > x, X(s) > y) - P(X(t) > x)P(X(s) > y)dxdy \geq 0. \quad (4)$$

Using transition probability function, $P(X(t) = j|X(s) = i)$ of the Markov process, we write (4) as,
$Cov(X(t), X(s)) =$

$$\int_R \int_R \sum_{\{i,j:X(s)=i>y,X(t)=j>x\}} [P(X(t) = j|X(s) = i) - P(X(t) = j)]P(X(s) = i)dxdy \geq 0 \quad (5)$$

Comparison of two Markov processes, $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$, only in terms of transition probabilities is not possible but comparison between covariances in terms of transition probabilities and state probabilities is more reasonable. The degree of association of two Markov processes can be compared using the following correlation function, $\rho_X(t,s)$.

In the case of the Markov process we have, $\rho_X(t,s) = \frac{Cov(X(t),X(s))}{\sqrt{Var(X(t)).Var(X(s))}} \geq 0$ implies association between $X(s)$ and $X(t)$. We use the correlation $\rho_X(t,s)$ as a function of transition probability function and state probabilities to measure the degree of association in time of the Markov process.

We compare the degree of association of two Markov processes using $\rho_X(t,s)$. This gives a stochastic ordering of two Markov processes based on strength of their association. Denote, $C_X(t,s) = \int_R \int_R [P(X(t) \geq x | X(s) \geq y) - P(X(t) \geq x)]P(X(s) \geq y)dxdy$,
$C_X(t,t) = \int_R \int_R [P(X(t) \geq max(x,y)) - P(X(t) \geq x)P(X(t) \geq y)]dxdy$
$\rho_X(t,s) = \frac{C_X(t,s)}{\sqrt{C_X(t,t).C_X(s,s)}}$. We propose the following definitions.

Definition 2: For two different Markov processes $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$, we say that association of $(X(t), X(s))$, $s < t$ is stronger than the association of $(Y(t), Y(s))$, $s < t$ if $\rho_X(t,s) \geq \rho_Y(t,s)$.

Definition 3: For two different Markov processes $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$, we say that association of X process is stronger than the association of Y process if $\forall s, t \in R$, $s < t$, $\rho_X(t,s) \geq \rho_Y(t,s)$. Some conditions of association in terms of the nondecreasing functions of the classes F and G and its distributional properties are given below. It provide a measure for the comparison of the degree of association of two system each consists of n associated components, see Prakash Rao and Dewan (2001).

Definition 4: A collection of random variables $\{X_n, n \geq 1\}$ is said to be associated if for every n and for every choice of coordinate-wise non-decreasing functions $h(\underline{x})$ and $g(\underline{x})$ from R^n to R,

$$Cov(h(\underline{X}), g(\underline{X})) \geq 0 \tag{6}$$

whenever it exist, where $\underline{X} = (X_1, ..., X_n)$.

Definition 5: The performance process of the ith component is a stochastic process $\{X_i(t), t \in \tau\}$ where for each fixed $t \in \tau, X_i(t)$ denotes the state of component i at time t. The joint performance process of the components is given by $\{\underline{X}(t), t \in \tau\} = \{(X_1(t), ..., X_n(t)), t \in \tau\}$.

Let $I = [t_A, t_B] \subset [0, \infty), \tau(I) = \tau \cap I$.

Definition 6: The joint performance process $\{\underline{X}(t),\ t \in \tau\}$ of the components is said to be associated in time interval I if and only if, for any integer m and $\{t_1, ..., t_m\} \subset \tau(I)$, the random variables in the array

$$\begin{array}{ccc} X_1(t_1) & ... & X_1(t_m) \\ ... & ... & ... \\ X_n(t_1) & ... & X_n(t_m) \end{array}$$

are associated.

For the component performance process $\{X_i(t), t \in \tau\}$, $i \in \{1, 2, ..., n\}$ and fixed $t_1 < ... < t_m$, let $h_i(\underline{X}_i) \in F, g_i(\underline{X}_i) \in G$ are nondecreasing function of random variables from R^m to R, where $\underline{X}_i = (X_i(t_1), X_i(t_2), ..., X_i(t_m))$. $X_i(t_1), X_i(t_2), ..., X_i(t_m)$ are associated if for every $h_i(\underline{x}_i), g_i(\underline{x}_i)$, $Cov(h_i(\underline{X}_i), g_i(\underline{X}_i)) \geq 0$ where $\mathrm{Cov}(\mathrm{h}_i(\underline{X}_i), g_i(\underline{X}_i)) = \int_R \int_R [P(h_i(\underline{X}_i) > x, g_i(\underline{X}_i) > y) - P(h_i(\underline{X}_i) > x)P(g_i(\underline{X}_i) > y)]dxdy$, $i \in \{1, 2, ..., n\}$. This implies that if

$$\int_R \int_R [P(h_i(\underline{X}_i) > x | g_i(\underline{X}_i) > y) - P(h_i(\underline{X}_i) > x)]P(g_i(\underline{X}_i) > y)dxdy \geq 0 \tag{7}$$

we have association of the component performance processes $\{X_i(t), t \in \tau\}$.

Definition 7: A Markov performance process $\{X_i(t), t \in \tau\}$ of component i is associated if $\int_R \int_R [P(h_i(\underline{X}_i) > x | g_i(\underline{X}_i) > y) - P(h_i(\underline{X}_i) > x)]P(g_i(\underline{X}_i) > y)dxdy \geq 0$ for every collection of random variables $\underline{X}_i = (X_i(t_1), ..., X_i(t_m))$ and every choice of coordinate wise nondecreasing function $h_i(\underline{x}_i)$ and $g_i(\underline{x}_i)$ from R^m to R.

In a similar way, we can find a condition for association of joint performance process of components, in terms of non-decreasing functions, which is quite desirable. In the following definition, we consider the nondecreasing functions $H \in F$ and $G \in G$ from R^{nm} to R.

Definition 8: The joint performance process of the components $\{\underline{\mathrm{X}}(\mathrm{t}),\mathrm{t} \in \tau\}$ $= \{(X_1(t), ..., X_n(t)), t \in \tau\}$ is associated in time if

$$\int_R \int_R [P(H(\underline{X}) > x | G(\underline{X}) > y) - P(H(\underline{X}) > x)]P(G(\underline{X}) > y)dxdy \geq 0$$

for every collection of random variables,

$$\underline{X} = (X_1(t_1), X_2(t_1), ..., X_n(t_1), X_1(t_2), ..., X_n(t_2), ..., X_1(t_m), ..., X_n(t_m))$$

and every choice of coordinate wise nondecreasing function $H(\underline{x})$ and $G(\underline{x})$ from R^{nm} to R.

The measure of degree of association of the system which consists of n associated components governed by Markov processes is given below.

Denote

$$C_{\underline{X}}(H, G) = \int_R \int_R [P(H(\underline{X}) > x | G(\underline{X}) > y) - P(H(\underline{X}) > x)]P(G(\underline{X}) > y)dxdy$$

$$C_{\underline{X}}(H, H) = \int_R \int_R [P(H(\underline{X}) > max\{x, y\}) - P(H(\underline{X}) > x)P(H(\underline{X}) > y)]dxdy$$

and

$\rho_{\underline{X}}(H,G) = \frac{C_{\underline{X}}(H,G)}{\sqrt{C_{\underline{X}}(H,H)C_{\underline{X}}(G,G)}}$

The degree of association of two performance processes $\{\underline{X}(t), t \geq 0\}$ and $\{\underline{Y}(t),\ t \geq 0\}$ of two systems can be compared using the measures $\rho_{\underline{X}}(H,G)$ and $\rho_{\underline{Y}}(H,G)$.

Definition 9: For two performance process $\{\underline{X}(t), t \geq 0\}$ and $\{\underline{Y}(t), t \geq 0\}$, of two systems consists of n associated components governed by the Markov processes $\{\underline{X}_i(t), t \geq 0\}$ and $\{\underline{Y}_i(t), t \geq 0\}$, $i = 1, 2, ..., n$ respectively, we say that association of X-system is stronger than the association of Y-system if $\forall m\,, n\, and\, H \in F$, $G \in G$, from R^{mn} to R, $\rho_{\underline{X}}(H,G) \geq \rho_{\underline{Y}}(H,G)$.

The proposed measures may help us (i) to suggest whether a Markov process is associated in time; and (ii) to asses the relative degree (or strength) of association of two different Markov processes, and (iii) to asses the relative strength of association of two performance process of two systems consists of n associated components which are governed by Markov processes.

3. A Weaker Condition for Association in Time of a Markov Process

A sufficient condition using transition probability function for association with the marginal Markovian processes is given by Hjort *et al.* (1985). We consider much weaker conditions for the Markov process to be associated in time. Let $\mathrm{P}^*_{ij}(s,t) = P(X(t) = j|X(s) \geq i)$, $s < t$and $\mathbf{P}^*(s,t) = \{P^*_{ij}(s,t)\}_{i,j\in\{0,1,...,M\}}$, $s < t$ Assume the existence of the following intensities

$$\mu^*_{ij}(s) = \begin{cases} lim_{h\to 0+}\frac{P^*_{ij}(s,s+h)}{h}, & i \neq j \\ lim_{h\to 0+}\frac{P^*_{ij}(s,s+h)-1}{h}, & i = j \end{cases}$$

Let $\mathrm{P}^*_{i,\geq j}(s,t) = P(X(t) \geq j|X(s) \geq i)$
$P^*_{i,<j}(s,t) = P(X(t) < j|X(s) \geq i)$
$\mu^*_{i,\geq j}(s) = \sum_{\nu=j}^{M} \mu^*_{i\nu}(s), i < j$
$\mu^*_{i,<j}(s) = \sum_{\nu=0}^{j-1} \mu^*_{i\nu}(s), i \geq j$

Now we consider the following definitions, see Barlow and Proschan (1975).

Definition 10: A random variable T is stochastically right tail increasing (st. RTI) in random variables $S_1, ..., S_k$ if $P(T > t|S_1 \geq s_1, ..., S_k \geq s_k)$ is nondecreasing in $s_1, s_2, ..., s_k$.

Definition 11: Random variables $T_1, ..., T_n$ are conditionally RTI in sequence if T_i is st. RTI in $T_1, ..., T_{i-1}$ for $i = 2, 3, ..., n$.

Definition 12: A process $\{X(t), t \geq 0\}$ is conditionally RTI in time if $\mathrm{P(X(t)}\geq j|X(s_1) \geq i_1, ..., X(s_n) \geq i_n)$ is nondecreasing in $i_1, ..., i_n$ for each j and for each choices of $s_1 < ... < s_n < t, n \geq 1$.

Manoharan (1995) proved the following result.

Theorem 1: If the random variables $T_1, T_2, ..., T_n$ are conditionally RTI in sequence, then they are associated.

Now using the Definition 12 and the above Theorem 1, we get the following result.

Theorem 2: If a stochastic process $\{X(t), t \geq 0\}$ is conditionally RTI in time, then it is associated in time.

A main result of this section which gives a weaker condition for the Markov process to be associated in time is given below.

Theorem 3: Let X be a continuous time Markov process with state space $\{0, 1, ..., M\}$. Assume $\mu^*_{ij}(s)$ to be continuous. Then each of the following three conditions are equivalent and implay that X is associated in time

(a) X is conditionally RTI in time.

(b) $P^*_{i,\geq j}(s, t)$ is nondecreasing in i for each j and for each choice of $s < t$.

(c) For each j and s

$\mu^*_{i,\geq j}(s)$ is nondecreasing in $i \in \{0, 1, ..., j-1\}$

$\mu^*_{i,<j}(s)$ is nonincreasing in $i \in \{j, j+1, ..., M\}$

Proof: In view of Theorem 2, it suffices to prove the equivalence of conditions (a), (b) and (c).

The equivalence of (a) and (b) follows from the Markov property of X.

Now to prove the equivalence of (b) and (c), note that statement (b) is equivalent to the following three conditions.

(i)$P(X(t) \geq j | X(s) \geq i) = P^*_{i,\geq j}(s, t)$ is nondecreasing in $i \in \{0, 1, 2..., j-1\}$

(ii)$P(X(t) < j | X(s) \geq i) = P^*_{i,<j}(s, t)$ is nonincreasing in $i \in \{j, j+1, ..., M\}$.

(iii)$P(X(t) \geq j | X(s) \geq j-1) \leq P(X(t) \geq j | X(s) \geq j)$

Thus if (b) holds then for $i < j$, $\mu^*_{i,\geq j}(s) = \sum_{\nu=j}^{M} \mu^*_{i\nu}(s)$ $= \sum_{\nu=j}^{M} lim_{h\to 0^+} \frac{P^*_{ij}(s,s+h)}{h} = lim_{h\to 0^+} \frac{P^*_{i,\geq j}(s,s+h)}{h}$ is nondecreasing in $i \in \{0, 1, ..., j-1\}$, and for $i \geq j$

$$\mu^*_{i,<j}(s) = \sum_{\nu=0}^{j-1} \mu^*_{i\nu}(s) = lim_{h\to 0^+} \frac{P^*_{i,<j}(s, s+h)}{h}$$

is nonincreasing in $i \in \{j, j+1, ..., M\}$. Hence (b) implies (c).

To show that (c) implies (b), let $\mathbf{M}^*$ denote the class of all stochastic matrices $\mathbf{P}^* = (P^*_{ij})_{i,j\in\{0,1,...,M\}}$ such that $\sum_{\nu=j}^{M} P^*_{i\nu}$ is nondecreasing in i for each j. In order to prove that X has property (b) it is enough to show that $\mathbf{P}^*(s, t) \in M^*$ for each choice $s < t$.

Define

$$a^*_{ij}(u) = \begin{cases} \mu^*_{ij}(u), & i \neq j \\ 1 - \sum_{j\neq i} \mu^*_{ij}(u), & i = j \end{cases}$$

and let $\mathbf{A}^*(u) = (a^*_{ij}(u))_{i,j\in\{0,1,\ldots,M\}}$. Also let $\mathbf{Q}^*(u) = \mathbf{A}^*(u) - \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Using the product integral representation (see Johansen (1977)) and the fact that the transition intensities are uniformly continuous on $[s,t]$, we have

$$\mathbf{P}^*(s,t) = lim_{n\to\infty} \prod_{j=0}^{n-1} [\mathbf{I} + \mathbf{Q}^*(s + (j/n)(t-s))(t-s)/n] \tag{8}$$

Note that $\mathbf{B}(u,h) = \mathbf{I} + \mathbf{Q}^*(u)h = (1-h)\mathbf{I} + \mathbf{A}^*(u)h$ is a stochastic matrix. Now if (c) is satisfied, $\sum_{\nu=j}^{M} b_{i\nu}(u,h) = \sum_{\nu=j}^{M} q^*_{i\nu}(u)h = \sum_{\nu=j}^{M} \mu^*_{i\nu}(u)h$ is nondecreasing in $i \in \{0,1,\ldots,j-1\}$, and $\sum_{\nu=0}^{j-1} b_{i\nu}(u,h) = \sum_{\nu=0}^{j-1} \mu^*_{i\nu}(u)h$ is nonincreasing in $i \in \{j, j+1, \ldots, M\}$. Also $\sum_{\nu=j}^{M} b_{j\nu}(u,h) - \sum_{\nu=j}^{M} b_{j-1,\nu}(u,h) = (1-h) + ha^*_{jj}(u) + h\sum_{\nu=j+1}^{M} a^*_{j\nu}(u) - h\sum_{\nu=j}^{M} a^*_{j-1,\nu}(u)$
$= (1-h) + h\left[1 - \sum_{l\neq j} \mu^*_{jl}(u)\right] + h\sum_{\nu=j+1}^{M} \mu^*_{j\nu}(u) - h\sum_{\nu=j}^{M} \mu^*_{j-1,\nu}(u)$
$= 1 + h\left[\sum_{\nu=j+1}^{M} \mu^*_{j\nu}(u) - \sum_{\nu=j}^{M} \mu^*_{j-1,\nu}(u) - \sum_{l\neq j} \mu^*_{jl}(u)\right]$ Now choose h small enough so that $\sum_{\nu=j}^{M} b_{j\nu}(u,h) - \sum_{\nu=j}^{M} b_{j-1,\nu}(u,h) \geq 0 \, \forall j = 1,2,\ldots,M$.Since μ^*_{ij}'s are bounded, we can choose h independent of u. Hence for sufficiently small h (independent of u),

$$\mathbf{B}(u,h) = \mathbf{I} + \mathbf{Q}^*(u)h$$

satisfies the conditions (i), (ii) and (iii), which means that $\mathbf{B}(u,h) \in M^*$. The class M^* being closed under multiplication and also under pointwise limits, we conclude from (8) that $P^*(s,t) \in M^*$. Hence (b) is true.

Remark 1: It can be easily seen that the conditions for the association of Markov process in Hjort *et al.* (1985) imply the conditions (a) to (c) of the Theorem 3 and hence the latter set gives a much weaker conditions for association in time of a Markov process.

Remark 2: For the binary reliability system (M=1) it is easily seen that condition (b) of the Theorem 3 is equivalent to $\mathrm{P}_{11}(s,t) + P_{00}(s,t) \geq 1 \; for\, each \quad s < t$ which is the sufficient condition for the association in time of X given by Esary and Proschan (1970). Furthermore, when the transition intensities are continuous, the condition (c) of the Theorem 3 is always satisfied and hence the corresponding Markov process is always associated.

Remark 3: One may have further extension of the conditions (a)-(c) of the Theorem 3 for a semi-Markov process by augmenting the waiting time variable to the state variable as dealt in Kuber and Dharmadhikari (1996).

4. Application

We consider the data set from medical field for the illustration of concept of measure of degree of association in Markov processes.

Example 1: We re-examine the data on an oral hygiene study, discussed in Das and Chattopadhyay (2004)(*cf.* Dharmadhikari and Dewan (2006)) for the illustration of the association of a vector valued process. The reduction in the amount of plaque on teeth is recorded. Each individual in the data was monitored for a couple of days. Two teeth were identified, one on the left lower canine which is in the left lower corner of a jaw, and one on molar at upper right jaw. The reduction in the thickness of plaque for subjects are usually recorded as belonging to four different categories, *viz*, no reduction, slight reduction, moderate reduction and vast reduction. To evaluate effectiveness of brushing, we use the proposed measures. To check whether it is possible to reduce the number of records per individual per day and there is some sort of dependence Das and Chattopadhyay (2004) developed a latent mixture regression model to study this categorical multivariate data. Table A.1 give a part of dental data analyzed. It gives stain on the same tooth at all four positions before and after brushing, respectively. Numbers under (P_1, P_2, P_3, P_4) indicate the amount of stain at each of the four positions on the selected tooth of an individual. The data in Table A.1 are conditionally increasing in its coordinates.

The state probabilities are given in Table A.2. The conditional probabilities $P(X(t) = j|X(s) = i)$ for $i, j \in \{0, 1, 2, 3\}$ for the four sets of data are calculated in Table A.3.

To get an ordering in terms of association we have to compute the measure of association. The values are obtained in Table A.4.

This shows that the data in the third (P_3) position is more associated. This information may be useful to medical practitioners.

5. Summary

The degree of association in time of a Markov process can be measures using proposed measures which are based on transition probability function. The measure can be used to compare two Markov process according to the degree of association. A weaker condition for association of a Markov process in time is derived. The proposed measure can be used in various areas such as engineering, medical, social science *etc.*

Acknowledgements

We are grateful to an anonymous referee who suggested many improvements and furthermore generously listed many useful references. Finally, we thank the Chair Editor for his encouragement, guidance and counsel.

References

Barlow, R. E., and Proschan, F., (1975). *Statistical Theory of Reliability and Life Testing.* New York: Holt, Rinehart and Winston.

Barlow, R. E., and Wu, A. S., (1978). Coherent system with multistate components. *Mathematics of Operations Research,* **3**, 275-281.

Dharmadhikari, A. D. and Dewan, I. (2006). Association in time of a vector valued process. *Statistics and Probability Letters,* **76**, 1147-1155.

Esary, J. D. and Proschan, F. (1970). A reliability bound for systems of maintained independent components. *Journal of the American Statistical Association*, **65**, 329-398.

Esary, J. D., Proschan, F. and Walkup, D. W. (1967). Association of random variables, with applications. *Annals of Mathematical Statistics*, **38**, 1466-1474.

Hjort, N. B., Natvig, B. and Funnemark, E.(1985). The association in time of a Markov process with applications to multistate reliability theory. *Journal of Applied Probability*, **22**, 473-479.

Johansen, S. (1977). *Product Integrals and Markov processes*, Preprint No. 3, University of Copenhagen, Institute of Mathematical Statistics.

Karlin, S. (1983). Association arrays in assessing forms of dependence between bivariate random variables. *Proceedings of the National Academy of Sciences*, USA, **80**, 647-651.

Kuber, M and Dharmadhikari, A. D. (1996). Association in time of a finite semi-Markov process. *Statistics and Probability Letters*, **26**, 125-133.

Manoharan, M. (1995). On a sufficient condition for association in time of a process, *Proceedings of the XV Indian Society of Probability and Statistics Conference* (M.S. University, Tirunelveli, India), 65-66.

Natvig, B. (1993). Strict and exact bounds for the availabilities in a fixed time interval for multistate monotone systems. *Scandanavian Journal of Statistics*, **20**, 171-175.

Prakasa Rao, B. L. S. and Dewan, I. (2001). *Associated sequences and related inference problems.* Elsevier, Shanbhag, D. N. and Rao, C. R. (Eds.), *Handbook of Statistics*, North-Holland, Amsterdam, **19**, 693-729.

Appendix A

Table A.1: Dental data stain before and after brushing

Individual	Before brushing				After brushing				Individual	Before brushing				After brushing			
	P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4		P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4
1	1	1	1	2	0	0	0	0	14	2	1	2	2	0	0	1	1
2	1	1	2	2	0	0	0	1	15	2	2	2	2	0	0	1	1
3	1	1	2	2	0	0	0	1	16	2	2	2	2	0	0	1	1
4	1	1	2	2	0	0	0	1	17	2	2	2	2	0	0	1	1
5	1	1	2	2	0	0	0	1	18	2	2	2	2	0	0	1	1
6	1	2	2	2	0	0	0	1	19	2	2	2	2	0	0	1	1
7	1	2	2	2	0	0	0	1	20	2	2	2	2	0	0	1	1
8	1	2	2	2	0	0	0	1	21	2	2	2	2	0	1	1	1
9	1	2	2	2	0	0	0	1	22	2	2	2	2	0	1	1	1
10	1	2	2	2	0	0	0	1	23	2	2	2	2	0	1	1	1
11	1	2	2	2	0	0	0	1	24	2	2	2	3	0	1	1	1
12	1	2	2	2	0	0	0	2	25	2	2	2	3	1	1	1	2
13	1	2	2	3	0	0	0	2									

Table A.2: State probabilities

P_1	P_2	P_3	P_4
$P(X(s)=1)=13/25$	$P(X(s)=1)=6/25$	$P(X(s)=1)=1/25$	$P(X(s)=1)=0$
$P(X(s)=2)=12/25$	$P(X(s)=2)=19/25$	$P(X(s)=2)=24/25$	$P(X(s)=2)=22/25$
$P(X(s)=3)=0$	$P(X(s)=3)=0$	$P(X(s)=3)=0$	$P(X(s)=3)=3/25$
$P(X(t)=0)=24/25$	$P(X(t)=0)=20/25$	$P(X(t)=0)=13/25$	$P(X(t)=0)=1/25$
$P(X(t)=1)=1/25$	$P(X(t)=1)=5/25$	$P(X(t)=1)=22/25$	$P(X(t)=1)=21/25$
$P(X(t)=3)=0$	$P(X(t)=3)=0$	$P(X(t)=3)=0$	$P(X(t)=2)=3/25$

Table A.3: The conditional probabilities $P(X(t)=j|X(s)=i)$ for $i,j \in \{0,1,2,3\}$

$X(t)$	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
$X(s)$																
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	$\frac{11}{12}$	$\frac{1}{12}$	0	0	$\frac{14}{19}$	$\frac{5}{19}$	0	0	$\frac{12}{24}$	$\frac{12}{24}$	0	0	$\frac{1}{22}$	$\frac{20}{22}$	$\frac{1}{22}$	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	0

Table A.4: Covariance

Position	P_1	P_2	P_3	P_4
$Cov(X(t), X(s))$	0	30/625	12/25	2/25